Coupling Wavelets and Finite Elements by the Mortar Method

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Abstract

We propose and analyse a mortar method with approximate constraint which we show to be particularly well suited for application in the framework of the wavelet/FEM coupling.

1 Introduction

One of the advantages of non conforming domain decomposition methods, is that they allow in principle to couple discretizations of different type. This feature is particulary attractive when considering discretizations whose applicability is, by nature, limited to tensor product domains. This is the case of wavelet type methods, which perform in a very promising way on academic examples, but whose application to real life problems is seriousely limited by the issue of geometry, among other things. Coupling (in a domain decomposition approach) with finite elements allows in principle to overcome this limitation.

In such methods, and in particular in the mortar method ([1]) which we are considering here, continuity across the interface is relaxed to a weak continuity constraint. In order to couple discretizations of different type the need arises eventually of applying the constraint operator, which implies the need of computing integrals of products of functions belonging to the two different discretizations. In particular, in the Wavelet/FEM coupling the problem arises of computing the integral of a wavelet type function times a piecewise polynomial defined on an unstructured grid. Unfortunately, due to the particular nature of wavelets, which are not known in closed form, there is up to date no way of computing such an integral exactely, and it is therefore necessary to approximate it somehow. The problem of the effect of using quadrature formulae for the constraint within the mortar method has already been considered by ([4]).

In this paper we propose a different way of approximating such integrals which is particularly well suited for the coupling of wavelet and finite elements.

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2 The mortar method with approximate constraint

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and consider the model problem: given $f \in L^2(\Omega)$ find $u : \Omega \to \mathbb{R}$ s.t.

$$-\nabla \cdot (a\nabla u) = f \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \Gamma = \partial\Omega, \tag{1}$$

where for simplicity we assume that the matrix a is constant symmetric positive definite. We consider here a very simple example of non conforming domain decomposition. More precisely consider a splitting of Ω in two subdomains as

$$\bar{\Omega} = \bar{\Omega}_+ \cup \bar{\Omega}_-, \qquad \text{with} \qquad \gamma = \partial\Omega_+ \cap \partial\Omega_-.$$

Denote by V_h^+ and V_h^-

$$V_h^+ \ \subset \ H^1_\Gamma(\Omega_+) = \{u \in H^1(\Omega_+) : \ u = 0 \text{ on } \Gamma \cap \Omega_+\} \tag{2}$$
$$V_h^- \ \subset \ H^1_\Gamma(\Omega_-) = \{u \in H^1(\Omega_-) : \ u = 0 \text{ on } \Gamma \cap \Omega_+\} \tag{3}$$

the two discrete spaces chosen for approximating u in Ω_+ and Ω_- respectively, and let $M_h \subset H^{-1/2}(\gamma)$, with $dim(M_h) = dim(V_h^-|_\gamma)$ be a suitable multiplier space — which in the mortar method is obtained from a subspace of $V_h^-|_\gamma$ with suitable modifications at the vertices of γ ([1]), or which coincides, in a more general formulation, with a suitable "dual space" of $V_h^-|_\gamma$ ([**?**]). In the classical formulation of the mortar method, the approximation of the solution of (1) is sought in the constrained space $\mathcal{X}_h$ defined as

$$\mathcal{X}_h = \{u : \ u|_{\Omega_+} \in V_h^+, \ u|_{\Omega_-} \in V_h^-, \ \int_\gamma [u]\lambda \, ds = 0 \ \ \forall \lambda \in M_h\},$$

where introducing the notation $u^+ = u|_{\Omega_+}$ and $u^- = u|_{\Omega_-}$, $[u] = u^+|_\gamma - u^-|_\gamma$ denotes the jump of the function u across the interface γ. The solution u to problem (1) is approximated by looking for u_h in $\mathcal{X}_h$ such that for all $v_h \in \mathcal{X}_h$ it holds

$$\int_{\Omega_+} a\nabla u_h \nabla v_h \, dx + \int_{\Omega_-} a\nabla u_h \nabla v_h \, dx = \int_\Omega f v_h.$$

In the solution of the linear system resulting from such problem the need arises eventually of computing the integrals appearing in the constraint

$$\int_\gamma (u_h^+ - u_h^-)\lambda \, ds = 0, \qquad \forall \lambda \in M_h. \tag{4}$$

Since in the mortar method the multiplier space M_h is strongly related to the "slave" space V_h^-, it is not reasonable to assume that the integrals of the products $u_h^-\lambda$ are computable in practice. This is not necessarily the case of the products $u_h^+\lambda$ where functions originating from totally unrelated spaces are involved. We will concentrate here on approximating this term in the constraint.

