The Mortar Wavelet Method

Silvia Bertoluzza * Valérie Perrier†

October 29, 1999

Abstract

This paper deals with the construction of wavelet approximation spaces, in the framework of the Mortar method.
We briefly describe the construction of a class of MultiResolution Analyses which are well suited for application by the Mortar method approach, and then we sketch the Mortar Wavelet method for the Laplace-Dirichlet problem for which we give an error estimate. Numerical tests show the feasibility of the method proposed.

1 Introduction

In order for wavelet methods to be applicable to real life problems, several issues remain to be faced. Among such issues the treatment of non trivial geometries is of particular relevance. As in the spectral method case, the domain decomposition approach seems particularly well suited to provide a satisfactory answer in such respect. Rather than resorting to conforming domain decomposition (for applications of the conforming approach in the wavelet framework see [1, 2]), we prefer to focus here on a non conforming approach, namely the Mortar Method ([3]), which allows in principle to couple wavelets to finite elements. Such coupling is particularly attractive: it would allow to decompose complicated domains into square regions to be treated by wavelet methods – with the advantage of using a high order scheme, simple adaptive strategies and optimal preconditioners – and smaller regions, well fitted to the geometry, to be treated by the finite element method.

*I.A.N.-C.N.R., v. Ferrata 1, Pavia (Italy). *Email: aivlis@ian.pv.cnr.it*
†LMC-IMAG, BP 53 - 38041 Grenoble cedex 9 (France). *Email: Valerie.Perrier@imag.fr*

2 Non conforming domain decomposition

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. We will consider a decomposition of Ω as the union of L subdomains Ω_ℓ, regular in shape,

$$\Omega = \bigcup_{\ell=1,L} \Omega_\ell,$$

which we will assume to be rectangular. We set

$$\Gamma_{\ell n} = \partial\Omega_n \cap \partial\Omega_\ell, \qquad\qquad S = \cup\Gamma_{\ell n},$$

and we denote by $\gamma_\ell^{(i)}$ $(i = 1, \ldots, 4)$ the i-th side of the ℓ-th domain: $\partial\Omega_\ell = \bigcup_{i=1}^4 \gamma_\ell^{(i)}$.

For simplicity we will assume here that the decomposition is *geometrically conforming*, that is each edge $\gamma_\ell^{(i)}$ coincides with $\Gamma_{\ell n}(= \partial\Omega_\ell \cup \partial\Omega_n)$ for some n, $1 \leq n \leq L$.

In such a framework, we will consider the following model problem. Given $f \in L^2(\Omega)$, find $u : \Omega \longrightarrow \mathbb{R}$ such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In order to split this problem, we start by introducing the following broken norm: for $u \in \prod H^1(\Omega_\ell)$ we write

$$\|u\|_{1,*} = \left(\sum_\ell \|u\|^2_{1,\Omega_\ell} \right)^{\frac{1}{2}}.$$

The space approximation is performed by introducing for each ℓ a family $\mathcal{V}_\delta^\ell$ of finite dimensional subspaces of $H^1(\Omega_\ell)$, whose elements satisfy an homogeneous boundary condition on the external boundary $\partial\Omega \cap \partial\Omega_\ell$. Given a finite dimensional subspace $M_\delta \subset L^2(S)$, we will consider the following approximation of $H^1_0(\Omega)$:

$$\mathcal{X}_\delta = \{v_\delta \in \prod_{\ell=1}^L \mathcal{V}_\delta^\ell, \quad \int_S [v_\delta]\lambda = 0, \ \forall \lambda \in M_\delta\}, \tag{2}$$

where $[u]$ denotes the jump of u across the skeleton S.

We can now introduce the following discrete problem:

Problem 2.1 (PD) *Find $u_\delta \in \mathcal{X}_\delta$, such that for all $v_\delta \in \mathcal{X}_\delta$,*

$$\sum_{\ell=1}^L \int_{\Omega_\ell} \nabla u_\delta \nabla v_\delta = \int_\Omega f v_\delta. \tag{3}$$

