Building Wavelets on $]0,1[$ at large scales

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Abstract We present a new approach to the construction of orthonormal wavelets on the interval which allows to overcome the "non interacting boundaries" restriction of existing constructions, and therefore to construct wavelets for $]0,1[$ also at large scales in such a way that, in the range of validity of the existing constructions, the two approaches give the same result.

Key Words. Multiresolution analysis, wavelets on interval.

1 Introduction

One of the main limitations to the full applicability of wavelet methods for solving partial differential equations, in a realistic general context, is the issue of non trivial geometries. Lately the research in this respect has been mainly directed to the use of a domain decomposition approach. This has been proposed in several papers in several forms (conforming methods [CTU97, CTU00, DS99, CM00] , non conforming methods [BP99]), the common features of which is the splitting of the non trivial domain as the union of subdomains, some or all of which are conformal images of $]0,1[^n$ and can therefore be discretized by tensor product wavelets, obtained starting from the construction of wavelet bases on the unit interval.

The use of wavelets in this context has uncovered a limitation of the existing construction of wavelets on the interval. We recall that basically all constructions of multiresolution on the interval, which are actually implemented, are based on suitably modifying the scaling functions that cross the boundaries, and they work under the assumption that the modifications made at boundary 0 do not interact with the ones made at boundary 1. This reduces to constructing the sequence of spaces $V_j^{]0,1[}$ (where the parameter j corresponds to a mesh-size 2^{-j}) roughly with the restriction $j \geq j_0$, the limit level j_0 depending on the type of wavelet on the line which is chosen as a starting point. In dimension d, this restriction fixes the minimum number of degrees of freedom to 2^{dj_0} (where in realistic situations

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j_0 may assume values of the size of $4 \sim 5$). In one or two dimensions, in a monodomain situation, this is acceptable, and wavelet methods based on such construction have been successfully implemented.

However, when we go to a domain decomposition framework in three dimensions, the situation becomes totally unacceptable. Just to make an example, for $j_0 = 5$, in three dimensions, one would be forced to use a minimum number of degrees of freedom per subdomain equals to $2^{3 \cdot 5} = 2^{15} = 32768$, and this also for small subdomains!

The aim of this paper is to provide a new approach to the construction of multiresolution on $]0,1[$ which allows to weaken such restriction by constructing $V_j^{]0,1[}$ for $j \geq \widehat{j_0}$, with $\widehat{j_0} \ll j_0$ (for Daubechies wavelets $\widehat{j_0} = 0$ independently of the order of the starting multiresolution on the line). For simplicity we will restrict ourselves to the orthonormal case, though in principle the approach we propose applies also to the biorthogonal framework.

For $j \geq j_0$ (small scales), the spaces $V_j^{]0,1[}$ constructed by this new approach coincide with the ones constructed in the previously mentioned papers. This is important, since it is not necessary to implement everything from scratch, in the case that a code for the latter is already available. However this approach, based on the use of discrete extension operators acting directly on the coefficient sequences rather than on the corresponding functions, allows easily to treat also values of j for which it is not possible to decouple the modifications for the left and right boundaries.
At large scales (small j) the resulting space $V_j^{]0,1[}$ will coincide with the space of polynomials of degree M_j ($M_j \sim 2^j - c$, c depending on the starting multiresolution analysis for $\mathbb{R}$). For intermediate values of j ($j \sim j_0 - 1$), the functions in $V_j^{]0,1[}$ will be globally supported functions which will allow to reconstruct polynomials of the same degree as the ones reconstructed by the original multiresolution.

2 Multiresolution on $]0,1[$

2.1 MRA on $L^2(\mathbb{R})$

The starting point to build all multiresolution analyses on $]0,1[$ is a multiresolution analysis on $\mathbb{R}$ [Mey90, Dau92]. In this section we briefly recall some definitions and properties that will be useful later on.

We assume that we are given an orthonormal, compactly supported, multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ on $\mathbb{R}$, that is a sequence of closed subspaces of $L^2(\mathbb{R})$, such that the following properties are satisfied:

i) the subspaces are nested: $V_j \subset V_{j+1}$ for all j;

ii) the union of the spaces is dense in $L^2(\mathbb{R})$ and the intersection is null:

$$\overline{\cup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}), \qquad \cap_{j \in \mathbb{Z}} V_j = \{0\};$$

iii) there exists a compactly supported *scaling function* $\varphi \in V_0$ such that, denoting by $\varphi_{j,k} = 2^{j/2}\varphi(2^j \cdot -k)$, the family $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j.

It is well known that properties i) and iii) imply the existence of reals h_k such that the following *refinement equation* holds:

$$\varphi(x) = \sqrt{2} \sum_{k=-\infty}^{+\infty} h_k \varphi(2x-k). \tag{1}$$

Without loss of generality, after possibly replacing φ with one of its integer translates, we can assume that the support of φ is "centered" around the origin: for some $N \in \mathbb{N}$

$$supp\varphi \subseteq [-N, N], \qquad |supp\varphi| \geq 2N-1,$$

which is equivalent to saying that

$$h_k = 0 \qquad \text{for all} \qquad k \notin [-N, N]\,. \tag{2}$$

Equation (1) can then be rewritten as

$$\varphi(x) = \sqrt{2} \sum_{k=-N}^{N} h_k \varphi(2x-k). \tag{3}$$

Let $P_j : L^2(\mathbb{R}) \longrightarrow V_j$ denote the L^2 orthogonal projection onto V_j

$$P_j f = \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{R}} f \varphi_{j,k}\, dx \right) \varphi_{j,k},$$

and let $W_j = (I - P_j)V_{j+1}$ be the orthogonal complement of V_j in V_{j+1}:

$$V_{j+1} = V_j \oplus W_j, \qquad W_j \perp V_j.$$

It is well known that $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$. Moreover, letting the *mother wavelet* $\psi \in W_0 \subset V_1$ be defined as

$$\psi(x) = \sqrt{2} \sum_{k=-N+1}^{N+1} g_k \varphi(2x-k) \qquad \text{with} \qquad g_k = (-1)^k h_{1-k}, \tag{5}$$

it is well known that the set $\{\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j. It is easy to check that

$$supp\,\psi \subset [1/2 - N, 1/2 + N]\,. \tag{6}$$

We make the further assumption that the space P_M of polynomials of degree less or equal than M is exactly reproduced by the set $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$; that is, if $p \in P_M$, it holds that

$$p(x) = \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}} p(x)\varphi(x-n)\, dx \right) \varphi(x-n). \tag{7}$$

We observe that the scalar products on the right hand side are well defined since $p(x) \in L^2_{loc}(\mathbb{R})$ and the scaling function φ is compactly supported. Moreover the sum converges pointwise since, for all $x \in \mathbb{R}$, only a finite number of terms is nonzero.

In the following we will make use of some well known properties of MRAs. In particular we will make use of the fact that a function is a polynomial if and only if its "scaling coefficients" constitute themselves a polynomial sequence. More precisely we recall that if, for $f \in L^2_{loc}(\mathbb{R})$, we denote by $f^j_k = \langle f, \varphi_{j,k} \rangle$ the $L^2(\mathbb{R})$ scalar product of f with the scaling function $\varphi_{j,k}$, the following lemma holds.

Lemma 2.1 *If f is a polynomial then the sequence $\{f^j_k\}_{k\in\mathbb{Z}}$ of its scaling coefficients is a polynomial of the same degree in the variable k. Conversely, if p is a polynomial of degree less or equal than M, then the function $f = \sum p(k)\varphi_{j,k} \in L^2_{loc}(\mathbb{R})$ is a polynomial of the same degree.*

In view of (1), an immediate consequence of lemma 2.1 is the following Corollary.

Corollary 2.1 *If the sequence f_k is a polynomial of degree less or equal than M in the variable k, then the sequence $b_k = \sum_l h_{2k-l} f_l$ is a polynomial in k of the same degree.*

2.2 The classical construction

Different ways have been proposed to adapt the multiresolution analysis to $L^2(]0,1[)$ [Mey90, CDV93, AHJP93, CL97, DKU99, TT99, MP95]. The most successful class of constructions is based on the idea of retaining all those scaling functions $\varphi_{j,k}$ such that $supp\varphi_{j,k} \subset [0,1]$, and adding suitable linear combinations of those scaling functions which cross the left (resp. right) boundary of the interval $]0,1[$, in such a way that a polynomial reproduction property analogous to (7) holds.