In order to do that, let us introduce two auxiliary spaces $U_\delta^- \subset L^2(\gamma)$ and $U_\delta^+ \subset L^2(\gamma)$ depending on a parameter δ, which we assume to have the same finite dimension and to satisfy

$$\inf_{\zeta\in U_\delta^-}\sup_{\eta\in U_\delta^+}\frac{\int_\gamma \zeta\eta\,ds}{\|\zeta\|_{H_{00}^{1/2}(\gamma)}\|\eta\|_{-1/2,\gamma}} \geq \alpha > 0. \tag{5}$$

Assume that the two auxiliary spaces are chosen in such a way that the integrals of the form $\int_\gamma \zeta\eta\,ds$ are computable provided either $\zeta \in V_h^+|_\gamma$ and $\eta \in U_\delta^+$ or $\zeta \in V_h^-|_\gamma$ and $\eta \in U_\delta^-$. For all $\zeta \in L^2(\gamma)$ let $P^-(\zeta) \in U_\delta^-$ be the unique element in U_δ^- such that

$$\int_\gamma P^-(\zeta)\eta\,ds = \int_\gamma \zeta\eta\,ds, \qquad \forall\eta \in U_\delta^+. \tag{6}$$

We propose here to approximate the integral of the product $u_h^+\lambda$ with the integral of $P^-(u_h^+)\lambda$ (where, by abuse of notation we will write u_h^+ instead of $u_h^+|_\gamma$). The constraint (4) is then replaced by the approximated constraint

$$\int_\gamma (P^-(u_h^+) - u_h^-)\lambda\,ds = 0, \qquad \forall\lambda \in M_h, \tag{7}$$

which corresponds to defining a new constrained space as

$$\mathcal{X}_h^* = \{u: \quad u|_{\Omega_+} \in V_h^+, \quad u|_{\Omega_-} \in V_h^-, \quad \int_\gamma (P^-(u^+) - u^-)\lambda\,ds = 0 \ \ \forall\lambda \in M_h\},$$

and approximating the solution to (1) by the solution of the following discrete problem: find $u_h \in \mathcal{X}_h^*$ such that for all $v_h \in \mathcal{X}_h^*$ it holds

$$\int_{\Omega_+} a\nabla u_h\nabla v_h\,dx + \int_{\Omega_-} a\nabla u_h\nabla v_h\,dx = \int_\Omega f v_h. \tag{8}$$

Denoting by $\|\cdot\|_{1,*}$

$$\|\cdot\|_{1,*} = \|\cdot\|_{1,\Omega_+} + \|\cdot\|_{1,\Omega_-}$$

the broken H^1 norms, we can prove the following bound.

Theorem 2.1 *Let the multiplier space M_h be chosen in such a way that the following assumptions are satisfied:*

(A1) *there exists a bounded projection $\pi : L^2(\gamma) \longrightarrow V_h^-|_\gamma$, such that for all $\eta \in L^2(\gamma)$ and for all $\lambda \in M_h$*

$$\int_\gamma (\eta - \pi\eta)\lambda = 0, \tag{9}$$

and for all $\eta \in H_{00}^{1/2}(\gamma)$

$$\|\pi\eta\|_{H_{00}^{1/2}(\gamma)} \lesssim \|\eta\|_{H_{00}^{1/2}(\gamma)}. \tag{10}$$