3 Approximation spaces in the wavelet context

The aim of this section is to introduce wavelet spaces which are well suited to be used in the framework described in the previous section. We begin with a couple of biorthogonal MultiResolution Analyses (MRA) of $L^2(0,1)$ ([4]), that is a couple of increasing sequences of finite dimensional subspaces $(V_j)_{j\geq j_0}$ and $\left(\tilde{V}_j\right)_{j\geq j_0}$,

$$
\begin{array}{rcl}
V_j & = & span < \varphi_{j,k}, \quad k=0,\cdots,2^j+1 > \ \subset L^2(0,1), \\
\tilde{V}_j & = & span < \tilde{\varphi}_{j,k}, \quad k=0,\cdots,2^j+1 > \ \subset L^2(0,1),
\end{array}
$$

whose respective union is dense in $L^2(0,1)$. The corresponding compactly supported scaling function bases $\{\varphi_{j,k}, k=0,\dots 2^j+1\}$ and $\{\tilde{\varphi}_{j,k}, k=0,\dots 2^j+1\}$, are assumed to be biorthogonal, *i.e.* they verify:

$$
\int_0^1 \varphi_{j,k}\tilde{\varphi}_{j,k'} \ = \ \delta_{kk'}, \quad \forall k,k'.
$$

We will make the following additional assumptions on V_j and $\tilde{V}_j$:

$$
\varphi_{j,k} \in H^R(0,1) \text{ and } \tilde{\varphi}_{j,k} \in H^{\tilde{R}}, \qquad j\geq j_0, \quad k=0,\dots,2^j+1,
$$

with $R>1$ and $\tilde{R}>0$, and polynomials up to order N and $\tilde{N}$ are included in V_{j_0} and $\tilde{V}_{j_0}$ respectively. In particular there exists coefficients a_n^k, $n=0,\cdots,N$ and $\tilde{a}_k^n$, $n=0,\cdots,\tilde{N}$ such that

$$
2^{j/2}(2^jx)^n = \sum_{k=0}^{2^j+1} a_k^n \ \varphi_{j,k}(x), \qquad 2^{j/2}(2^jx)^n = \sum_{k=0}^{2^j+1} \tilde{a}_k^n \ \tilde{\varphi}_{j,k}(x).
$$

Finally we can also suppose that all scaling functions of V_j vanish at the edges 0 and 1, except one function at each edge. For example we will assume:

$$
\begin{array}{l}
\varphi_{j,0}(0) \neq 0 \quad \text{and} \quad \varphi_{j,2^j+1}(1) \neq 0 \ , \quad \forall j \geq j_0, \\
\varphi_{j,k}(0) = 0 \quad \text{and} \quad \varphi_{j,k}(1) = 0 \ , \quad \forall k=1,\dots,2^j.
\end{array}
$$

It is well known that the above assumptions imply that the projectors $P_j : L^2(0,1) \to V_j$ and $\tilde{P}_j : L^2(0,1) \to \tilde{V}_j$ defined by

$$
P_jf = \sum_{k=0}^{2^j+1} < f,\tilde{\varphi}_{jk} > \varphi_{jk}, \qquad \tilde{P}_jf = \sum_{k=0}^{2^j+1} < f,\varphi_{jk} > \tilde{\varphi}_{jk},
$$

satisfy the following direct estimates for all $u\in H^t(0,1)$:

$$\|u - P_j u\|_{s,]0,1[} \lesssim 2^{-j(t-s)} \|u\|_{t,]0,1[} \qquad \text{if } s \le R, \qquad s < t \le N+1$$
$$\|u - \tilde{P}_j u\|_{s,]0,1[} \lesssim 2^{-j(t-s)} \|u\|_{t,]0,1[} \qquad \text{if } s \le \tilde{R}, \qquad s < t \le \tilde{N}+1$$

In order to build a suitable multiplier space M_δ to be used in the definition (2) of $\mathcal{X}_\delta$, we construct a second multiscale analysis $\tilde{M}_j$, whose basis is biorthogonal to the interior scaling functions $\{\varphi_{j,k}, k = 1, \cdots, 2^j\}$. More precisely set

$$\begin{array}{rcll} \tilde{\phi}_{j,k} & = & \tilde{\varphi}_{j,k} + c_k \ \tilde{\varphi}_{j,0} \ , & for\ k = 1, \tilde{N} \\ \tilde{\phi}_{j,k} & = & \tilde{\varphi}_{j,k} \ , & for\ k = \tilde{N}+1, 2^j - \tilde{N} \\ \tilde{\phi}_{j,k} & = & \tilde{\varphi}_{j,k} + d_k \ \tilde{\varphi}_{j,2^j+1} \ , & for\ k = 2^j - \tilde{N}+1, 2^j \end{array}$$

with

$$\begin{array}{ll} c_k = \alpha_k/\alpha_0, \quad k = 1, \tilde{N}-1 & c_{\tilde{N}} = 1/\alpha_0 \\ d_k = \beta_k/\beta_0, \quad k = 2^j - \tilde{N}+2, 2^j & d_{2^j-\tilde{N}+1} = 1/\beta_0 \end{array}$$

where $(\alpha_k)_{k=0,\tilde{N}}$ and $(\beta_k)_{k=0,\tilde{N}}$ are respectively the solutions of the two following linear systems

$$\sum_{k=0}^{\tilde{N}-1} \tilde{a}_k^n \alpha_k = \tilde{a}_{\tilde{N}}^n, \quad \text{and} \quad \sum_{k=2^j-\tilde{N}+2}^{2^j+1} \tilde{a}_k^n \beta_k = \tilde{a}_{2^j-\tilde{N}+1}^n, \quad \forall n = 0, \tilde{N}-1.$$

