Wavelet/FEM Coupling by the Mortar Method

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Abstract. We propose and analyze in an abstract framework a mortar method with approximate constraint, based on replacing the "master side" function appearing in the weak continuity condition with its projection on a suitably chosen auxiliary space. We show how to choose the auxiliary space in such a way that such a technique can be applied for computing the weak continuity constraints arising in the framework of the wavelet/FEM coupling.

1 Introduction

One of the advantages of non conforming domain decomposition methods, is that they allow in principle to couple discretizations of different type. This feature is particularly attractive when considering discretizations whose applicability is, by nature, limited to tensor product domains. This is the case of wavelet type methods. Wavelets perform in a very promising way on academic examples. For instance they allow the design of simple and efficient adaptive schemes and the stiffness matrices resulting from their application are well conditioned (after a suitable rescaling). Their application to real life problems is however seriously limited by the issue of geometry, among other things. Coupling (in a domain decomposition approach) with finite elements allows in principle to overcome this limitation.

In non conforming domain decomposition methods, and in particular in the mortar method ([1]) which we are considering here, continuity across the interface is relaxed to a weak continuity constraint. In order to couple discretizations of different type the need arises eventually of applying the constraint operator, which implies the need of computing integrals of products of functions belonging to the two different discretizations. In particular, in the Wavelet/FEM coupling the problem arises of computing the integral of a wavelet type function times a piecewise polynomial defined on an unstructured grid. Unfortunately, due to the particular nature of wavelets, which are not known in closed form, there is up to date no way of computing such an integral exactly, and it is therefore necessary to approximate it somehow. Clearly the approximation of such an integral in the constraint results in general in a different definition of the weak continuity constraint, and therefore in a different approximation space with respect to the one obtained when computing the integrals exactly. Such change has an effect on both consistency and approximation errors ([5]). Moreover it is well known that when considering wavelets, standard quadrature formulae are not well suited, since wavelets have lower regularity than the one needed to match the potential accuracy of the method.

In this paper we propose, in an abstract framework, a different way of approximating such integrals which is particularly well suited for the coupling of wavelet and finite elements. This consists in approximating, within the integral, a product of the form $u_h^+\lambda_h$ (u_h^+ being the trace on the so called "master" or "mortar" side of the piecewise H^1 function u_h, and λ_h being the multiplier) with the product $P^-(u_h^+)\lambda_h$, where P^- is an $H^{1/2}_{00}$ bounded projector onto a suitable auxiliary space U_h^-. We provide an abstract error estimate for the resulting modified mortar method and we show how to choose the auxiliary space and the projector when considering the coupling of wavelets with finite elements.

2 The Mortar Method with Approximate Constraint

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and consider the model problem: given $f \in L^2(\Omega)$ find $u : \Omega \to \mathbb{R}$ s.t.

$$-\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{1}$$

where for simplicity we assume that the matrix a is constant symmetric positive definite. We consider here a very simple example of non conforming domain decomposition. More precisely consider a geometrically conforming splitting of Ω in two polygonal subdomains as

$$\bar{\Omega} = \bar{\Omega}_+ \cup \bar{\Omega}_-, \qquad \text{with} \qquad \gamma = \partial\Omega_+ \cap \partial\Omega_-,$$

in such a way that Ω_+ and Ω_- share one common edge γ.

Denote by V_h^+ and V_h^-

$$V_h^+ \subset H^1_\Gamma(\Omega_+) = \{u \in H^1(\Omega_+) : \quad u = 0 \text{ on } \Gamma \cap \Omega_+\} \tag{2a}$$
$$V_h^- \subset H^1_\Gamma(\Omega_-) = \{u \in H^1(\Omega_-) : \quad u = 0 \text{ on } \Gamma \cap \Omega_+\} \tag{2b}$$

the two discrete spaces chosen for approximating u in Ω_+ and Ω_- respectively. Without loss of generality, we can assume the side of γ coming from Ω_+ to be the master side of the side of γ coming from Ω_-. Let $M_h \subset H^{-1/2}(\gamma)$, with $dim(M_h) = dim(V_h^-|_\gamma)$, be a suitable multiplier space — which in the mortar method is obtained from a subspace of $V_h^-|_\gamma$ with suitable modifications at the vertices of γ ([1]), or which coincides, in a more general formulation, with a suitable "dual space" of $V_h^-|_\gamma$ ([3,9]). In the classical formulation of the mortar method, the approximation of the solution of (1) is sought in the constrained space $\mathcal{X}_h$ defined as

$$\mathcal{X}_h = \{u : \quad u|_{\Omega_+} \in V_h^+, \quad u|_{\Omega_-} \in V_h^-, \quad \int_\gamma [u]\lambda \, ds = 0 \quad \forall \lambda \in M_h\},$$

where, introducing the notation $u^+ = u|_{\Omega_+}$ and $u^- = u|_{\Omega_-}$, $[u] = u^+|_\gamma - u^-|_\gamma$ denotes the jump of the function u across the interface γ. The solution u to problem (1) is approximated by looking for u_h in $\mathcal{X}_h$ such that for all $v_h \in \mathcal{X}_h$ it holds

$$\int_{\Omega_+} a\nabla u_h \cdot \nabla v_h \, dx + \int_{\Omega_-} a\nabla u_h \cdot \nabla v_h \, dx = \int_\Omega f v_h.$$