(A2) *there exists a discrete lifting* $R_h : V_h^-|_\gamma \longrightarrow V_h^-$ *such that for all* $\eta \in V_h^-|_\gamma$

$$\|R_h\eta\|_{1,\Omega_-} \lesssim \|\eta\|_{H_{00}^{1/2}(\gamma)}. \tag{11}$$

Moreover let the two auxiliary spaces U_δ^+ *and* U_δ^- *be chosen in such a way that the following Jackson type inequality holds for some* $\tilde{R}, R \geq 1/2$*: for all* r, $1/2 \leq r \leq R$ *(resp. for all* $\tilde{r}$, $-1/2 \leq \tilde{r} \leq \tilde{R}$*)*

$$\eta \in H_0^r(\gamma) \qquad \Longrightarrow \qquad \inf_{\eta_\delta \in U_\delta^-} \|\eta - \eta_\delta\|_{1/2,\gamma} \lesssim \delta^{r-1/2}\|\eta\|_{r,\gamma}, \tag{12}$$

$$\eta \in H^{\tilde{r}}(\gamma) \qquad \Longrightarrow \qquad \inf_{\eta_\delta \in U_\delta^+} \|\eta - \eta_\delta\|_{-1/2,\gamma} \lesssim \delta^{\tilde{r}+1/2}\|\eta\|_{\tilde{r},\gamma}, \tag{13}$$

Then, if u_h *is the solution of problem (8), and if the true solution* u *of problem (1) verifies* $u \in H^s(\Omega)$ *for some* s, $2 \leq s \leq \min\{\tilde{R} + 3/2, R + 1/2\}$*, the following error estimate holds:*

$$\begin{aligned} \|u - u_h\|_{1,*} \quad \lesssim \quad & \delta^{s-1}\|u\|_{s,\Omega} + \inf_{\lambda \in M_h} \|\partial_{\nu_a} u - \lambda\|_{H^{-1/2}(\gamma)} \\ & + \inf_{v_h \in V_h^+} \|u - v_h\|_{1,\Omega_+} + \inf_{v_h \in V_h^-} \|u - v_h\|_{1,\Omega_-} \end{aligned} \tag{14}$$

where ∂_{ν_a} *denotes the trace on* γ *of outer co-normal derivative to the subdomain* Ω_+*.*

Proof: Let $\mathcal{X}^* = \{u \in H^1_\Gamma(\Omega_+) \times H^1_\Gamma(\Omega_-)$ s.t. (7) holds $\}$, and let $u_* \in \mathcal{X}^*$ be the unique element such that for all $v \in \mathcal{X}^*$ it holds that

$$\int_{\Omega_+} a\nabla u_* \nabla v \, dx + \int_{\Omega_-} a\nabla u_* \nabla v \, dx = \int_\Omega f v_h.$$

Since $\mathcal{X}_h^* \subset \mathcal{X}^*$, it is not difficult to check that it holds for all $v_h \in \mathcal{X}_h$

$$\|u_* - u_h\|_{1,*} \lesssim \|u - u_*\|_{1,*} + \|u - v_h\|_{1,*}$$

which implies

$$\|u - u_h\|_{1,*} \lesssim \|u - u_*\|_{1,*} + \inf_{v_h \in \mathcal{X}_h^*} \|u - v_h\|_{1,*}. \tag{15}$$

Let now u be the solution of (1). For all $v \in H^1_\Gamma(\Omega_+) \times H^1_\Gamma(\Omega_-)$ it holds

$$\int_{\Omega_+} a\nabla u \nabla v \, dx + \int_{\Omega_-} a\nabla u \nabla v \, dx = \int_\Omega f v - \int_\gamma \partial_{\nu_a} u[v] \, ds.$$

Then for any $\tilde{\eta}_\delta \in U_\delta^+$ and $\lambda_h \in M_h$, using (7) and (6) we can bound the first term on the right hand side of (15) as:

$$\begin{aligned} \|u - u_*\|_{1,*}^2 \quad \lesssim \quad & \left| \int_\gamma \partial_{\nu_a} u[u_*] \, ds \right| && (16) \\ \lesssim \quad & \int_\gamma \partial_{\nu_a} u(u_*^+ - P^-(u_*^+)) + \int_\gamma \partial_{\nu_a} u(P^-(u_*^+) - u_*^-) && (17) \\ = \quad & \int_\gamma (\partial_{\nu_a} u - \tilde{\eta}_\delta)(u_*^+ - P^-(u_*^+)) + \int_\gamma (\partial_{\nu_a} u - \lambda_h)(P^-(u_*^+) - u_*^-) && (18) \\ \lesssim \quad & \|\partial_{\nu_a} u - \tilde{\eta}_\delta\|_{-1/2,\gamma} \|(u_*^+ - P^-(u_*^+)\|_{H_{00}^{1/2}(\gamma)} && (19) \\ & + \|\partial_{\nu_a} u - \lambda_h\|_{-1/2,\gamma} \|P^-(u_*^+) - u_*^-\|_{H_{00}^{1/2}(\gamma)} && (20) \end{aligned}$$