For the purpose of comparing such constructions with the one proposed here, let us briefly review their main features. To fix the ideas we refer to the work by P. Monasse and V. Perrier [MP95]. For $j \geq j_0 = [log_2 4N]$, the vector space $V_j^{]0,1[}$ is obtained as the span of the following functions:

i) $2^{j/2}\widetilde{\varphi}^0_k(2^j x)|_{]0,1[}$, for $k = 0, \cdots, M$,

ii) $\varphi_{j,k}$, for $k = N, \cdots, 2^j - N$,

iii) $2^{j/2}\widetilde{\varphi}^1_k(2^j(x-1))|_{]0,1[}$, for $k = 0, \cdots, M$,

where the edge scaling functions $\widetilde{\varphi}^0_k$ and $\widetilde{\varphi}^1_k$ are defined as a linear combinations of the $\varphi_{j,k}$ crossing the respective boundaries (0 or 1) with suitable polynomial coefficients: for $k = 0, \cdots, M$

$$\widetilde{\varphi}^0_k(x) = \sum_{l=-N+1}^{N-1} P^0_k(l)\varphi(x-l),$$

$$\widetilde{\varphi}_k^1(x) = \sum_{l=-N+1}^{N-1} P_k^1(l)\varphi(x-l),$$

where $P_0^0, \cdots, P_M^0$ and $P_0^1, \cdots, P_M^1$ are suitable bases for the space of polynomials of degree M. The $\varphi_{j,k}$ for $k = N, \cdots, 2^j - N$ are the so called *interior scaling functions* (coinciding with those scaling functions in V_j whose support is included in $[0,1]$), while the $\widetilde{\varphi}_k^0$ and the $\widetilde{\varphi}_k^1$ for $k = 0, \cdots, M$ are the so called *edge scaling functions* at the boundaries 0 and 1 respectively. An easy calculation yields the following identities:

$$2^{j/2}\widetilde{\varphi}_k^0(2^j x) = \sum_{l=-N+1}^{N-1} P_k^0(l)\varphi_{j,l}(x) \tag{8}$$

$$2^{j/2}\widetilde{\varphi}_k^1(2^j(x-1)) = \sum_{l=2^j-N+1}^{2^j+N-1} \check{P}_k^1(l)\varphi_{j,l}(x) \tag{9}$$

with $\check{P}_k^1(x) = P_k^1(x - 2^j)$.

It is beyond the goal of this paper to throughly review the properties of such bases, for which we refer to the papers mentioned above.

2.3 The new construction

In order to construct a MRA on $]0,1[$, we start again from a MRA on $L^2(\mathbb{R})$ verifying the assumptions of Section 2.1. Let us for convenience introduce the notation V_j^{loc}

$$V_j^{loc} = span < \varphi_{j,k}, \quad k \in \mathbb{Z} >^{L^2_{loc}(\mathbb{R})} = \left\{ \sum_k c_k \varphi_{j,k}, \quad c_k \in \mathbb{R}\ \forall k \in \mathbb{Z} \right\}.$$

The idea is to define our MRA as the restriction to $]0,1[$ of the subspace V_j^* of V_j^{loc} of functions, the sequence of whose coefficients has, across the boundaries, a polynomial behavior. The degree of the polynomial is set depending on j in such a way that, as we will see, the subspace is well defined even at much larger scale than $j = j_0$. More precisely we give the following definition:

Definition. *For every j, let*

$$N_j = 2^j - 2N + 2M + 3,$$

$$M_j = min\{M, N_j - 1\},$$

and set

$$V_j^* := \{ f = \sum_{k\in\mathbb{Z}} f_k^j \varphi_{jk} \quad : \quad \exists\ p_l\,,\, p_r \in P_{M_j} \quad s.t$$

$$f_k^j = p_l(k) \quad \forall\ k \in\ ]-\infty, N-1], \tag{10}$$

$$f_k^j = p_r(k) \quad \forall\ k \in\ [2^j - N + 1, +\infty[\,\}. \tag{11}$$

Remark 2.1 *V_j^* is well defined for all $j \geq \widehat{j_0}$, where $\widehat{j_0}$ is the smallest j such that $N_j - 1 > 0$; that is*

$$\widehat{j_0} = [log_2(2(N-M)-1)] .$$

We observe that, unlike j_0, $\widehat{j_0}$ does not depend directly on N, but on the difference $N-M$; if we consider, for example, Daubechies wavelets we have $N-M=1$, so that we can construct V_j^* for all j's such that $j \geq 0$.

Remark 2.2 *The two polynomials p_l and p_r in the definition of V_j^* are not necessarily independent; actually, it is not difficult to check (see Proposition 2.4) that for $j \leq [\log_2(2(N-1)-M)]$ the two polynomials will always coincide.*

Remark 2.3 *The parameters N_j and M_j will respectively be the dimension of V_j^* and the degree of polynomials exactly reproduced in V_j^*.*

The candidate to form a multiresolution analysis on $]0,1[$ is the sequence of spaces $V_j^*|_{]0,1[} \subseteq L^2(]0,1[)$. The nestedness property will be a trivial consequence of the following proposition.

Proposition 2.1 *The sequence $\{V_j^*\}_{j \geq \widehat{j_0}}$ satisfies*

$$V_j^* \subset V_{j+1}^* \qquad \textit{for all } j \geq \widehat{j_0}.$$

Proof: Let $f \in V_j^*$, $f = \sum_k f_k^j \varphi_{j,k}$. Using equation (3) we obtain that

$$f = \sum_k f_k^{j+1} \varphi_{j+1,k},$$

where the the sequence $\{f_k^{j+1}\}_k$ satisfies

$$f_k^{j+1} = \frac{1}{\sqrt{2}} \sum_n h_{k-2n} f_n^j.$$

We can restrict the above sum to those values of n such that $h_{k-2n} \neq 0$, that is, by the property (2), to n such that $(k-N)/2 \leq n \leq (k+N)/2$; we now observe that $k \leq N-1$ implies that $n \leq N-1$, and then, by the definition of V_j^*, we have that

$$f \in V_j^* \Longrightarrow \qquad f_n^j = p_l(n) \qquad \forall\, n \leq N-1,$$

with $p_l \in P_{M_j}$. Using Lemma (2.1) it results that $f_k^{j+1} = p(k)$ for some polynomial $p \in P_{M_j} \subseteq P_{M_{j+1}}$.

In a similar way, it is easy to show that, if $k \geq 2^{j+1} - N + 1$,

$$f_k^{j+1} = q(k)$$

for some polynomial $q \in P_{M_j} \subseteq P_{M_j+1}$. By definition of V_{j+1}^* this implies that $f \in V_{j+1}^*$. $\square$

Using lemma 2.1 it is also not difficult to prove the following proposition.

Proposition 2.2 *The space V_j^* contains the polynomials of degree lower or equal than M_j.*

We now need to check that defining a multiresolution analysis on the interval by taking the restriction to $]0,1[$ of the spaces V_j^* is consistent with the classical definition of $V_j^{]0,1[}$, as reported in section 2.2, that is for $j \geq j_0$ it holds that $V_j^*|_{]0,1[} \equiv V_j^{]0,1[}$. In order to do that, and also to construct a basis for the spaces $V_j^*|_{]0,1[}$ and later on for the complement spaces, it will be convenient to introduce some operator allowing to select a particular function in V_j^* "corresponding" to a given function in V_j^{loc}.

It is easy to see that, for all the values of $j \geq \widehat{j_0}$, the set of coefficients $\{f_k^j\}_{k\in\mathbb{Z}}$ of a function $f \in V_j^*$ is uniquely determined by the finite set $\{f_k^j\}_{k\in I_j}$ with I_j defined by

$$I_j = \{N-M-1, \cdots, 2^j - N + M + 1\};$$

the remaining coefficients $\{f_k^j\}_{k\notin I_j}$ can be obtained via an "extension" operator E_j which extrapolates the sequence $\{f_k^j\}_{k\in I_j}$ on the left and on the right by suitable polynomials of degree M_j. More precisely, given any vector $c = \{c_k\}_{k\in I_j} \in \mathcal{S}(I_j)$ ($\mathcal{S}(\mathcal{I})$ denoting the space of real valued sequences with indexes in the set $\mathcal{I}$), let $p_l(c;\cdot)$ and $p_r(c;\cdot)$ be the polynomials assuming the values $c_{N-M-1}, \cdots, c_{N-M-1+M_j}$ at the points $N-M-1, \cdots, N-M-1+M_j$ and $c_{2^j-N+M+1-M_j}, \cdots, c_{2^j-N+M+1}$ at the points $2^j-N+M+1-M_j, \cdots, 2^j-N+M+1$ respectively. More precisely, set

$$p_l(c;x) = \sum_{m=N-M-1}^{N-M-1+M_j} c_m L_{M_j,m}(x),$$

with

$$L_{M_j,m}(x) = \prod_{\substack{i=N-M-1\\ i\neq m}}^{N-M-1+M_j} \frac{x-i}{m-i},$$

and

$$p_r(c;x) = \sum_{m=2^j-N+M+1-M_j}^{2^j-N+M+1} c_m L_{M_j,m}(x),$$

with

$$L_{M_j,m}(x) = \prod_{\substack{i=2^j-N+M+1-M_j\\ i\neq m}}^{2^j-N+M+1} \frac{x-i}{m-i}.$$

The linear extension operator $E_j : \mathcal{S}(I_j) \longrightarrow \mathcal{S}(\mathbb{Z})$ is then defined as follows: for $c = (c_k)_{k\in I_j} \in \mathcal{S}(I_j)$ set

$$(E_j c)_k = \begin{cases} p_l(c;k) & \forall k,\ k \leq N-M-2 \\ c_k & \forall k,\ N-M-1 \leq k \leq 2^j - N + M + 1 \\ p_r(c;k) & \forall k,\ k \geq 2^j - N + M + 2. \end{cases} \tag{12}$$

Via E_j, we can define an operator

$$\mathcal{E}_j : V_j^{loc} \longrightarrow V_j^*$$

which associates to every function in V_j^{loc} a "corresponding" function in V_j^*: given $f = \sum f_k^j \varphi_{j,k} \in V_j^{loc}$, we define $\mathcal{E}_j f$ as the unique element in V_j^* whose k-th coefficient coincide with the k-th coefficient of f for all $k \in I_j$. More precisely, if $\underline{f}^j = \left(f_k^j\right)_{k \in I_j}$ denotes the vector of "relevant" scaling coefficients of the function f ("relevant" meaning "corresponding to a k in I_j", that is corresponding to a degree of freedom for V_j^*), $\mathcal{E}_j : V_j^{loc} \longrightarrow V_j^*$ is defined as follows:

$$\mathcal{E}_j(f) = \sum_{k=-\infty}^{+\infty} (E_j \underline{f}^j)_k \varphi_{j,k}|_{]0,1[}. \tag{13}$$

Remark 2.4 *It is not difficult to check that*

$$V_j^* = \mathcal{I}m(\mathcal{E}_j), \tag{14}$$

that is every function in V_j^* is the image of a function in V_j^{loc} via the operator $\mathcal{E}_j$. This is a trivial consequence of the fact that for all $f \in V_j^*$ it holds $\mathcal{E}_j(f) = f$. Moreover it is also immediate to prove that

$$\mathcal{E}_j(V_j^{loc}) = \mathcal{E}_j(span < \varphi_{j,k}, \quad k \in I_j >).$$

It is now not difficult to prove the following proposition, showing that, for large values of j, the space $V_j^*|_{]0,1[}$ turns out to be exactly $V_j^{]0,1[}$ as defined in section 2.2.