In the solution of the linear system resulting from such problem the need arises eventually of computing the integrals appearing in the constraint

$$\int_\gamma (u_h^+ - u_h^-)\lambda \, ds = 0, \qquad \forall \lambda \in M_h. \tag{3}$$

Since in the mortar method the multiplier space M_h is strongly related to the "slave" space V_h^-, it is reasonable to assume that the integrals of the products $u_h^-\lambda$ are computable in practice. This is not necessarily the case of the products $u_h^+\lambda$, where functions originating from totally unrelated spaces are involved. We will concentrate here on approximating this term in the constraint.

In order to do that, let us introduce two auxiliary spaces $U_\delta^- \subset L^2(\gamma)$ and $U_\delta^+ \subset L^2(\gamma)$ depending on a parameter δ, which we assume to have the same finite dimension and to satisfy

$$\inf_{\zeta \in U_\delta^-} \sup_{\eta \in U_\delta^+} \frac{\int_\gamma \zeta\eta\, ds}{\|\zeta\|_{H^{1/2}_{00}(\gamma)} \|\eta\|_{-1/2,\gamma}} \geq \alpha > 0. \tag{4}$$

Assume that the two auxiliary spaces are chosen in such a way that the integrals of the form $\int_\gamma \zeta\eta\, ds$ are computable provided either $\zeta \in V_h^+|_\gamma$ and $\eta \in U_\delta^+$ or $\zeta \in V_h^-|_\gamma$ and $\eta \in U_\delta^-$. For all $\zeta \in L^2(\gamma)$ let $P^-(\zeta) \in U_\delta^-$ be the unique element in U_δ^- such that

$$\int_\gamma P^-(\zeta)\eta\, ds = \int_\gamma \zeta\eta\, ds, \qquad \forall \eta \in U_\delta^+. \tag{5}$$

Remark that, thanks to property (4) the projector P^- is well defined and it verifies

$$\|P^-(\zeta)\|_{H^{1/2}_{00}(\gamma)} \lesssim \|\zeta\|_{H^{1/2}_{00}(\gamma)}.$$

We propose here to approximate the integral of the product $u_h^+\lambda$ with the integral of $P^-(u_h^+)\lambda$ (where, by abuse of notation we will write u_h^+ instead of $u_h^+|_\gamma$). The constraint (3) is then replaced by the approximated constraint

$$\int_\gamma (P^-(u_h^+) - u_h^-)\lambda\, ds = 0, \qquad \forall \lambda \in M_h, \tag{6}$$

which corresponds to defining a new constrained space as

$$\mathcal{X}_h^* = \{u : \ u|_{\Omega_+} \in V_h^+, \ \ u|_{\Omega_-} \in V_h^-, \ \ \int_\gamma (P^-(u^+) - u^-)\lambda\, ds = 0 \ \ \forall \lambda \in M_h\},$$

and approximating the solution to (1) by the solution of the following discrete problem: find $u_h \in \mathcal{X}_h^*$ such that for all $v_h \in \mathcal{X}_h^*$ it holds

$$\int_{\Omega_+} a\nabla u_h \cdot \nabla v_h\, dx + \int_{\Omega_-} a\nabla u_h \cdot \nabla v_h\, dx = \int_\Omega f v_h. \tag{7}$$

Denoting by $\|\cdot\|_{1,*}$

$$\|\cdot\|_{1,*} = \|\cdot\|_{1,\Omega_+} + \|\cdot\|_{1,\Omega_-}$$

the broken H^1 norm, we can prove the following bound.

Theorem 1. *Let the multiplier space M_h be chosen in such a way that the following assumptions are satisfied:*

(A1) *there exists a bounded projection $\pi : L^2(\gamma) \longrightarrow V_h^-|_\gamma$, such that for all $\eta \in L^2(\gamma)$ and for all $\lambda \in M_h$*

$$\int_\gamma (\eta - \pi\eta)\lambda = 0, \tag{8}$$

and for all $\eta \in H^{1/2}_{00}(\gamma)$

$$\|\pi\eta\|_{H^{1/2}_{00}(\gamma)} \lesssim \|\eta\|_{H^{1/2}_{00}(\gamma)}. \tag{9}$$