By triangular inequality, we can bound

$$\|u_*^+ - P^-(u_*^+)\|_{H_{00}^{1/2}(\gamma)} \leq \|u_*^+ - u\|_{H_{00}^{1/2}(\gamma)} + \|u - P^- u\|_{H_{00}^{1/2}(\gamma)} + \|P^-(u - u_*^+)\|_{H_{00}^{1/2}(\gamma)}$$

and

$$\|P^-(u_*^+) - u_*^-\|_{H_{00}^{1/2}(\gamma)} \leq \|P^-(u_*^+ - u)\|_{H_{00}^{1/2}(\gamma)} + \|P^- u - u\|_{H_{00}^{1/2}(\gamma)} + \|u - u_*^-\|_{H_{00}^{1/2}(\gamma)},$$

from which, using the boundedness of P^- as well as the trace theorem we obtain

$$\begin{aligned} \|u_*^+ - P^-(u_*^+)\|_{H_{00}^{1/2}(\gamma)} &\leq \|u - u_*\|_{1,*} + \|u - P^- u\|_{H_{00}^{1/2}(\gamma)}, \\ \|P^-(u_*^+) - u_*^-\|_{H_{00}^{1/2}(\gamma)} &\leq \|u - u_*\|_{1,*} + \|u - P^- u\|_{H_{00}^{1/2}(\gamma)}. \end{aligned}$$

Then, using the arbitrariness of λ_h and $\tilde{\eta}_\delta$ as well as inequalities (12) and (13), it is not difficult to conclude by standard reasoning that

$$\|u - u_*\|_{1,*} \lesssim \delta^{s-1}\|u\|_{s,\Omega} + \inf_{\lambda \in M_h} \|\partial_{\nu_a} u - \lambda\|_{H^{-1/2}(\gamma)}. \tag{21}$$

In order to bound the second term in (15) let $v_h^+ \in V_h^+$ and $v_h^- \in V_h^-$ be respectively approximations to $u|_{\Omega_+}$ and $u|_{\Omega_-}$. The function $v_h \in L^2(\Omega)$ defined as $v_h|_{\Omega_+} = v_h^+$ and $v_h|_{\Omega_-} = v_h^-$ does not satisfy the constraint and therefore it does not belong to $\mathcal{X}_h^*$. Let $\check{v}_h \in \mathcal{X}_h^*$ be defined by

$$\check{v}_h|_{\Omega_+} = v_h^+, \qquad \check{v}_h|_{\Omega_-} = v_h^- - R_h\left[\pi\left(v_h^- - P^-(v_h^+)\right)\right].$$

It is not difficult to chack that $\check{v}_h$ satisfies (7) and therefore $\check{v}_h \in \mathcal{X}_h^*$. We have

$$\|u - \check{v}_h\|_{1,*} \lesssim \|u - v_h\|_{1,*} + \|\pi\left(v_h^- - P^-(v_h^+)\right)\|_{H_{00}^{1/2}(\gamma)}.$$

Now, adding and subtracting $u|_\gamma$ and $P^-(u|_\gamma)$, by triangular inequality, using the boundedness of P^- and the trace theorem we obtain

$$\|\pi\left(v_h^- - P^-(v_h^+)\right)\|_{H_{00}^{1/2}(\gamma)} \lesssim \|u - v_h\|_{1,*} + \|u - P^-(u)\|_{H_{00}^{1/2}} \lesssim \|u - v_h\|_{1,*} + \delta^{s-1}\|u\|_{1,\Omega}.$$

Again, by the arbitrariness of v_h we conclude that

$$\inf_{v_h \in \mathcal{X}_h^*} \|u - \check{v}_h\|_{1,*} \lesssim \inf_{v_h \in V_h^+ \times V_h^-} \|u - v_h\|_{1,*} + \delta^{s-1}\|u\|_{1,\Omega},$$

which, together with (21) and (15) yields (14). □

Remark 2.1 *The extremely simple configuration considered (only two subdomains), hides some of the issues related to the analysis of the mortar method in more general configurations — namely the treatment of cross points. Moreover, in this simple case the solution obtained with the method proposed coincides with the one obtained using the three fields formulation (see [3]) with a suitable choice of the discretization spaces. However, the approach used and the results obtained in this paper carry over to more complex cases (with the presence of cross-points), where the method obtained does not fall into the three fields formulation framework, with, in the worse case, a loss of a logarithmic factor in the error estimate.*