Proposition 2.3 *For all j such that $j \geq j_0$ it holds*

$$V_j^*|_{]0,1[} \equiv V_j^{]0,1[}.$$

Proof: Since $V_j^*|_{]0,1[}$ is the image of a linear operator applied to a space of dimension equals to $\#I_j$ we have that

$$dim(V_j^*|_{]0,1[}) \leq \#I_j = 2^j - 2N + 2M + 3 = N_j = dim(V_j^{]0,1[}).$$

Then, it is sufficient to prove that $V_j^{]0,1[} \subseteq V_j^*|_{]0,1[}$, or equivalently that every basis function of $V_j^{]0,1[}$ belongs to $V_j^*|_{]0,1[}$.
It is easy to see that the interior scaling functions satisfy (10) and (11) for $p_l = p_r = 0$, while for the edge scaling functions, the thesis is a simple consequence of (8) and (9). $\square$

Thanks to the previous proposition, the following definition is then consistent with the classical one.

Definition. *For all $j \geq \widehat{j_0}$, we define $V_j^{]0,1[}$ to be the restriction of V_j^* to the unit interval:*

$$V_j^{]0,1[} := V_j^*|_{]0,1[}.$$

Propositions 2.1 and 2.2 yield then trivially the following corollary.

Corollary 2.2 *The sequence* $\{V_j^{]0,1[}\}_{j\geq\widehat{j_0}}$ *satisfies*

$$V_j^{]0,1[} \subset V_{j+1}^{]0,1[} \qquad \textit{for all } j \geq \widehat{j_0}.$$

Moreover $V_j^{]0,1[}$ *contains the polynomials of degree lower or equal than* M.

For $j \geq j_0$ the spaces $V_j^{]0,1[}$ have been studied in several papers and their properties are well understood. Let us then give a closer look at the structure of $V_j^{]0,1[}$ for small values of j. The following proposition holds.

Proposition 2.4 *Let* j *be such that* $\widehat{j_0} \leq j \leq [log_2(2(N-1)-M)]$*; then* $V_j^{]0,1[}$ *coincides with the space of polynomials of degree* M_j.

Proof: According to the definition of V_j^*, the two polynomial sequences p_l and p_r are uniquely determined by their M_j+1 values at the nodes $k = N-1-M_j, \cdots, N-1$ and $k = 2^j - N + 1, \cdots, 2^j - N + 1 + M_j$ respectively. If $j \leq [log_2(2(N-1)-M)]$ (which is equivalent to $N_j - 1 \leq M$), then $2^j - N + 1 + M_j \leq N - 1$; this implies that

$$p_l(k) = f_k^j = p_r(k),$$

that is the two polynomials coincide. Letting then $p = p_l = p_r$, we have then

$$f_k^j = p(k) \quad \text{for all}\, k, \quad \text{with}\, deg(p) = M_j$$

and, from Lemma 2.1, it follows that f is a polynomial of degree M_j. □

Remark 2.5 *For the values of* j *such that* $[log_2(2(N-1)-M)] < j < j_0$*, the functions of* $V_j^{]0,1[}$ *are globally supported and they have a polynomial behavior near the boundaries* 0 *and* 1.

2.4 A basis for $V_j^{]0,1[}$

A Riesz's basis $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{k\in I_j}$ for $V_j^{]0,1[}$ can be constructed by simply applying the operator $\mathcal{E}_j$ defined in (13) to a suitable subset of the basis $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ of V_j. More precisely, we introduce the following definition.

Definition. *For every* $k = I_j$ *let* $\breve{\varphi}_{j,k}^{]0,1[} \in V_j^{]0,1[}$ *be defined as follows:*

$$\breve{\varphi}_{j,k}^{]0,1[} = \mathcal{E}_j \varphi_{j,k}|_{]0,1[}. \tag{15}$$

In particular, denoting by $\delta^{j,k} = (\delta^{j,k}_n)_{n\in I_j} \in \mathcal{S}(I_j)$

$$\delta^{j,k}_n = \begin{cases} 0 & \text{if } n \neq k \\ 1 & \text{if } n = k, \end{cases}$$

the vector of length $2^j - 2N + 2M + 3$, whose components are all zero but the k-th, which takes value 1, by definition the functions $\breve{\varphi}^{]0,1[}_{j,k}$ are a linear combination of the functions $\varphi_{j,m}|_{]0,1[}$ with coefficients $\eta^{j,k}_m$ obtained by extrapolating $\{\delta^{j,k}_n\}_{n\in I_j}$:

$$\breve{\varphi}^{]0,1[}_{j,k} = \sum_{m=-N}^{N+2^j} \eta^{j,k}_m \varphi_{j,m}|_{]0,1[} \qquad \text{with} \qquad \eta^{j,k} = (E_j \delta^{j,k}), \tag{16}$$

where we consider in the sum only those values of m for which $\varphi_{j,m}|_{]0,1[}$ does not vanish. The following proposition holds:

Proposition 2.5 *The set $\{\breve{\varphi}^{]0,1[}_{j,k}\}_{k\in I_j}$ is a Riesz basis for $V^{]0,1[}_j$.*

Proof: We first observe that the set $\{\breve{\varphi}^{]0,1[}_{j,k}\}_{k\in I_j}$ generates $V^{]0,1[}_j$. In fact, thanks to Remark (2.4), every $g \in V^{]0,1[}_j$ is the restriction of the image by the operator $\mathcal{E}_j$ of a corresponding function f in *span* $< \varphi_{j,k}, \quad k \in I_j >$:

$$g = \mathcal{E}_j f|_{]0,1[} = \sum_{m=-N}^{N+2^j} (E_j \underline{f}^j)_m \varphi_{j,m}|_{]0,1[},$$

where we recall that $\underline{f}^j = (f^j_k)_{k\in I_j}$ is the vector of those scaling coefficients of f corresponding to an index $k \in I_j$. $\underline{f}^j$ can also be written as

$$\underline{f}^j = \sum_{k=N-M-1}^{2^j-N+M+1} f^j_k \delta^{j,k}.$$

Now, using the linearity of the operator E_j, it follows that

$$\begin{aligned} g &= \sum_{m=-N}^{N+2^j} \left(\sum_{k=N-M-1}^{2^j-N+M+1} f^j_k (E_j \delta^{j,k}) \right)_m \varphi_{j,m}|_{]0,1[} = \\ &= \sum_{k=N-M-1}^{2^j-N+M+1} f^j_k \sum_{m=-N}^{N+2^j} \eta^{j,k}_m \varphi_{j,m}|_{]0,1[} = \sum_{k=N-M-1}^{2^j-N+M+1} f^j_k \breve{\varphi}^{]0,1[}_{j,k}. \end{aligned}$$

Let us now prove that the set $\{\breve{\varphi}^{]0,1[}_{j,k}\}_{k\in I_j}$ is also linearly independet. Suppose in fact that for some scalars α_k it holds

$$\sum_{k\in I_j} \alpha_k \breve{\varphi}^{]0,1[}_{j,k} = 0.$$

This rewrites

$$\sum_{m=-N+1}^{2^j+N-1} \alpha_m \varphi_{j,m}|_{]0,1[} = 0 \tag{17}$$

where, for $m \notin I_j$, α_m is given by

$$\begin{aligned}
\alpha_m &= \left(\sum_{k \in I_j} \eta_m^{j,k} \alpha_k\right), \qquad && m = -N+1, \ldots, N-M-2 \\
\alpha_m &= \left(\sum_{k \in I_j} \eta_m^{j,k} \alpha_k\right), \qquad && m = 2^j - N + M + 2, \ldots, 2^j + N - 1.
\end{aligned}$$

Now, it is well known (see for instance [Lem91]) that the set $\{\varphi_{j,k}, \quad k = -N + 1, \ldots, 2^j + N - 1\}$ is linearly independent in $L^2(]0,1[)$. Then equation (17) implies that

$$\alpha_m = 0 \qquad \text{for all } m \in [-N+1, 2^j + N - 1] \supset I_j.$$

In particolar $\alpha_m = 0$ for all $m \in I_j$, hence the set $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{k \in I_j}$ is linearly independent.