Then we set

$$\tilde{M}_j = span < \tilde{\phi}_{j,k}, \quad k = 1, \ldots, 2^j > .$$

The following theorem was proven in [5]

Theorem 3.1 *For all j the set $\{\tilde{\phi}_{j,k}, k = 1, \cdots, 2^j\}$ is biorthogonal to the set $\{\varphi_{j,k}, k = 1, \cdots, 2^j\}$, that is it holds*

$$\int_{]0,1[} \varphi_{j,k}(x) \tilde{\phi}_{j,n}(x)\, dx = \delta_{nk}.$$

Moreover the projector $\tilde{\pi}_j : L^2(0,1) \to \tilde{M}_j$ defined by

$$\tilde{\pi}_j f = \sum_{k=1}^{2^j} < f, \varphi_{jk} > \tilde{\phi}_{jk},$$

verifies the following direct estimate: for all $u \in H^t(0,1)$,

$$\|u - \tilde{\pi}_j u\|_{s,]0,1[} \lesssim 2^{-j(t-s)} \|u\|_{t,]0,1[} \qquad if\ s \le \tilde{R}, \qquad s < t \le \tilde{N}+1$$

As usual in the unit square $]0,1[^2$ we will define the approximation spaces by tensor product:

$$\mathcal{V}_j = V_j \otimes V_j = span < \varphi_{j,k} \otimes \varphi_{j,k'} \ , \ k, k' = 0, \dots, 2^j + 1 > \qquad (4)$$

The family $(\mathcal{V}_j)_{j \geq j_0}$ constitutes a MRA of $L^2(]0,1[^2)$. The two-dimensional biorthogonal projection on $\mathcal{V}_j$ will be denoted by $\mathcal{P}_j$. Two-dimensional wavelets are constructed (as usual) by tensor products of one-dimensional bases. The direct inequalities are still valid in dimension 2 ([4]). In particular, for all $u \in H^s(]0,1[^2)$, $1 < s \leq N + 1$ it holds:

$$\|u - \mathcal{P}_j(u)\|_{1,]0,1[^2} \lesssim 2^{-j(s-1)} \|u\|_{s,]0,1[^2}$$

4 The Mortar Wavelet Method

The discrete space $\mathcal{X}_\delta$ of (2) will be then defined as follows.

Let $F^\ell :]0,1[^2 \to \Omega_\ell$ denote for each ℓ the affine transformation mapping the reference square onto the subdomain Ω_ℓ. The discrete spaces $\mathcal{V}_\delta^\ell$ in Ω_ℓ can be defined by

$$\mathcal{V}_\delta^\ell = F_\ell(\mathcal{V}_{j_\ell}), \qquad j_\ell \geq j_0.$$

Following ([3, 6]), for defining the multiplier space M_δ by the Mortar method approach, we start by choosing a splitting of the skeleton S as the disjoint union of a certain number of subdomain sides $\gamma_\ell^{(i)}$, usually called "mortars" or "slave sides". More precisely, we choose an index set $I \subset \{1, \dots, L\} \times \{1, \dots, 4\}$ such that,

$$S = \bigcup_{(\ell,i) \in I} \gamma_\ell^{(i)}, \qquad \begin{array}{c} (\ell_1, i_1), (\ell_2, i_2) \in I, \\ (\ell_1, i_1) \neq (\ell_2, i_2) \end{array} \Rightarrow \gamma_{\ell_1}^{(i_1)} \cap \gamma_{\ell_2}^{(i_2)} = \emptyset. \qquad (5)$$

We can now define M_δ as follows:

$$M_\delta|_{\gamma_\ell^{(i)}} = F_\ell^i(\tilde{M}_{j_\ell}), \qquad (\ell, i) \in I,$$

where $F_\ell^i :]0,1[\to \gamma_\ell^{(i)}$ denotes the restriction of F_ℓ to the counterimage $F_\ell^{-1}(\gamma_\ell^{(i)})$ of $\gamma_\ell^{(i)}$.