3 Wavelet/FEM Coupling

Let us now consider the case of Wavelet/FEM coupling. In order to get two suitable auxiliary spaces, we will in such a case need a couple of biorthogonal multiresolution analyses $\{V_j\}_{j\geq j_0}$ and $\{\tilde{V}_j\}_{j\geq j_0}$ of $L^2(\gamma)$ with the following characteristics ([5, **?**]).

- $V_j \subset H^1(\gamma)$ is the subspace of P1 finite elements on the uniform grid $\mathcal{G}_j$ obtained by splitting γ into 2^j equal segments;
- $\tilde{V}_j \subset H^1(\gamma)$ is a subspace having the same dimension as V_j, which is *biorthogonal* to V_j in the following sense: denoting by $e_{j,k}$ $(k = 0, \ldots, 2^j)$ the nodal basis function in V_j corresponding to the k-th point in the grid $\mathcal{G}_j$, the space $\tilde{V}_j$ has a Riesz's basis $\{\tilde{e}_{j,k},\ k = 0, \ldots, 2^j\}$ which satisfies

$$\int_\gamma e_{j,k}\tilde{e}_{j,k'} = \delta_{kk'}, \quad \forall k, k' = 0, \ldots, 2^j;$$

- the functions $\tilde{e}_{j,k}$ can be obtained as linear combination of the restriction to γ (identified through a suitable mapping with the interval $(0, 1)$), of the translates and contracted (with a contraction factor 2^j) of a compactly supported function $\tilde{e}$, which we assume to be *refinable*, *i.e.* to verify, for suitable values of the coefficients h_k,

$$\tilde{e}(s) = \sum_{k=0}^{N} h_k \tilde{e}(2s - k);$$

- $\tilde{V}_j$ satisfies a Strang-Fix condition of order M, that is it contains polynomials up to degree $M - 1$, while, of course, V_j contains polynomials of order 1.

Let $V_j^0 = V_j \cap H_0^1(\gamma)$ and $\tilde{V}_j^0 = \tilde{V}_j \cap H_0^1(\gamma)$, it is possible to construct two subspaces $V_j^* \subset V_j$ and $\tilde{V}_j^* \subset \tilde{V}_j$ satisfying

$$dim(V_j^*) = dim\tilde{V}_j^0, \qquad dim(\tilde{V}_j^*) = dimV_j^0,$$

$$\inf_{\eta\in V_j^0} \sup_{\zeta\in\tilde{V}_j^*} \frac{\int_\gamma \eta\zeta\, ds}{\|\eta\|_{H_{00}^{1/2}(\gamma)}\|\zeta\|_{-1/2,\gamma}} \geq \alpha_1, \qquad \inf_{\eta\in\tilde{V}_j^0} \sup_{\zeta\in V_j^*} \frac{\int_\gamma \eta\zeta\, ds}{\|\eta\|_{H_{00}^{1/2}(\gamma)}\|\zeta\|_{-1/2,\gamma}} \geq \alpha_2,$$

and in such a way that they satisfy a Strang-Fix condition with the same order as V_j and $\tilde{V}_j$ respectively. Moreover it is possible to construct Riesz's bases $e_{j,k}^*$ and $\tilde{e}_{j,k}^*$ for V_j^* and $\tilde{V}_j^*$ respectively in such a way that the two following biorthogonality relations hold:

$$\int_\gamma e_{j,k}\tilde{e}_{j,k'}^* = \delta_{k,k'}, \qquad \int_\gamma \tilde{e}_{j,k}e_{j,k'}^* = \delta_{k,k'}, \qquad \forall k, k' = 1, \ldots, 2^j - 1. \tag{22}$$

For a way of constructing such spaces see [**?**]

Thanks to the refinability of the function $\tilde{e}$, it is possible, using the techniques of [6, 2], to compute integrals of the product of a wavelet type function times any function in $\tilde{V}_j$ (and therefore in $\tilde{V}_j^0$ and in $\tilde{V}_j^*$), while the product of a function in V_j, V_j^0 and V_j^* with a finite element type function can be computed by standard techniques, already implemented in the mortar method for finite elements with non-matching grids.