For $j \geq j_0$, the fact that the set $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{k \in I_j}$ is a Riesz basis for $V_j^{]0,1[}$ can be proven by some nowadays standard arguments. For $\widehat{j_0} \leq j < j_0$ this descends from the fact that $\| \sum_{k \in I_j} f_k^j \breve{\varphi}_{j,k}^{]0,1[} \|_{L^2}^2$ and $\sum_{k \in I_j} |f_k^j|^2$ are norms on the finite dimensional space $V_j^{]0,1[}$, hence they are equivalent. □

Remark 2.6 *It is not difficult to check that the above reasoning also yields that for all functions $f \in V_j^*$ it holds*

$$f|_{]0,1[} = \sum_{k \in I_j} \left(\int_{\mathbb{R}} f \varphi_{j,k} \right) \breve{\varphi}_{j,k}^{]0,1[},$$

that is for such functions, the coefficients of their restriction to $]0,1[$ with respect to the basis $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{k \in I_j}$, can be retrieved by integrating over the whole $\mathbb{R}$ the function f times the original basis function on the line $\varphi_{j,k}$. A consequence is that, if the basis functions $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{k \in I_j}$ were to be used in a biorthogonal framework, some of the steps of the corresponding biorthogonal wavelet transform (namely the recostruction step that allows the computation of the coefficients in $V_{j+1}^{]0,1[}$ of a function in $V_j^{]0,1[}$) would reduce to performing an extension E_j and then the usual FWT on the line. We will not exploit this feature in the present paper, where we rather concentrate on obtaining an orthonormal setting.

Remark 2.7 *For $j \geq j_0$, the linear independency of the functions $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{k \in I_j}$ could also be proven by a dimensional argument. In fact we have a generator system of cardinality $\#(I_j) = dim V_j^{]0,1[}$. This implies that the generator system is a basis. However this argument cannot be applied for $j < j_0$, since in such case we only know that $dim V_j^{]0,1[} \leq \#(I_j)$.*

If $j \geq j_0$, the basis $\{\breve{\varphi}_{j,k}^{]0,1[}, \quad k \in I_j\}$ just constructed has the the same structure as the basis obtained (before orthonormalizing) using the classical constructions. In particular it is easy to see that, for k such that $N \leq k \leq 2^j - N$, the scaling functions $\breve{\varphi}_{j,k}^{]0,1[}$ verify $\breve{\varphi}_{j,k}^{]0,1[} \equiv \varphi_{j,k}$, since the extension operator has no effects on the corresponding scaling coefficients. Such functions are usually called *interior*, while we will refer to the functions interacting with the left boundary ($k = N - M - 1, \cdots, N - 1$), resp. with the right boundary ($k = 2^j - N + 1, \cdots, 2^j - N + M + 1$), as *left boundary* (resp. *right boundary*) scaling functions. The following proposition holds.

Proposition 2.6 *For $j \geq j_0$ the interior scaling functions $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{k=N}^{2^j-N}$ are orthogonal to the left boundary scaling functions $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{k=N-M-1}^{N-1}$ and to the right boundary scaling functions $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{2^j-N+1}^{2^j-N+M+1}$.*

Proof: Let k be such that $N - M - 1 \leq k \leq N - 1$. It is easy to see that, for such values of k, the sum in (16) becomes

$$\breve{\varphi}_{j,k}^{]0,1[} = \sum_{m=-N}^{N-1} \eta_m^{j,k} \varphi_{j,m}|_{]0,1[},$$

since the remaining component of the extended coefficients are either zero or correspond to scaling functions $\varphi_{j,m}$ verifying $\varphi_{j,m}|_{]0,1[} = 0$. Therefore, for l such that $N \leq l \leq 2^j - N$ we have

$$\langle \breve{\varphi}_{j,k}^{]0,1[}, \breve{\varphi}_{j,l}^{]0,1[} \rangle = \langle \breve{\varphi}_{j,k}^{]0,1[}, \varphi_{j,l} \rangle = \sum_{m=-N}^{N-1} \eta_m^{j,k} \langle \varphi_{j,m}, \varphi_{j,l} \rangle = 0.$$

The second part of the proof is analogous to the first one. □

The boundary scaling functions are scale invariant, as the following proposition states.

Proposition 2.7 *If $j \geq j_0$, for the values of k such that $N - M - 1 \leq k \leq N - 1$ (left boundary functions) it holds that*

$$\breve{\varphi}_{j,k}^{]0,1[}(x) = 2^{(j-j_0)/2} \breve{\varphi}_{j_0,k}^{]0,1[}(2^{j-j_0} x), \tag{18}$$

while for the values of k such that $2^j - N + 1 \leq k \leq 2^j - N + M + 1$ (right boundary functions) it holds that

$$\breve{\varphi}_{j,k}^{]0,1[}(x) = 2^{(j-j_0)/2} \breve{\varphi}_{j_0,k-2^j+2^{j_0}}^{]0,1[}(2^{j-j_0}(x-1)+1). \tag{19}$$

Proof: We will prove the scale invariance only for the left boundary functions, since for the right ones the proof is essentially the same.
Since $j \geq j_0$, and k verifies $N - M - 1 \leq k \leq N - 1$, it follows that

$$\breve{\varphi}_{j,k}^{]0,1[} = \sum_{m=-N}^{N-1} \eta_m^{j,k} \varphi_{j,m}|_{]0,1[}, \tag{20}$$

where we observe that

$$\eta_m^{j,k} = (E_j \delta^{j,k})_m = \prod_{i=N-M-1}^{N-1} \frac{m-i}{k-i},$$

is independent of j, since the polynomial extension is scale invariant. In particular we have that

$$\breve{\varphi}_{j+1,k}^{]0,1[} = \sum_{m=-N}^{N-1} \eta_m^{j+1,k} \varphi_{j+1,m}|_{]0,1[} = \sum_{m=-N}^{N-1} \eta_m^{j,k} \varphi_{j+1,m}|_{]0,1[}. \tag{21}$$

Therefore, simply recalling that

$$\varphi_{j,l}(x) = 2^{j/2}\varphi(2^j x - l) \quad \text{and} \quad \varphi_{j+1,l}(x) = 2^{(j+1)/2}\varphi(2^{j+1}x - l),$$

comparing (20) and (21) and setting $y = 2x$, yields

$$\breve{\varphi}_{j+1,k}^{]0,1[}(y) = \sqrt{2}\breve{\varphi}_{j,k}^{]0,1[}(2y).$$

By induction on j it is then not difficult to conclude that (18) holds. $\square$

An orthonormal basis $\{\varphi_{j,k}^{]0,1[}\}_{k\in I_j}$ for $V_j^{]0,1[}$ can be obtained by simply applying an orthonormalization procedure to the basis $\{\breve{\varphi}_{j,k}^{]0,1[}\}_{k\in I_j}$. Since for $j \geq j_0$, $V_j^{]0,1[}$ is nothing else than the space defined by classical definition, any of the procedures (preserving localization) of the papers quoted at the beginning of the section can be used. As far as $j < j_0$ is concerned, we observe that the corresponding space V_j^* is, by nature, global. The support of all elements of V_j^* is in fact the entire interval $]0,1[$. Therefore, there is no particular need to use some specific orthonormalization procedure (since localization is not achievable), and then any approach, like for instance the application of a Gram-Schmidt procedure can be applied. Figure 1 shows some examples of orthonormalized scaling functions obtained by a Daubechies MRA.