(A2) *there exists a discrete lifting* $R_h : V_h^-|_\gamma \longrightarrow V_h^-$ *such that for all* $\eta \in V_h^-|_\gamma$

$$\|R_h \eta\|_{1,\Omega_-} \lesssim \|\eta\|_{H^{1/2}_{00}(\gamma)}. \tag{10}$$

Moreover let the two auxiliary spaces U_δ^+ *and* U_δ^- *be chosen in such a way that the following Jackson type inequality holds for some* $\tilde{R}, R \geq 1/2$: *for all* r, $1/2 \leq r \leq R$ *(resp. for all* $\tilde{r}$, $-1/2 \leq \tilde{r} \leq \tilde{R}$*)*

$$\eta \in H^r_0(\gamma) \quad \Longrightarrow \quad \inf_{\eta_\delta \in U_\delta^-} \|\eta - \eta_\delta\|_{1/2,\gamma} \lesssim \delta^{r-1/2} \|\eta\|_{r,\gamma}, \tag{11a}$$

$$\eta \in H^{\tilde{r}}(\gamma) \quad \Longrightarrow \quad \inf_{\eta_\delta \in U_\delta^+} \|\eta - \eta_\delta\|_{-1/2,\gamma} \lesssim \delta^{\tilde{r}+1/2} \|\eta\|_{\tilde{r},\gamma}. \tag{11b}$$

Then, if u_h *is the solution of problem (7), and if the true solution* u *of problem (1) verifies* $u \in H^s(\Omega)$ *for some* s, $2 \leq s \leq \min\{\tilde{R} + 3/2, R + 1/2\}$, *the following error estimate holds:*

$$\|u - u_h\|_{1,*} \lesssim \delta^{s-1} \|u\|_{s,\Omega} + \inf_{\lambda \in M_h} \|\partial_{\nu_a} u - \lambda\|_{H^{-1/2}(\gamma)}$$
$$+ \inf_{v_h \in V_h^+} \|u - v_h\|_{1,\Omega_+} + \inf_{v_h \in V_h^-} \|u - v_h\|_{1,\Omega_-} \tag{12}$$

where ∂_{ν_a} *denotes the trace on* γ *of the outer co-normal derivative to the subdomain* Ω_+.

Proof. Let

$$\mathcal{X}^* = \{u \in H^1_\Gamma(\Omega_+) \times H^1_\Gamma(\Omega_-) \ \text{ s.t. (6) holds } \},$$

and let $u_* \in \mathcal{X}^*$ be the unique element such that for all $v \in \mathcal{X}^*$ it holds that

$$\int_{\Omega_+} a \nabla u_* \cdot \nabla v \, dx + \int_{\Omega_-} a \nabla u_* \cdot \nabla v \, dx = \int_\Omega f v.$$

Let us denote by $a^* : H^1_\Gamma(\Omega_+) \times H^1_\Gamma(\Omega_-) \to \mathbb{R}$ the bilinear form

$$a^*(v, w) = \int_{\Omega_+} a \nabla v \cdot \nabla w \, dx + \int_{\Omega_-} a \nabla v \cdot \nabla w \, dx.$$

We first observe that, since both $\partial\Omega_+ \setminus \gamma$ and $\partial\Omega_- \setminus \gamma$ have strictly positive length (since γ coincides with just one edge of both Ω_+ and Ω_-), a^* is coercive on $H^1_\Gamma(\Omega_+) \times H^1_\Gamma(\Omega_-)$ with respect to the broken norm $\|\cdot\|_{1,*}$.

Since $\mathcal{X}_h^* \subset \mathcal{X}^*$, and since for all $w_h \in \mathcal{X}_h^*$ we have, by Galerkin orthogonality, $a^*(u_* - u, w_h) = 0$, it is not difficult to check that it holds for all $v_h \in \mathcal{X}_h^*$

$$\|u_* - u_h\|^2_{1,*} \lesssim a(u_* - u_h, u_* - u_h) = a(u_* - u_h, u_* - u + u - v_h)$$

which yields

$$\|u_* - u_h\|_{1,*} \lesssim \|u - u_*\|_{1,*} + \|u - v_h\|_{1,*}.$$

The arbitrariness of v_h implies

$$\|u - u_h\|_{1,*} \lesssim \|u - u_*\|_{1,*} + \inf_{v_h \in \mathcal{X}_h^*} \|u - v_h\|_{1,*}. \tag{13}$$

Since u is the solution of (1), for all $v \in H^1_\Gamma(\Omega_+) \times H^1_\Gamma(\Omega_-)$ it holds

$$\int_{\Omega_+} a\nabla u \cdot \nabla v\, dx + \int_{\Omega_-} a\nabla u \cdot \nabla v\, dx = \int_\Omega fv - \int_\gamma \partial_{\nu_a} u[v]\, ds.$$