We remark that unlike the classical Mortar method, the spaces $\tilde{M}_{j_\ell}$ used for defining on each mortar side the space M_δ are not included in the trace spaces $\mathcal{V}_\delta^\ell|_{\gamma_\ell^{(i)}}$, but in the corresponding dual spaces.

With such a choice it is possible to prove the following theorem[5].

Theorem 4.1 *There exists a unique solution of problem (PD), satisfying the following error estimate: if the solution u of problem (1) satisfies $u|_{\Omega_\ell} \in H^s(\Omega_\ell)$, $\forall \ell = 1, \cdots, L$,*

$$\begin{aligned} \|u - u_\delta\|_{1,*} \lesssim & \left(\sum_{\ell=1}^L 2^{-2j_\ell(s-1)} \|u\|^2_{s,\Omega_\ell}\right)^{1/2} \\ & + \left(\sum_{\ell=1}^L 2^{-2j_\ell(s-1)} \left\| \frac{\partial u}{\partial \nu_\ell} \right\|^2_{s-3/2,\partial\Omega_\ell}\right)^{1/2}, \end{aligned}$$

where ν_ℓ denotes the outer unit normal to the subdomain Ω_ℓ.

5 Implementation

For each subdomain Ω_ℓ we will denote by R^ℓ the stiffness matrix relative to the discretization of the Laplace operator in $\mathcal{V}^\ell_\delta$: more precisely, setting for each $\mathbf{k} = (k_1, k_2) \in \{0, \cdots, 2^j + 1\}^2$

$$\begin{aligned} \Phi_{j,\mathbf{k}}(x, y) &= \phi_{j,k_1}(x)\phi_{j,k_2}(y), && (x, y) \in]0, 1[^2, \\ \Phi^\ell_{j,\mathbf{k}}(x, y) &= \Phi_{j,k}(F_\ell^{-1}(x, y)), && (x, y) \in \Omega_\ell, \end{aligned}$$

we write

$$R^\ell_{\mathbf{n},\mathbf{k}} = \int_{\Omega_\ell} \nabla\Phi^\ell_{j_\ell,\mathbf{k}} \nabla\Phi^\ell_{j_\ell,\mathbf{n}}.$$

An element u_δ of $\mathcal{X}_\delta$ has the form

$$u_\delta = (u^\ell_\delta)_{\ell=1,L}, \qquad \text{with} \qquad u^\ell_\delta = \sum_{\mathbf{k}} u^\ell_{\mathbf{k}}\, \Phi^\ell_{j_\ell,\mathbf{k}},$$

where the coefficients $(u^\ell_{\mathbf{k}})$ must satisfy the discrete equivalent of the jump constraint on the interface S.

The actual degrees of freedom, which we will denote by $\mathbf{u}^M$, are all the coefficients $u^\ell_{\mathbf{k}}$, $\mathbf{k} = (k_1, k_2) \in \{1, \cdots, 2^j\}^2$, corresponding to basis functions vanishing on $\partial\Omega_\ell$, and those coefficients $u^\ell_{\mathbf{k}}$, k_1 and/or k_2 being either 0 or $2^{j_\ell} + 1$, corresponding to either a vertex of Ω_ℓ or to a "non mortar" side ($\gamma^{(i)}_\ell$, $(\ell, i) \notin I$, see (5)).

The value of those coefficients $u^\ell_{\mathbf{k}}$ (k_1 or k_2 being either 0 or $2^{j_\ell}+1$) corresponding to basis functions vanishing at the vertices of Ω_ℓ and "living" on mortar sides ($\gamma^{(i)}_\ell$, $(\ell, i) \in I$) is uniquely determined by the remaining coefficients through the jump condition. If we denote these last constrained coefficients by $\mathbf{u}^S$, we will have that

$$\mathbf{u} = P \begin{pmatrix} \mathbf{u}^M \\ \mathbf{u}^S \end{pmatrix}, \qquad \mathbf{u}^S = C\mathbf{u}^M,$$