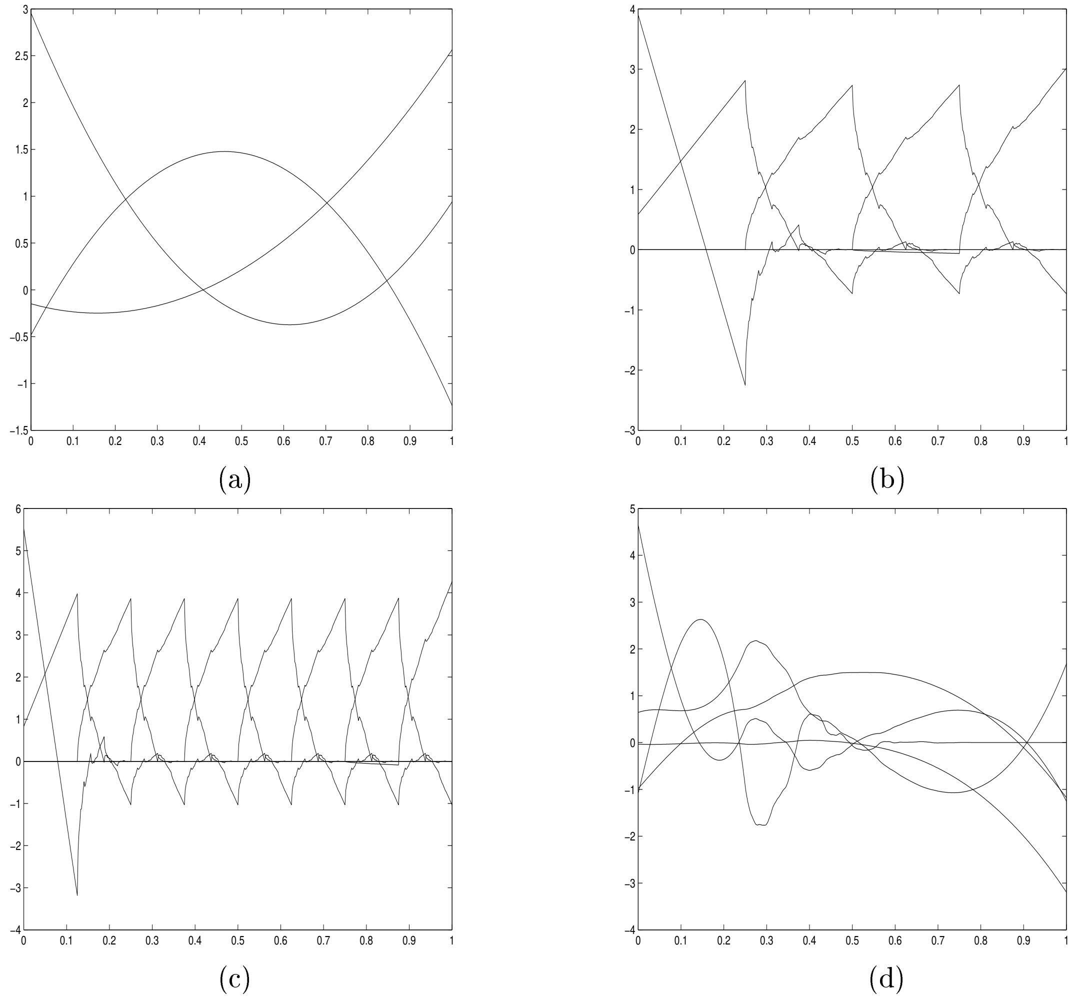
3 Wavelets

The aim of this section is the construction of an orthonormal wavelet basis for $L^2(]0,1[)$ corresponding to the multiresolution analysis just introduced. Let $P_j^{]0,1[} : L^2(\mathbb{R}) \longrightarrow V_j^{]0,1[}$ denote the $L^2(]0,1[)$ orthogonal projection onto $V_j^{]0,1[}$, and let $W_j^{]0,1[}$ be the complement space of $V_j^{]0,1[}$ in $V_{j+1}^{]0,1[}$:

$$W_j^{]0,1[} = (I - P_j^{]0,1[})V_{j+1}^{]0,1[},$$

$$W_j^{]0,1[} \oplus V_j^{]0,1[} = V_{j+1}^{]0,1[}, \quad \text{and} \quad V_j^{]0,1[} \perp W_j^{]0,1[}.$$

For $j \geq j_0$ several constructions of bases for $W_j^{]0,1[}$, basically consisting in retaining "interior" wavelets and adding the projection on $W_j^{]0,1[}$ of the right number of suitably chosen scaling functions of $V_{j+1}^{]0,1[}$, are available.



(a) (b) (c) (d)

Figure 1: Orthonormal Daubechies scaling functions; (a) Db3 scaling functions for $j = 1$. For such value of j, $V_j^{]0,1[}$ is the space of polynomials of degree 2. Figures (b) and (c) show Db2 scaling functions for $j = 2$ and $j = 3$ respectively. This is the case $j \geq j_0$, so that the construction consists in retaining the *interior* functions and modifying only those one crossing the boundaries. (d) Db4 scaling functions for $j = 2$; this is the case the functions are globally supported on]0,1[.

Using a standard basis completion argument, for $j < j_0$ it is always possible to choose a subset of 2^j functions $\{\varphi_{j+1,n}^{]0,1[}\}_{n \in I_j^w}$ (with $I_j^w \subset I_{j+1}$, $\#I_j^w = 2^j$) out of the basis $\{\varphi_{j+1,k}^{]0,1[}\}_{k \in I_{j+1}}$ in such a way that, together with the basis $\{\check{\varphi}_{j,k}^{]0,1[}\}_{k \in I_j}$, they form a new basis for $V_{j+1}^{]0,1[}$:

$$V_{j+1}^{]0,1[} = span < \varphi_{j+1,n}^{]0,1[}, \quad n \in I_j^w, \qquad \check{\varphi}_{j,k}^{]0,1[}, \quad k \in I_j > .$$

Then it is trivial to show that the set of functions

$$\check{\psi}_{j,k}^{]0,1[} = \varphi_{j+1,k}^{]0,1[} - P_j^{]0,1[} \varphi_{j+1,k}^{]0,1[}, \quad k \in I_j^w$$

forms a basis for $W_j^{]0,1[}$, which again can be orthonormalized by any technique (recall that for $j \leq j_0$ all functions are global by nature, so there is no need to look for a special technique).

Alternatively we propose, in the following section, a way of constructing a basis for $W_j^{]0,1[}$ that unifies the case $j \geq j_0$, and $j < j_0$. Numerical evidence seems to indicate that the technique proposed has good stability properties (the basis functions have an L^∞ bound with a reasonable constant). Unfortunately a number of assumptions on the linear independence of certain (small) sets of functions are needed in order for such a construction to work. These have to be verified "a posteriori" case by case. In all the cases we tested such assumptions did hold. We want to underline that, in the case in which such assumptions did not hold, one can always resort to the previous wavelet constructions for $j \geq j_0$ and to the basis completion technique mentioned above for $\widehat{j_0} \leq j < j_0$.

3.1 A wavelet basis

In order to construct a basis for $W_j^{]0,1[}$, we start by considering the wavelet functions $\psi_{j,k} \in W_j$. We recall that, using the two-scale equation (5), $\psi_{j,k} \in W_j \subset V_{j+1}$ can be written as follows:

$$\psi_{j,k} = \sum_n g_n \varphi_{j+1,2k+n}.$$

By applying the extension operator $\mathcal{E}_{j+1}$ (13) we define for each $k \in \mathbb{Z}$ a function $\vartheta_{j,k}^{]0,1[} \in V_{j+1}^{]0,1[}$ by

$$\vartheta_{j,k}^{]0,1[} = \mathcal{E}_{j+1} \psi_{j,k}|_{]0,1[}.$$

It is easy to see that

$$\vartheta_{j,k}^{]0,1[} = \sum_{m \in I_{j+1}} g_{m-2k} \check{\varphi}_{j+1,m}^{]0,1[}.$$

Now, letting

$$I_j^w = \{0, \cdots, 2^j - 1\}$$

for every $k \in I_j^w$ we define

$$\check{\psi}_{j,k}^{]0,1[} = \vartheta_{j,k}^{]0,1[} - P_j^{]0,1[}\vartheta_{j,k}^{]0,1[}.$$

We point out that for $j \geq j_0$, such construction gives us essentially the same basis as the one constructed in the papers quoted at the beginning of section 2.2. In particular, for $k = N, \cdots, 2^j - N$, the functions $\check{\psi}_{j,k}^{]0,1[}$ coincide with the so called *interior wavelets*:

$$\check{\psi}_{j,k}^{]0,1[} = \psi_{j,k},$$

where, for such values of k it holds $\operatorname{supp} \psi_{j,k} \subset [0,1]$. Furthermore, it is easy to show that the boundary wavelets satisfy a scaling invariance property, as stated by the following proposition:

Proposition 3.1 *If $j \geq j_0$, for the values of k such that $0 \leq k \leq N-1$ (left boundary wavelets) it holds that*

$$\check{\psi}_{j,k}^{]0,1[}(x) = 2^{(j-j_0)/2}\check{\psi}_{j_0,k}^{]0,1[}(2^{j-j_0}x), \tag{22}$$

while for the values of k such that $2^j - N + 1 \leq k \leq 2^j - 1$ (right boundary wavelets) it holds that

$$\check{\psi}_{j,k}^{]0,1[}(x) = 2^{(j-j_0)/2}\check{\psi}_{j_0,k-2^j+2^{j_0}}^{]0,1[}(2^{j-j_0}(x-1)+1). \tag{23}$$

The following orthogonality property holds.

Proposition 3.2 *For $j \geq j_0$ the* interior wavelets $\{\check{\psi}_{j,k}^{]0,1[}\}_{k=N}^{2^j-N}$ *are orthogonal to the* left boundary wavelets $\{\check{\psi}_{j,k}^{]0,1[}\}_{k=0}^{N-1}$ *and to the* right boundary wavelets $\{\check{\psi}_{j,k}^{]0,1[}\}_{k=2^j-N+1}^{2^j-1}$.

Proof: We will prove only that the interior wavelets are orthogonal to the left boundary wavelets, that is that

$$\langle \check{\psi}_{j,k}^{]0,1[}, \check{\psi}_{j,l}^{]0,1[} \rangle = 0,$$

for all $k = 0, \cdots, N-1$ and $l = N \cdots, 2^j - N$, since for the right boundary wavelets the proof is essentially the same.

By the definition of $\check{\psi}_{j,k}^{]0,1[}$, it follows that

$$\langle \check{\psi}_{j,k}^{]0,1[}, \check{\psi}_{j,l}^{]0,1[} \rangle = \langle \vartheta_{j,k}^{]0,1[}, \psi_{j,l} \rangle - \langle P_j^{]0,1[}\vartheta_{j,k}^{]0,1[}, \psi_{j,l} \rangle = \int_{\mathbb{R}} \vartheta_{j,k}^{]0,1[}\psi_{j,l}\, dx,$$

where the last identity descends from the observation that, on one hand, $P_j^{]0,1[}\vartheta_{j,k}^{]0,1[}$ is the restriction to $]0,1[$ of a V_j function (which is $L^2(\mathbb{R})$-orthogonal to $\psi_{j,l}$) and, on the other hand, thanks to the property of the support of $\psi_{j,l}$, the integral over $]0,1[$ can be replaced by an integral over $\mathbb{R}$.