Then for any $\tilde\eta_\delta \in U^+_\delta$ and $\lambda_h \in M_h$, using (6) and (5) we can bound the first term on the right hand side of (13) as:

$$\begin{aligned}
\|u - u_*\|^2_{1,*} &\lesssim \left| \int_\gamma \partial_{\nu_a} u[u_*]\, ds \right| \\
&\lesssim \int_\gamma \left|\partial_{\nu_a} u(u^+_* - P^-(u^+_*))\right| + \int_\gamma \left|\partial_{\nu_a} u(P^-(u^+_*) - u^-_*)\right| \\
&= \int_\gamma \left|(\partial_{\nu_a} u - \tilde\eta_\delta)(u^+_* - P^-(u^+_*))\right| \\
&\qquad + \int_\gamma \left|(\partial_{\nu_a} u - \lambda_h)(P^-(u^+_*) - u^-_*)\right| \\
&\lesssim \|\partial_{\nu_a} u - \tilde\eta_\delta\|_{-1/2,\gamma} \|(u^+_* - P^-(u^+_*)\|_{H^{1/2}_{00}(\gamma)} \\
&\qquad + \|\partial_{\nu_a} u - \lambda_h\|_{-1/2,\gamma} \|P^-(u^+_*) - u^-_*\|_{H^{1/2}_{00}(\gamma)}.
\end{aligned}$$

By triangular inequality, we can bound

$$\begin{aligned}
\|u^+_* - P^-(u^+_*)\|_{H^{1/2}_{00}(\gamma)} \leq{}& \|u^+_* - u\|_{H^{1/2}_{00}(\gamma)} \\
&+ \|u - P^- u\|_{H^{1/2}_{00}(\gamma)} + \|P^-(u - u^+_*)\|_{H^{1/2}_{00}(\gamma)}
\end{aligned}$$

and

$$\begin{aligned}
\|P^-(u^+_*) - u^-_*\|_{H^{1/2}_{00}(\gamma)} \leq{}& \|P^-(u^+_* - u)\|_{H^{1/2}_{00}(\gamma)} + \|P^- u - u\|_{H^{1/2}_{00}(\gamma)} \\
&+ \|u - u^-_*\|_{H^{1/2}_{00}(\gamma)},
\end{aligned}$$

from which, using the boundedness of P^- as well as the trace theorem we obtain

$$\begin{aligned}
\|u^+_* - P^-(u^+_*)\|_{H^{1/2}_{00}(\gamma)} &\leq \|u - u_*\|_{1,*} + \|u - P^- u\|_{H^{1/2}_{00}(\gamma)}, \\
\|P^-(u^+_*) - u^-_*\|_{H^{1/2}_{00}(\gamma)} &\leq \|u - u_*\|_{1,*} + \|u - P^- u\|_{H^{1/2}_{00}(\gamma)}.
\end{aligned}$$

Then, using the arbitrariness of λ_h and $\tilde\eta_\delta$ as well as inequalities (11a) and (11b), it is not difficult to conclude by standard reasoning that

$$\|u - u_*\|_{1,*} \lesssim \delta^{s-1} \|u\|_{s,\Omega} + \inf_{\lambda \in M_h} \|\partial_{\nu_a} u - \lambda\|_{H^{-1/2}(\gamma)}. \tag{14}$$

In order to bound the second term in (13), let $v^+_h \in V^+_h$ and $v^-_h \in V^-_h$ be respectively approximations to $u|_{\Omega_+}$ and $u|_{\Omega_-}$. The function $v_h \in L^2(\Omega)$ defined as $v_h|_{\Omega_+} = v^+_h$ and $v_h|_{\Omega_-} = v^-_h$ does not satisfy the constraint and therefore it does not belong to $\mathcal{X}^*_h$. Let $\check v_h \in \mathcal{X}^*_h$ be defined by

$$\check v_h|_{\Omega_+} = v^+_h, \qquad \check v_h|_{\Omega_-} = v^-_h - R_h \left[\pi\left(v^-_h - P^-(v^+_h)\right)\right].$$

It is not difficult to check that $\check{v}_h$ satisfies (6) and therefore $\check{v}_h \in \mathcal{X}_h^*$. We have

$$\|u - \check{v}_h\|_{1,*} \lesssim \|u - v_h\|_{1,*} + \|\pi \left(v_h^- - P^-(v_h^+)\right)\|_{H^{1/2}_{00}(\gamma)}.$$

Now, adding and subtracting $u|_\gamma$ and $P^-(u|_\gamma)$, by triangular inequality, using the boundedness of P^- and the trace theorem we obtain

$$\begin{aligned}\|\pi \left(v_h^- - P^-(v_h^+)\right)\|_{H^{1/2}_{00}(\gamma)} &\lesssim \|u - v_h\|_{1,*} + \|u - P^-(u)\|_{H^{1/2}_{00}} \\ &\lesssim \|u - v_h\|_{1,*} + \delta^{s-1}\|u\|_{1,\Omega}.\end{aligned}$$

Again, by the arbitrariness of v_h we conclude that

$$\inf_{v_h \in \mathcal{X}_h^*} \|u - \check{v}_h\|_{1,*} \lesssim \inf_{v_h \in V_h^+ \times V_h^-} \|u - v_h\|_{1,*} + \delta^{s-1}\|u\|_{1,\Omega},$$

which, together with (14) and (13) yields (12).