For using such spaces for coupling wavelets and finite elements in the mortar method we distinguish two cases.

FEM master / Wavelet slave. In this case we set V_h^+ to be a finite element space on an unstructured, non uniform grid while V_h^- and M_h are two wavelet type spaces. The approximate integration is done by setting $U_\delta^+ = V_j^0$ and $U_\delta^- = \tilde{V}_j^*$.

Wavelet master / FEM slave. In this case we set V_h^+ to be a wavelet type space, while V_h^- and M_h are two finite element spaces defined on unstructured, non uniform grid, the grid for M_h being the trace on γ of the grid for V_h^-. The approximate integration is this time performed by setting $U_\delta^+ = \tilde{V}_j^0$ and $U_\delta^- = V_j^*$.

Once $P^-(v_h^+)$ is known, the space U_δ^- is chosen in both cases in such a way that the integrals of the product $\lambda_h P^-(v_h^+)$ can be computed. We then only need to compute the $P^-(v_h^+)$. This can be done by taking advantage of the biorthogonality property (22). In fact it is not difficult to see that in the first case (FEM master / Wavelet slave), we have

$$P^- u = \sum_{k=1}^{2^j-1} \left(\int_\gamma u e_{j,k}^* \, ds \right) \tilde{e}_{j,k},$$

while in the second case (Wavelet master / FEM slave) we have

$$P^- u = \sum_{k=1}^{2^j-1} \left(\int_\gamma u \tilde{e}_{j,k}^* \, ds \right) e_{j,k}.$$

Again, in both cases the two auxiliary spaces have been chosen in such a way that the two integrals defining the projectors are computable. Moreover, biorthogonality implies that *no linear system* has to be solved in order to compute the auxiliary projector.

By applying Theorem 2.1 we can finally estimate the effect of using the approximate integration technique proposed in the previous section. In both cases we get the following bound: if $u \in H^s(\Omega)$ with $2 \le s \le T$ it holds

$$\begin{aligned} \|u - u_h\|_{1,*} \lesssim\ & 2^{-j(s-1)} \|u\|_{s,\Omega} + \inf_{\lambda \in M_h} \|\partial_{\nu_a} u - \lambda\|_{H^{-1/2}(\gamma)} \\ & + \inf_{v_h \in V_h^+} \|u - v_h\|_{1,\Omega_+} + \inf_{v_h \in V_h^-} \|u - v_h\|_{1,\Omega_-}. \end{aligned} \tag{23}$$

where, depending on the choice of the master and slave, the limit T in the bound is respectively $T = \min\{7/2, M + 1/2\}$ for the 'FEM master' case and $T = \min\{5/2, M + 3/2\}$ for the 'Wavelet master' case.

References

[1] C. Bernardi, Y. Maday, and A.T. Patera. A new nonconforming approach to domain decomposition: The mortar element method. In H. Brezis & J.-L. Lions eds., editor, *Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar*, volume XI of *Notes Math. Ser. 299*, pages 13–51. 1994.

[2] S. Bertoluzza, C. Canuto, and K. Urban. On the adaptive computation of integrals of wavelets. *Appl. Numer. Math.*, 1999. to appear.

[3] F. Brezzi and D. Marini. A three-field domain decomposition method. In A. Quarteroni, J. Periaux, Y.A. Kuznetsov, and O.B. Widlund, editors, *Domain Decomposition Methods in Science and Engineering*, volume 157 of *American Mathematical Society, Contemporary Mathematics*, pages 27–34, 1994.

[4] L. Cazabeau, C. Lacour, and Y. Maday. Numerical quadratures and mortar methods. In Bristeau et al., editor, *Computational Sciences for the 21st Century*, pages 119–128. Wiley and Sons, 1997.

[5] A. Cohen, I. Daubechies, and P. Vial. Wavelets on the interval and fast wavelet transforms. *ACHA*, 1:54–81, 1993.

[6] W. Dahmen and C.A. Michelli. Using the refinement equation for evaluating integrals of wavelets. *SIAM J. Numer. Anal.*, 30:507–537, 1993.