By the definition of the extension operator $\mathcal{E}_j$, for $k < N$ we have that

$$\mathcal{E}_{j+1}\psi_{j,k} = \psi_{j,k} + \sum_{m=-\infty}^{N-M-2} d_m \varphi_{j+1,m},$$

where d_m are suitable coefficinents, whose value is irrelevant for the proof.

We next observe that $m < 2l + 1 - N$ implies that

$$\int_{\mathbb{R}} \psi_{j,l}\varphi_{j+1,m}\, dx = g_{m-2l} = 0.$$

Finally, $l > N - 1$ implies on one hand that

$$\langle \vartheta_{j,k}^{]0,1[}, \psi_{j,l} \rangle = \int_{\mathbb{R}} \mathcal{E}_{j+1}\psi_{j,k}\psi_{j,l} = \sum_{m=-\infty}^{N-M-2} d_m \int_{\mathbb{R}} \varphi_{j+1,m}\psi_{j,l},$$

while $m < N - M - 2$ implies that $m < 2l + 1 - N$, which implies the thesis. □

Unfortunately, we are unable to prove (in general) that the set $\{\check{\psi}_{j,k}^{]0,1[}\}_{k \in I_j^w}$ is linearly independent. We will then have to make some minimal hypotheses, and verify them a posteriori. More precisely we make the following assumptions:

#1. The functions

$$\check{\psi}_{j_0,0}^{]0,1[}, \cdots, \check{\psi}_{j_0,N-1}^{]0,1[}$$

are linearly independent, as well as

$$\check{\psi}_{j_0,2^{j_0}-N+1}^{]0,1[}, \cdots, \check{\psi}_{j_0,2^{j_0}-1}^{]0,1[}.$$

#2. For every j, $j = \widehat{j_0}, \cdots, j_0 - 1$, the set $\{\check{\psi}_{j,k}^{]0,1[}\}_{k \in I_j^w}$ is linearly independent.

Under such assumptions it is not difficult to prove the following proposition:

Proposition 3.3 *For all $j \geq \widehat{j_0}$ The set $\{\check{\psi}_{j,k}^{]0,1[}\}_{k \in I_j^w}$ forms a basis for the space $W_j^{]0,1[}$.*

Proof: Thanks to the scale invariance properties (22) and (23) assumption #1 yields, also for all $j \geq j_0$, the linear independece of the set

$$\check{\psi}_{j,0}^{]0,1[}, \cdots, \check{\psi}_{j,N-1}^{]0,1[},$$

as well as of the set

$$\check{\psi}_{j,2^j-N+1}^{]0,1[}, \cdots, \check{\psi}_{j,2^j-1}^{]0,1[}.$$

Therefore, for every $j \geq j_0$, the complete set $\{\check{\psi}_{j,k}^{]0,1[}\}_{k \in I_j^w}$ is linearly independent, since the interior wavelets $\{\check{\psi}_{j,k}^{]0,1[}\}_{k=N}^{2^j-N}$ are linearly independent and, according to Proposition 3.2, they are orthogonal to the boundary wavelets $\{\check{\psi}_{j,k}^{]0,1[}\}_{k=0}^{N-1}$ and $\{\check{\psi}_{j,k}^{]0,1[}\}_{k=2^j-N+1}^{2^j-1}$. Using assumption #2, we then obtain that the set $\{\check{\psi}_{j,k}^{]0,1[}\}_{k\in I_j^w}$ is linearly independent for all $j \geq \widehat{j_0}$. Finally, since $dim(W_j^{]0,1[}) = \#I_j^w$, it follows that the set $\{\check{\psi}_{j,k}^{]0,1[}\}_{k\in I_j^w}$ forms a basis. □

The considerations at page 13 on the orthonormalization of the set $\{\check{\varphi}_{j,k}^{]0,1[},\ k \in I_j\}$ carry over unchanged to the orthonormalization of the set $\{\check{\psi}_{j,k}^{]0,1[},\ k \in I_j^w\}$. We will denote by $\{\psi_{j,k}^{]0,1[}\}$ the corresponding set of orthonormalized wavelet functions. Figure 2 show the orthonormal wavelets corresponding to the orthonormal scaling functions of Figure 1.

4 Fast wavelet transform

Let us now briefly describe the Fast Wavelet Transform algorithm, which allows the computation of the coefficients

$$f_k^j = \langle f, \varphi_{j,k}^{]0,1[}\rangle, \quad \text{and} \quad d_k^j = \langle f, \psi_{j,k}^{]0,1[}\rangle$$

of the $L^2(]0,1[)$ projections

$$P_j^{]0,1[}f = \sum_{k\in I_j} f_k^j \varphi_{j,k}^{]0,1[}, \qquad \text{and} \qquad (P_{j+1}^{]0,1[} - P_j^{]0,1[})f = \sum_{k\in I_j^w} d_k^j \psi_{j,k}^{]0,1[}$$

of a function $f \in L^2(]0,1[)$ onto $V_j^{]0,1[}$ and $W_j^{]0,1[}$ directly from the coefficients

$$f_k^{j+1} = \langle f, \varphi_{j+1,k}^{]0,1[}\rangle$$

of its projection

$$P_{j+1}^{]0,1[} = \sum_{k\in I_{j+1}} f_k^{j+1} \varphi_{j+1,k}$$

onto $V_{j+1}^{]0,1[}$.

For $j \geq j_0$, when the spaces constructed in this paper coincide with the ones already studied in the previous constructions, the considerations of the corresponding papers trivially carry over to our case: the FWT takes the form

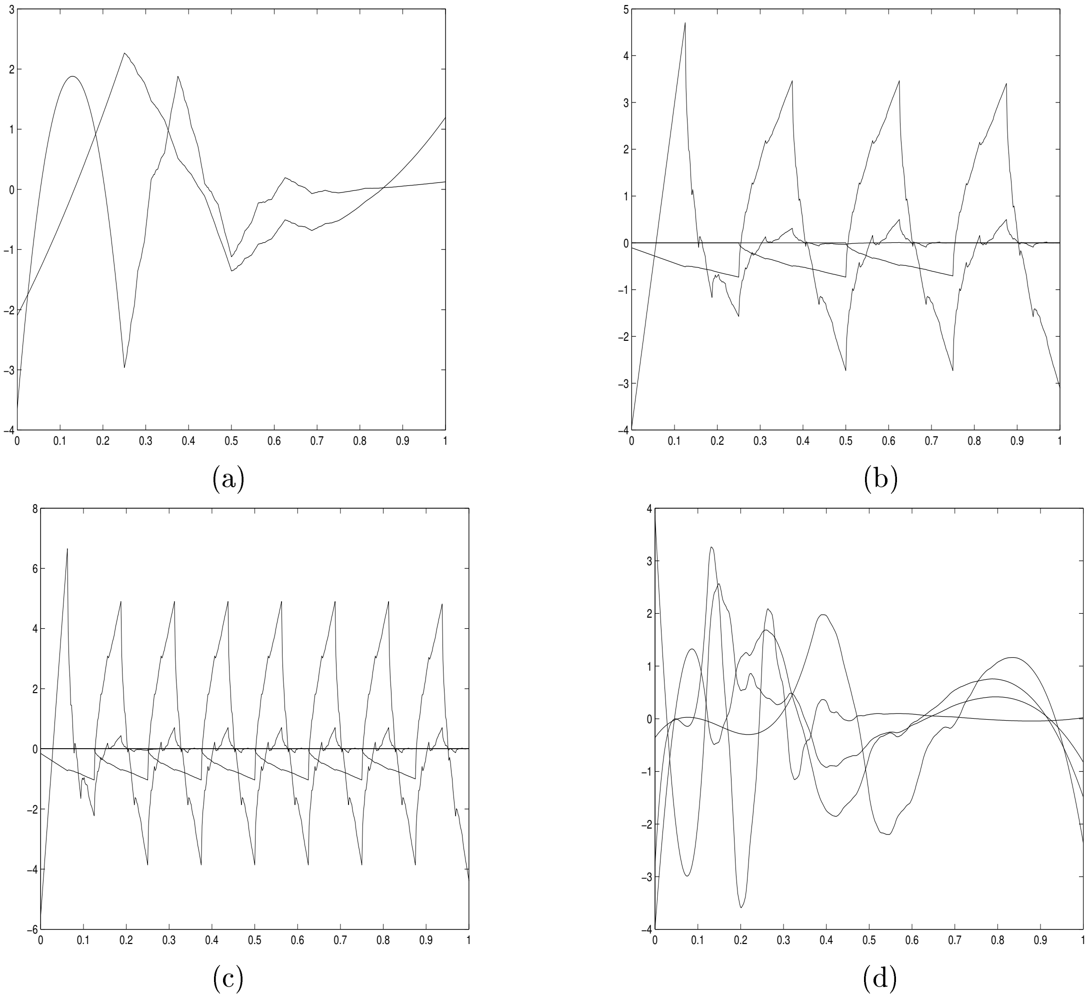


Figure 2: Orthonormal Daubechies wavelet functions; (a) Wavelet base corresponding to the Db3 MRA for $j = 1$. Figures (b) and (c) show the wavelet basis corresponding to the Db2 MRA for $j = 2$ and $j = 3$ respectively. In both cases $j \geq j_0$ so that the new construction coincides with the classical one. (d) Db4 wavelet base for $j = 2$: the functions are globally supported.

$$f_k^j = \begin{cases} \sum_{l\in I_{j+1}} a_{k,n}^{left} f_l^{j+1} & k = N-M-1,\cdots,N-1 \\ \sum_l h_{l-2k} f_l^{j+1} & k = N,\cdots,2^j - N \\ \sum_{l\in I_{j+1}} a_{k,l}^{right} f_l^{j+1} & k = 2^j - N+1,\cdots,2^j - N + M + 1, \end{cases}$$

$$d_k^j = \begin{cases} \sum_{l\in I_{j+1}} b_{k,n}^{left} f_l^{j+1} & k = 0,\cdots,N-1 \\ \sum_l g_{l-2k} f_l^{j+1} & k = N,\cdots,2^j - N \\ \sum_{l\in I_{j+1}} b_{k,l}^{right} f_l^{j+1} & k = 2^j - N+1,\cdots,2^j - 1, \end{cases}$$

where the coefficients $a_{k,n}^{left}$, $a_{k,n}^{right}$, $b_{k,n}^{left}$ and $b_{k,n}^{right}$ are independent of the scale j. In particular, we recall that for the values of k corresponding to the interior functions, the transform has the same form than the FWT in $L^2(\mathbb{R})$.