Remark 1. The extremely simple configuration considered (only two subdomains), hides some of the issues related to the analysis of the mortar method in more general configurations — namely the treatment of cross points and the coercivity of the broken bilinear form on the constrained space. Moreover, in this simple case the solution obtained with the method proposed coincides with the one obtained using the three fields formulation (see [2]) with a suitable choice of the discretization spaces. However, the approach used and the results obtained in this paper carry over to more complex cases (with the presence of cross-points), where the method obtained does not fall into the three fields formulation framework, with, in the worse case, a loss of a logarithmic factor in the error estimate.

3 Wavelet/FEM Coupling

Let us now consider the case of the coupling of wavelets and finite elements. As already pointed out, the use of both kinds of discretization can be particularly useful in a domain decomposition framework, giving the possibility of using wavelet bases in some rectangular shaped subdomains, and finite element discretizations to catch more complicated geometries. Weak continuity across the interface requires the jump of the solution to be orthogonal to the multiplier space, which translates in requiring

$$\int_\gamma u_h^+ \lambda \, ds = \int_\gamma u_h^- \lambda \, ds \qquad \forall \lambda \in M_h,$$

where u_h^+ is a piecewise polynomial function and λ a wavelet type function or, conversely, u_h^+ is a wavelet and λ a piecewise polynomial. Unfortunately, according to the particular nature of wavelets (whose expression is not known directly but only via the so called *refinement equation*), in both cases it is not possible to compute the above integral $\int_\gamma u_h^+ \lambda$ exactly.

In this section we will describe how to overcome such a problem by building up the two auxiliary spaces introduced in Section 2 to design the approximate constraint operator in the particular case of Wavelet/FEM coupling.

3.1 Building the Auxiliary Spaces

We recall that in general a wavelet type function ζ on $(0,1)$ at level j ($h = 2^{-j}$ corresponding to the mesh-size) (be it an element of the trace space or of the corresponding multiplier space) can be written as the restriction to $(0,1)$ of a linear combination of the dilated and translated of a single *refinable* function φ

$$\zeta(x) = \sum_{\ell} \zeta_{j,\ell}\varphi(2^j x - \ell)|_{(0,1)}. \tag{15}$$

The *refinability* of the function φ is by definition the existence of coefficients $\{g_l\}_{l\in\mathbb{Z}}$ such that

$$\varphi(s) = \sum_{k=0}^{K_1} g_k \varphi(2s-k).$$

In order to construct suitable auxiliary spaces, we will need a couple of biorthogonal multiresolution analyses $\{V_j\}_{j\geq j_0}$ and $\{\tilde{V}_j\}_{j\geq j_0}$ of $L^2(\gamma)$ with the following characteristics ([6,8]):

- $V_j \subset H^1(\gamma)$ is the subspace of B-splines of order n (with $n \geq 1$) on the uniform grid $\mathcal{G}_j$ obtained by splitting γ into 2^j equal segments;

- $\tilde{V}_j \subset H^1(\gamma)$ is a subspace having the same dimension as V_j, which is *biorthogonal* to V_j in the following sense: denoting by $\{e_{j,k}\ ,\ k = 0,\ldots,2^j\}$ the B-splines basis function in V_j , the space $\tilde{V}_j$ has a basis $\{\tilde{e}_{j,k},\ k = 0,\ldots,2^j\}$ which satisfies

$$\int_\gamma e_{j,k}\tilde{e}_{j,k'} = \delta_{kk'}, \quad \forall k,k' = 0,\ldots,2^j.$$

  Moreover, $\{\tilde{e}_{j,k}\}$ is a Riesz's basis for $\tilde{V}_j$, that is the following norm equivalence holds uniformly in j:

$$\|\sum_{k=0}^{2^j} c_k \tilde{e}_{j,k}\|_{0,\gamma} \simeq \left(\sum_{k=0}^{2^j} |c_k|^2\right)^{1/2}.$$