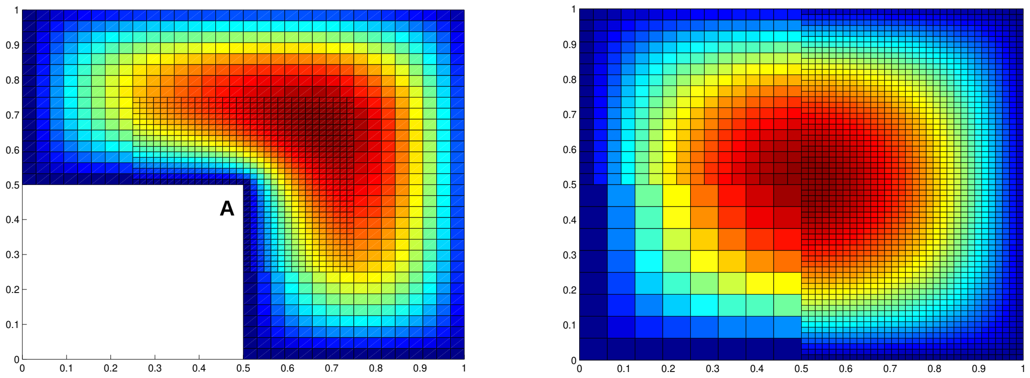

Figure 1: On the left, the solution of problem (6) on an L-shaped domain with $L = 12$ subdomains, $j = 4$ on the three subdomains adjacent to the corner A, $j = 3$ in the other subdomains. On the right, solution of problem (6) on a square domain with $L = 4$ subdomains; in the four subdomains j takes respectively the values 3, 4, 5 and 5.

(where P is a suitable permutation matrix and C is the matrix expressing the constraint).

It is well known that applying the stiffness matrix R corresponding to problem (3) to the vector $\mathbf{u}^M$ of degrees of freedom can then be rewritten as

$$R\mathbf{u}^M = \left[\begin{array}{cc} I & C^T \end{array} \right] P^T \begin{pmatrix} R^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R^L \end{pmatrix} P \left[\begin{array}{c} I \\ C \end{array} \right] \mathbf{u}^M.$$

The multiplication by the matrix C takes a particularly simple form in the wavelet context. On one mortar side $\gamma_\ell^{(i)} = \Gamma_{\ell n}$, which for instance we can assume to be the lower side of Ω_ℓ and the upper side of Ω_n, the jump condition would imply

$$\begin{aligned} \sum_{k=1}^{2^{j_\ell}} u^\ell_{(k,0)} \phi_{j_\ell,k}(x) &= \pi_{j_\ell} \Big(\sum_{k=0}^{2^{j_n+2}} u^n_{(k,2^{j_n}+1)} \phi_{j_n,k}(x) \\ &\qquad - u^\ell_{(0,0)} \phi_{j_\ell,0}(x) - u^\ell_{(2^{j_\ell}+1,0)} \phi_{j_\ell,2^{j_\ell}+1}(x) \Big). \end{aligned}$$

The coefficients of the left hand side can be retrieved by means of a Fast Wavelet Transform (FWT) in $O(2^{\max\{j_\ell,j_n\}})$ operations. In other words, multiplying $\mathbf{u}^M$ by C reduces to applying a sequence of FWTs on the mortar sides.

We tested such an approach on a very simple model problem, namely

$$-\Delta u = 1 \tag{6}$$

on an L-shaped and on a square domain respectively, with homogeneous boundary conditions. In figure 1 we report the results of such tests. We used Daubechies D-3 orthonormal compactly supported scaling functions, for which $N = \tilde{N} = 3$. In both cases all the subdomains are squares of the same size.

In agreement with the theory the tests clearly show that, though different levels of discretization are used in the different subdomains, and though no strong continuity is imposed at the interfaces, the solution is correctly calculated by the method proposed.

Acknowledgements

Work partially supported by Laboratoire ASCI (Univ. Paris XI), by the EC-TMR Network ERB-FMRX-98-0184 *"Wavelets in Numerical Simulation"*, Contract N. ERB-FMRX-98-0184, and carried out in the framework of CEMRACS98.

References

[1] C. Canuto, A. Tabacco, and K. Urban. The wavelet element method. part I: Construction and analysis. *Appl. Comput. Harm. Anal.*, 6, (1999).

[2] W. Dahmen and R. Schneider. Composite wavelet bases for operator equations. preprint TU-Chemnitz, (1997), to appear in *Math. Comp.*

[3] C. Bernardi, Y. Maday, and A.T. Patera. A new nonconforming approach to domain decomposition: the mortar element method. in *Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar* (1992).

[4] A. Cohen. Numerical analysis of wavelet methods. in *Handbook in Numerical Analysis, volume VII*, P.G. Ciarlet and J.L. Lions, editors. Elsevier Science Publishers, North Holland, (1999), to appear.

[5] S. Bertoluzza and V. Perrier. The Mortar method in the wavelet context. Technical report, LAGA n 99-17, (1999).

[6] F. Ben Belgacem and Y. Maday. Non conforming spectral method for second order elliptic problems in 3d. *East-West J. Numer. Math.*, (1994).