For $j < j_0$ the trivial matrix-vector multiplication form of the transform

$$f_k^j = \sum_{l\in I_{j+1}} a_{k,n}^j f_l^{j+1}, \qquad \text{and} \qquad d_k^j = \sum_{l\in I_{j+1}} b_{k,n}^j f_l^{j+1},$$

with

$$a_{k,l}^j = \langle \varphi_{j+1,l}^{]0,1[}, \varphi_{j,k}^{]0,1[} \rangle, \qquad \text{and} \qquad b_{k,l}^j = \langle \varphi_{j+1,l}^{]0,1[}, \psi_{j,k}^{]0,1[} \rangle$$

for every $k \in I_j$ and $l \in I_{j+1}$, cannot be further simplified. The matrices

$$A^j = \left(a_{k,l}^j \right) \qquad \text{and} \qquad B^j = \left(b_{k,l}^j \right)$$

depend on j and they will then have to be pre-computed and stored, and large scale steps of the fast wavelet transform will simply be matrix-vector multiplications.

Analogous considerations hold for the inverse fast wavelet transform, allowing to deduce the coefficients f_k^{j+1}, $k \in I_{j+1}$, directly from the coefficients f_k^j, $k \in I_j$ and d_k^j, $k \in I_j^w$. In particular, for $j < j_0$, setting $\underline{f}^{j+1} = (f_k^{j+1})_{k\in I_{j+1}}$, $\underline{f}^j = (f_k^j)_{k\in I_j}$, and $\underline{d}^j = (d_k^j)_{k\in I_j^w}$, using the fact that the FWT operator is, in the case here considered, orthogonal, we have

$$\underline{f}^{j+1} = \begin{bmatrix} A^j \\ B^j \end{bmatrix}^{-1} \begin{pmatrix} \underline{f}^j \\ \underline{d}^j \end{pmatrix} = (A^j)^T \underline{f}^j + (B^j)^T \underline{d}^j. \tag{24}$$

Remark 4.1 *The last expression in equation 24 can also be directly derived from the trivial equality*

$$P_{j+1}^{]0,1[} f = P_j^{]0,1[} f + (P_{j+1}^{]0,1[} - P_j^{]0,1[}) f.$$

5 Multiresolution on $]0,1[$ with boundary conditions.

In this section we briefly describe how to incorporate boundary conditions in the construction of $V_j^{]0,1[}$. Assuming from now on that $M \geq 1$ and that $\varphi \subset H^1(\mathbb{R})$, let us define the space

$$V_j^0 := V_j^{]0,1[} \cap H_0^1(]0,1[)$$

as the set of functions f belonging to $V_j^{]0,1[}$ and satisfying the homogeneous boundary conditions $f(0) = f(1) = 0$. Before presenting the construction of a basis for V_j^0, we establish some preliminary results.

Proposition 5.1 *The set* $\mathcal{B} = \{1-x, x, \{\check{\varphi}_{j,k}^{]0,1[}\}_{k\in I_j^0}\}$, *with*

$$I_j^0 = \{N-M, \cdots, 2^j - N + M\},$$

is a basis for $V_j^{]0,1[}$.

Proof: Let $\{\check{\varphi}_{j,k}^{]0,1[}\}_{k\in I_j}$ be the basis for $V_j^{]0,1[}$ defined in (15); by Corollary 2.2, the functions $1-x$ and x belong to $V_j^{]0,1[}$, so they can be expanded in terms of the basis functions $\{\check{\varphi}_{j,k}^{]0,1[}\}_{k\in I_j}$:

$$1 - x = \sum_{k\in I_j} c_k^{1-x} \check{\varphi}_{j,k}^{]0,1[}$$

and

$$x = \sum_{k\in I_j} c_k^{x} \check{\varphi}_{j,k}^{]0,1[},$$

for some coefficients c_k^{1-x} and c_k^x. Using Remark 2.6, it follows that

$$c_k^{1-x} = \int_{\mathbb{R}} (1-x)\varphi_{j,k}\,dx \quad \text{and} \quad c_k^x = \int_{\mathbb{R}} x\varphi_{j,k}\,dx$$

so that

$$1 - x = \sum_{k\in I_j} (\alpha + \beta k)\check{\varphi}_{j,k}^{]0,1[} \tag{25}$$

and

$$x = \sum_{k\in I_j} (1 - \alpha - \beta k)\check{\varphi}_{j,k}^{]0,1[}, \tag{26}$$

for some constants α and β depending on j. Using equations (25) and (26), we deduce that

$$span\{\mathcal{B}\} = span\{1, \sum_{k\in I_j} k\check{\varphi}_{j,k}^{]0,1[}, \{\check{\varphi}_{j,k}^{]0,1[}\}_{k\in I_j^0}\}.$$

Now, the set $\{1, \sum_{k\in I_j} k\check{\varphi}_{j,k}^{]0,1[}, \{\check{\varphi}_{j,k}^{]0,1[}\}_{k\in I_j^0}\}$ is a basis for $V_j^{]0,1[}$ since the matrix that represents the change of coordinates with respect to the basis $\{\check{\varphi}_{j,k}^{]0,1[}\}_{k\in I_j}$ is invertible, as one can easily verify. Therefore it follows that $\mathcal{B}$ too is a basis for $V_j^{]0,1[}$, and the thesis is proved. $\square$

Let now $I_L : C^0([0,1]) \longrightarrow \mathbb{P}_1$ be the linear interpolation operator

$$I_L(f)(x) = f(0)(1-x) + f(1)x,$$

which associates to every continuous function f the corresponding polynomial of degree 1, interpolating f at the two points 0 and 1, and let $\breve{\varphi}^0_{j,k}$, $k \in I^0_j$, be defined by

$$\breve{\varphi}^0_{j,k} = \breve{\varphi}^{]0,1[}_{j,k} - I_L \breve{\varphi}^{]0,1[}_{j,k}.$$

Proposition 5.2 *The set $\{\breve{\varphi}^0_{j,k}\}_{k \in I^0_j}$ is a basis for V^0_j.*

Proof: The operator $(1 - I_L) : V_j^{]0,1[} \to V_j^0$ is surjective since $(1 - I_L)|_{V_j^0}$ coincides with the identity. Therefore, since $(1-I_L)x = (1-I_L)(1-x) = 0$, the set $\{(1-I_L)\breve{\varphi}^{]0,1[}_{j,k},\ \ k \in I^0_j\}$ generates V^0_j. Moreover it is a basis since $dim(V^0_j) = \#I^0_j$. $\square$

The validity of the following corollary is easily checked.

Corollary 5.1 *The set $\{x, 1-x, \breve{\varphi}^0_{j,k},\ \ k \in I^0_j\}$ is a basis for $V_j^{]0,1[}$.*

We can then construct an orthonormal basis for V^0_j by applying any orthonormalization procedure to the set $\{\breve{\varphi}^0_{j,k}\}_{k \in I^0_j}$. We thus obtain a set $\{\varphi^0_{j,k}\}_{k \in I^0_j}$ satisfying:

$$\int_0^1 \varphi^0_{j,k}\varphi^0_{j,n} = \delta_{n,k}, \qquad \text{and} \qquad \varphi^0_{j,k}(0) = \varphi^0_{j,k}(1) = 0$$

(see Figure 3). Clearly, for $j \geq j_0$ we will employ one of the localization preserving orthonormalization techniques proposed in the papers mentioned at the beginning of section 2.2.