- the functions $\tilde{e}_{j,k}$ can be obtained as linear combination of the restriction to γ (identified through a suitable mapping with the interval $(0,1)$), of the translates and dilates (with a dilation factor 2^j) of a compactly supported function $\tilde{e}$,

$$\tilde{e}_{j,k} = \sum_{\ell} c^j_{k,\ell}\tilde{e}(2^j x - \ell)|_{(0,1)},$$

  which we assume to be itself *refinable*, *i.e.* to verify, for suitable values of the coefficients h_k,

$$\tilde{e}(s) = \sum_{k=0}^{K_2} h_k \tilde{e}(2s-k);$$

- all the scaling functions of V_j and $\tilde V_j$ vanish at the edges 0 and 1, except one function at each edge, that is

$$e_{j,0}(0) \neq 0 \quad \text{and} \quad e_{j,2^j}(1) \neq 0 \quad \forall j \geq j_0,$$
$$e_{j,k}(0) = 0 \quad \text{and} \quad e_{j,k}(1) \ = 0 \quad \forall j \geq j_0 \quad \text{and} \quad \forall k = 1, \cdots, 2^{j-1},$$

and

$$\tilde e_{j,0}(0) \neq 0 \quad \text{and} \quad \tilde e_{j,2^j}(1) \neq 0 \quad \forall j \geq j_0,$$
$$\tilde e_{j,k}(0) = 0 \quad \text{and} \quad \tilde e_{j,k}(1) \ = 0 \quad \forall j \geq j_0 \quad \text{and} \quad \forall k = 1, \cdots, 2^{j-1}.$$

- $\tilde V_j$ satisfies a Strang-Fix condition of order M, that is it contains polynomials up to degree $M-1$, while, of course, V_j contains polynomials of order n. In particular, this implies that there exists coefficients a^l_k and b^l_k, $l = 0, \cdots, n$ and $\tilde a^l_k$ and $\tilde b^l_k$, $l = 0, \cdots, M-1$ such that we will have for all $j \geq j_0$ and for $l = 0, \cdots, n$

$$2^{j/2}(2^j x)^l = \sum_{k=0}^{2^j} a^l_k e_{j,k}(x), \tag{16a}$$
$$2^{j/2}(2^j(1-x))^l = \sum_{k=0}^{2^j} b^l_k e_{j,k}(x), \tag{16b}$$

and for $l = 0, \cdots, M-1$

$$2^{j/2}(2^j x)^l = \sum_{k=0}^{2^j} \tilde a^l_k \tilde e_{j,k}(x), \tag{17a}$$
$$2^{j/2}(2^j(1-x))^l = \sum_{k=0}^{2^j} \tilde b^l_k \tilde e_{j,k}(x). \tag{17b}$$

Let us now denote by $V^0_j = V_j \cap H^1_0(\gamma)$ and $\tilde V^0_j = \tilde V_j \cap H^1_0(\gamma)$ the space of functions of V_j and $\tilde V_j$ respectively, vanishing at the boundaries of the interval

$$V^0_j = span < e_{j,k}, k = 1, \cdots, 2^j - 1 >$$
$$\tilde V^0_j = span < \tilde e_{j,k}, k = 1, \cdots, 2^j - 1 > .$$

Following [3], it is possible to construct two subspaces $V^*_j \subset V_j$ and $\tilde V^*_j \subset \tilde V_j$ which will play the role of one of the two auxiliary spaces U^+_δ and U^-_δ in the case of coupling wavelets and finite elements.

Let us briefly review the construction: let the vectors $(\alpha_k)_{k=0,\cdots,M-2}$ and $(\beta_k)_{k=2^j-M+2,\cdots,2^j}$ denote respectively the solutions of the two following linear systems:

$$\sum_{k=0}^{M-2} \tilde a^l_k \alpha_k = \tilde a^l_{M-1} \qquad \forall\, l = 0, \cdots, M-2 \tag{18a}$$
$$\sum_{k=2^j-M+2}^{2^j} \tilde b^l_k \beta_k = \tilde b^l_{2^j-M+1} \qquad \forall\, l = 0, \cdots, M-2. \tag{18b}$$

The solvability of the above linear system is discussed in ([3]). The space $\tilde{V}_j^*$ is obtained by defining

$$\tilde{V}_j^* = span < \tilde{e}_{j,k}^*, k = 1, \cdots, 2^j - 1 >$$

where

$$\begin{aligned} \tilde{e}_{j,k}^* &= \tilde{e}_{j,k} + c_k \tilde{e}_{j,0}, && \text{for } k = 1, \cdots, M-1 \\ \tilde{e}_{j,k}^* &= \tilde{e}_{j,k}, && \text{for } k = M, \cdots, 2^j - M \\ \tilde{e}_{j,k}^* &= \tilde{e}_{j,k} + d_k \tilde{e}_{j,2^j}, && \text{for } k = 2^j - M + 1, \cdots, 2^j - 1 \end{aligned}$$

with

$$\begin{aligned} c_k &= -\frac{\alpha_k}{\alpha_0}, \quad k = 1, \cdots, M-2 && c_{M-1} = \frac{1}{\alpha_0} \\ d_k &= -\frac{\beta_k}{\beta_{2^j}}, \quad k = 2^j - M + 2, \cdots, 2^j - 1 && d_{2^j - M + 1} = \frac{1}{\beta_{2^j}}. \end{aligned}$$

The same construction can be carried out for the space

$$V_j^* = span < \tilde{e}_{j,k}^*, k = 1, \cdots, 2^j - 1 >$$

with the obvious switching of the dual "ingredients".

The following theorem has been proven in [3]

Theorem 2. *The spaces V_j^* and $\tilde{V}_j^*$ have the same dimension as $\tilde{V}_j^0$, and V_j^0 respectively. They satisfy the* inf-sup *conditions*

$$\inf_{\eta \in V_j^0} \sup_{\zeta \in \tilde{V}_j^*} \frac{\int_\gamma \eta\zeta \, ds}{\|\eta\|_{H_{00}^{1/2}(\gamma)} \|\zeta\|_{-1/2,\gamma}} \geq \alpha_1$$

$$\inf_{\eta \in \tilde{V}_j^0} \sup_{\zeta \in V_j^*} \frac{\int_\gamma \eta\zeta \, ds}{\|\eta\|_{H_{00}^{1/2}(\gamma)} \|\zeta\|_{-1/2,\gamma}} \geq \alpha_2,$$

and a Strang-Fix condition with the same order as V_j and $\tilde{V}_j$ respectively. Moreover the two Riesz's bases $e_{j,k}^*$ and $\tilde{e}_{j,k}^*$ are constructed in such a way that the two following biorthogonality relations hold:

$$\int_\gamma e_{j,k} \tilde{e}_{j,k'}^* = \delta_{k,k'}, \qquad \int_\gamma \tilde{e}_{j,k} e_{j,k'}^* = \delta_{k,k'}, \qquad \forall k, k' = 1, \ldots, 2^j - 1. \tag{19}$$

Thanks to the refinability of the functions $\tilde{e}$ and φ, it is possible to compute integrals of the product of a wavelet type function ζ of the form (15) with any function in $\tilde{V}_j$ (and therefore in $\tilde{V}_j^0$ and in $\tilde{V}_j^*$) by reducing it to the solution of an eigenvector problem associated with the coefficients of the corresponding refinement equations ([7]). In fact, by linearity of the integral, the product of ζ times any function in $\tilde{V}_j$ reduces to computing several integrals of the form

$$\begin{aligned} \int_0^1 \tilde{e}(2^j x - l)\varphi(2^j x - k)(x) dx &= \int_0^{2^j} \tilde{e}(y)\varphi(y + l - k) dy \\ &= \frac{1}{2^j} \sum_{m=0}^{2^j} \int_m^{m+1} \tilde{e}(y)\varphi(y + l - k) dy. \end{aligned}$$

Let us briefly recall how such integrals can be computed. Defining

$$I(m,k)=\int_m^{m+1}\tilde{e}(x)\varphi(x-k),$$

and recalling that the characteristic function of the interval $(0,1)$ satisfies itself a refinement equation, namely

$$\chi_{(0,1)}(x)=\sum_{n=0}^{1}\chi_{(0,1)}(2x-n),$$

we can write

$$\begin{aligned}
I(m,k) &= \sum_{n=0}^{1}\sum_{p=0}^{K_2}\sum_{r=0}^{K_1} h_p g_r \int_{\mathbb{R}} \chi_{(0,1)}(2x-2m-n)\tilde{e}(2x-p)\varphi(2x-2k-r)dx \\
&= \frac{1}{2}\sum_{n=0}^{1}\sum_{p=0}^{K_2}\sum_{r=0}^{K_1} h_p g_r \int_{\mathbb{R}} \chi_{(0,1)}(y+p-n-2m)\tilde{e}(y)\varphi(y+p-2k-r)dy \\
&= \frac{1}{2}\sum_{n=0}^{1}\sum_{p=0}^{K_2}\sum_{r=0}^{K_1} h_p g_r I(2m+n-p,2k+r-p) \\
&= \frac{1}{2}\sum_{n=2m-p}^{2m-p+1}\sum_{p=0}^{K_2}\sum_{r=2k-p}^{2k-p+K_2} h_p g_{r+p-2k} I(n,r).
\end{aligned}$$

Since both φ and $\tilde{e}$ have compact support, $I(m,k)$ does not vanish only for a finite number of couples (m,k). It is not difficult to realize that the original problem has then been reduced to the solution of an eigenvector equation for the multi-indexed vector $(I(m,k))_{m,k}$, that has to be solved once for all. Under suitable mild assumptions on the functions $\tilde{e}$ and φ, we can assume that the function I satisfies a normalization hypothesis $\sum_m\sum_k I(m,k)=1$, so that such eigenvector problem has a unique solution.