A particular orthonormal basis for $V_j^{]0,1[}$ can then be obtained by orthonormalizing the two functions x and $1-x$ with respect to $\{\varphi^0_{j,k}\}_{k \in I^0_j}$:

$$\varphi^0_{j,N-M-1}(x) = \frac{x - \sum_{k \in I^0_j} \langle x, \varphi^0_{j,k}\rangle \varphi^0_{j,k}(x)}{\|x - \sum_{k \in I^0_j} \langle x, \varphi^0_{j,k}\rangle \varphi^0_{j,k}(x)\|_{L^2(0,1)}}$$

and

$$\varphi^0_{j,2^j-N+M+1}(x) = \frac{(1-x) - \sum_{k \in I^{0\prime}_j} \langle 1-x, \varphi^0_{j,k}\rangle \varphi^0_{j,k}(x)}{\|(1-x) - \sum_{k \in I^{0\prime}_j} \langle 1-x, \varphi^0_{j,k}\rangle \varphi^0_{j,k}(x)\|_{L^2(0,1)}},$$

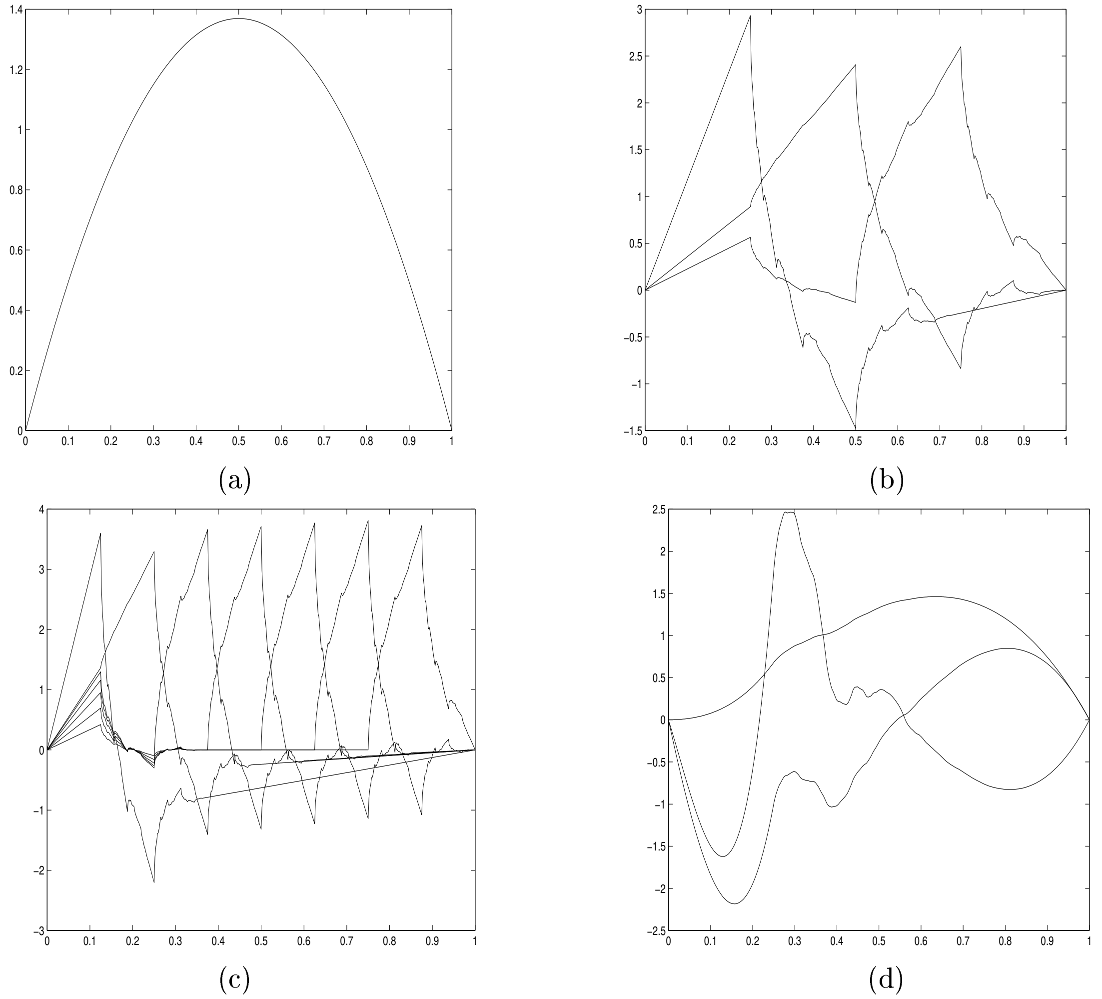
where $I_j^{0'} = \{N - M - 1\} \cup I_j^0$. With this procedure we end up with an orthonormal basis for $V_j^{]0,1[}$ such that

$$\varphi^0_{j,N-M-1}(1) \neq 0 \qquad \varphi^0_{j,N-M-1}(0) = 0$$

and

$$\varphi^0_{j,2^j-N+M+1}(1) \neq 0 \qquad \varphi^0_{j,2^j-N+M+1}(0) \neq 0.$$

In particular non homogeneous boundary conditions are very easily imposed.



(a) (b) (c) (d)

Figure 3: Orthonormal Daubechies scaling functions with homogeneous boundary conditions; (a) A base for $V_j^{]0,1[}$ corresponding to Db3 MRA for $j = 1$. $V_j^{]0,1[}$ is the space of polynomials of degree 2 satisfying homogeneous boundary conditions. (b) The space $V_j^{]0,1[}$ corresponding to Db2 for $j = 2$. (c) $V_j^{]0,1[}$ corresponding to Db2 for $j = 3$. (d) The basis functions coming from Db4 MRA for $j = 2$ are globally supported.

The argument discussed at the beginning of section 3 with respect to the construction of wavelets for $W_j^{]0,1[}$, carry over to the construction of an orthonormal basis for the orthogonal complement W_j^0 of V_j^0 in V_{j+1}^0. In order to apply also in this case the idea proposed in section 3.1, we introduce the orthogonal projection $P_j^0 : L^2(\mathbb{R}) \longrightarrow V_j^0$ onto V_j^0. Therefore W_j^0 is defined as

$$W_j^0 = (I - P_j^0)V_{j+1}^0.$$

In particular, a basis for W_j^0 is defined as follows: for every $k = 0, \cdots, 2^j - 1$

$$\check{\psi}_{j,k}^0 = \vartheta_{j,k}^0 - P_j^0 \vartheta_{j,k}^0,$$

where $\vartheta_{j,k}^0 = ((1 - I_L)\mathcal{E}_j \psi_{j,k})|_{]0,1[}$. Figure 4 shows the basis $\{\psi_{j,k}^0\}$ for W_j^0, obtained after orthonormalization.

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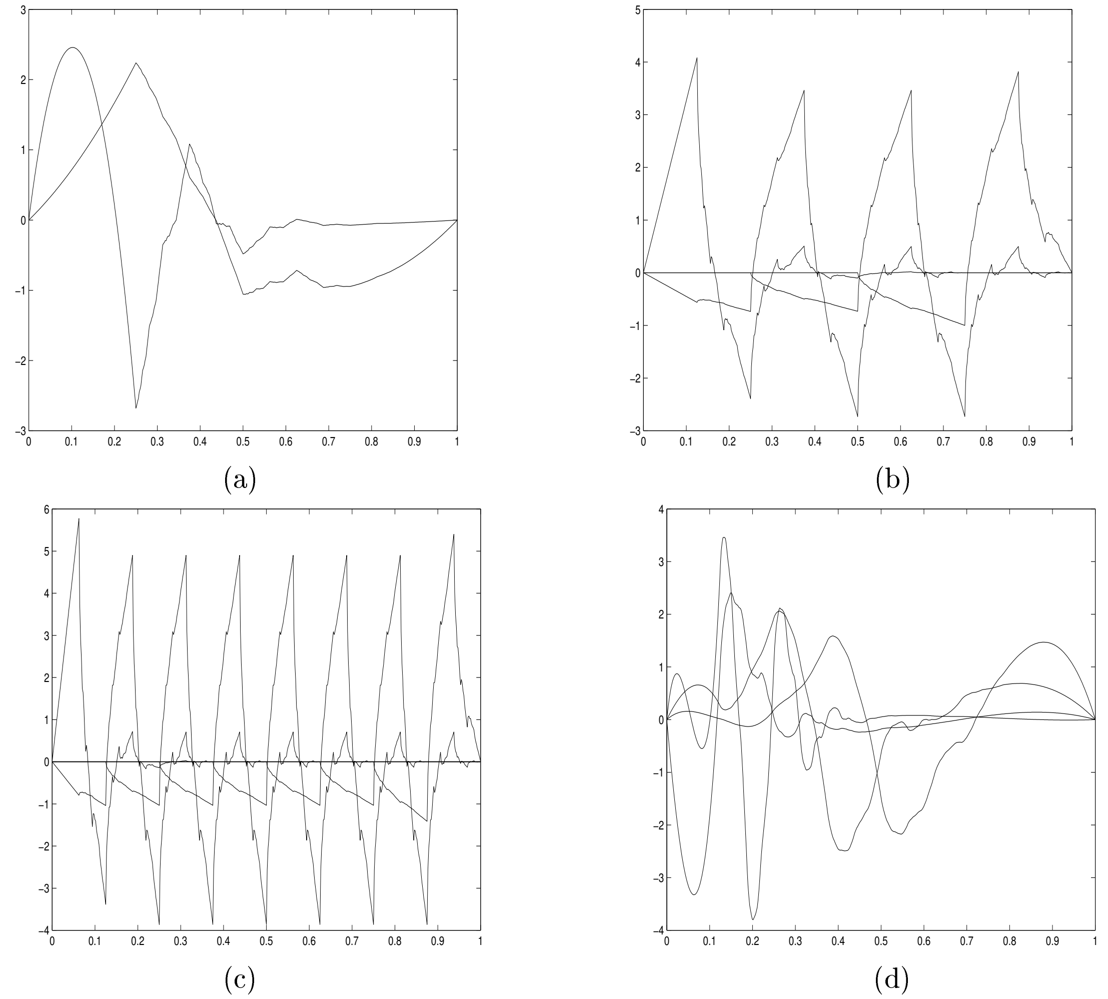

Figure 4: Orthonormal Daubechies wavelet functions with homogeneous boundary conditions; (a) Db3 j=1 (b) Db2 j=2 (c) Db2 j=3 (d) Db4 j=2

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