3.2 FEM Master / Wavelet Slave

For choosing the auxiliary spaces U_δ^+ and U_δ^- to use for coupling wavelets and finite elements in the mortar method, we distinguish two cases: *FEM master / Wavelet slave* and *Wavelet master / FEM slave*. In both cases, U_δ^+ and U_δ^- must be chosen in such a way that $P^-(u_h^+)$ and the integrals of the product $\lambda_h P^-(u_h^+)$ can be computed. Let us first consider the case of a finite element master side coupled with a wavelet type slave side. In this case we assume V_h^+ to be a finite element space of polynomials of degree $N-1$ on an unstructured, non uniform grid, and we denote by h_+ the maximum size of the discretization length, while V_h^- and M_h are two wavelet type spaces at level j. Referring to the choice of M_h as proposed in [3], that is M_h is a suitable dual space of $V_h^-|_\gamma$, let us assume that M_h and V_h^- both contain polynomials of the same degree $L-1$. It is easy to check that the appropriate choice of the auxiliary spaces is obtained by setting $U_\delta^+=V_j^*$, the space of B-splines of order n satisfying non homogeneous boundary conditions, and $U_\delta^-=\tilde{V}_j^0$. In fact, this is the case where u_h^+ is a FEM type function, whose projection on U_δ^-, defined by (5), can be computed by taking advantage of the biorthogonality property (19) in the following way:

$$P^- u_h^+ = \sum_{k=1}^{2^j-1} \left(\int_\gamma u_h^+ e_{j,k}^* \, ds \right) \tilde{e}_{j,k}.$$

Recall now that $\tilde{e}_{j,k}$ is a linear combination of the translates and dilates of the refinable function $\tilde{e}$ and its integral with a wavelet type function can be computed using the previously mentioned technique, while e_{jk}^* is a B-spline whose integral against the elements of $V_h^+|_\gamma$ can be computed exactly. By applying Theorem 1 we can then estimate the effect of using the approximate integration technique proposed in Section 2. We have the following

Corollary 1. *If $u \in H^s(\Omega)$ with $2 \le s \le T$ it holds*

$$\|u - u_h\|_{1,*} \lesssim \left(2^{-j(s-1)} + h_+^{s-1} \right) \|u\|_{s,\Omega}, \tag{20}$$

where the limit T in the bound is $T = \min\{n + 5/2, M + 1/2, L, N\}$.

Proof. Using the abstract inequality (12), it is easy to see that in this case, setting $\delta = 2^{-j}$, two Jackson's type inequalities of the form (11a) and (11b) hold, with $R = M$ and $\tilde{R} = n+1$. Then we can estimate

$$\|u - u_h\|_{1,*} \lesssim 2^{-j(s-1)} \|u\|_{s,\Omega} + 2^{-j(s-1)} \|\partial_{\nu_a} u\|_{s-3/2,\gamma}$$

$$+ h_+^{(s-1)} \|u\|_{s,\Omega_+} + 2^{-j(s-1)} \|u\|_{s,\Omega_-}$$

provided that $s \le N$ and $s \le L$. Furthermore, using a classical trace theorem we can write

$$\|u - u_h\|_{1,*} \lesssim 2^{-j(s-1)} \|u\|_{s,\Omega} + 2^{-j(s-1)} \|u\|_{s,\Omega_-}$$

$$+ h_+^{(s-1)} \|u\|_{s,\Omega_+} + 2^{-j(s-1)} \|u\|_{s,\Omega_-},$$

which easily implies (20).

3.3 Wavelet Master / FEM Slave

In this case we set V_h^+ to be a wavelet type space, while V_h^- and M_h are two finite element spaces defined on unstructured, non uniform grids of mesh size equals to h_-, the grid for M_h being the trace on γ of the grid for V_h^-. In analogy with the *FEM master / Wavelet slave* case, the approximate integration is this time performed by setting $U_\delta^+ = \tilde{V}_j^*$ and $U_\delta^- = V_j^0$. The projection $P^-(u_h^+)$ in this case takes the form

$$P^- u_h^+ = \sum_{k=1}^{2^j-1} \left(\int_\gamma u_h^+ \tilde{e}_{j,k}^* \, ds \right) e_{j,k},$$

and weak continuity leads to the computation of FEM/FEM integrals.
In this case it is easy to prove that Corollary 1 becomes

Corollary 2. *If $u \in H^s(\Omega)$ with $2 \le s \le T$ it holds*

$$\|u - u_h\|_{1,*} \lesssim \left(2^{-j(s-1)} + h_-^{s-1} \right) \|u\|_{s,\Omega}, \tag{21}$$

with $T = \min\{n + 3/2, M + 3/2, L, N\}$.

Remark 2. In both cases *FEM master / Wavelet slave* and *Wavelet master / FEM slave*, the two auxiliary spaces have been chosen in such a way that the two integrals defining the projectors are computable. Moreover, biorthogonality implies that *no linear system* has to be solved in order to compute the auxiliary projector.

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