# Wavelet Stabilization and Preconditioning for Domain Decomposition*

Silvia Bertoluzza Angela Kunoth

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**Abstract**

The aim of this paper is to propose a stabilization technique in order to circumvent the two *inf–sup conditions* needed for stability of the discretization of the Three Fields Formulation for domain decomposition methods [BM93, BM94]. Realization of this technique in terms of wavelets is presented. Furthermore, the resulting discrete problem is shown to admit an asymptotically optimal preconditioner also based on wavelets.

**Key words:** Three Fields Formulation, domain decomposition, biorthogonal wavelets, stabilization, preconditioning.
**AMS subject classification:** 65N55.

## 1 Introduction

In this paper we deal with domain decomposition of elliptic problems by the Three Fields Formulation, developed in [BM93, BM94]. Differently from other non-conforming formulations [BMP94, Dor89] of domain decomposition, weak continuity is not imposed by requiring that the jump across the interface is orthogonal to a suitable multiplier space, but by introducing the space $\Phi$ of traces of functions in $H^1(\Omega)$ on the interface $\Sigma$. The elements of $\Phi$ are to be used as a Dirichlet boundary condition — imposed by means of Lagrange multipliers — for the functions in the subdomains.

Existence and uniqueness of the solution of the resulting variational domain decomposition formulation of a given elliptic second order partial differential equation relies on the validity of two *inf–sup conditions* relating the interface trace space with the space of Lagrange multipliers, and the latter with the space of functions in the subdomains.

By applying a Schur complement technique, the solution of such a problem can be reduced (both at the continuous and at the discrete level) to the solution of an equation on the "trace" unknown, corresponding to the *Steklov–Poincaré* operator. Computing

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the action of such operator on a given function reduces to solving independent Dirichlet problems on the subdomains, a task which can be carried out in parallel.

Clearly, in order to design efficient solvers for this domain decomposition formulation there is the need of a good preconditioner for the interface problem. The Steklov-Poincaré operator is a first order elliptic operator, and it is well known that if the "trace" space $\Phi$ is discretized by means of suitable wavelet functions, asymptotically optimal diagonal preconditioners do exist.

On the other hand, in order for the discrete problem to be well posed, the approximation spaces used for discretizing the functions in the subdomains, the Lagrange multipliers, and the "traces" respectively, need to satisfy suitable compatibility conditions, whose fulfillment is not always straightforward. The use of wavelet type bases on the interface would then put some heavy restriction on the choice of the discretizations in the subdomains. This seriously limits one of the main advantages of this formulation: the possibility to use in principle very different discretizations on each subdomain.

One possibility is to circumvent such restrictions by applying a stabilization technique ([BFHR97]). Different stabilization techniques for the three fields domain decomposition method have been proposed in [BBM92, BFMR97]. In choosing here a specific stabilization technique, one has to keep in mind the goal of applying in the end a wavelet preconditioner to the discrete interface problem. This leads us to discard the techniques proposed in [BBM92], for which the stabilized operator is continuous and coercive with respect to two *non equivalent* norms, whereas wavelet preconditioners apply to operators satisfying

$$\langle Av, v \rangle \sim \|v\|^2,$$

that is, stability and continuity are required with respect to the same norm.

The stabilization technique we propose here consists in adding a residual term, measured in a natural scalar product (of type $H^{1/2}(\Sigma)$). Such stabilization acts only on the interface unknowns, without changing the problems in the subdomains. Therefore, in the implementation of such formulation one can use one's favorite solver for the problem in the subdomains without any need for modification. Moreover, we show that, under very mild compatibility conditions on the Lagrange multiplier spaces — which can basically be reduced to asking that each multiplier space contains the space of piecewise constants on the coarse grid composed by the "elements" $\partial\Omega^k \cap \partial\Omega^\ell$ — if stable discretizations are used for each of the subdomain problems, the stabilized formulation is boundedly invertible for any choice of the discretization for the "trace" space.

Also for this formulation the solution can be reduced to the inversion of a Steklov-Poincaré type operator, which is shown to be continuous and coercive with respect to the natural norm on the "trace" space. Then, wavelet preconditioners can be applied, provided wavelet bases exist which induce on such a space an equivalent norm, expressed in terms of the wavelet coefficients.

In Section 2 we introduce the new continuous stabilized formulation for the problem considered. In Section 3 the discretization of such a problem is discussed with particular emphasis on the use of wavelet bases for the interface problem, for which an asymptotically optimal preconditioner is described.

Different constructions of biorthogonal wavelets by conforming domain decomposition [CM98, CTU99, DS97, DS98] are available, for which one can prove that the above mentioned norm equivalence holds [DK], and which are then well suited to be applied in our case. Anyway, for our purpose we can also make use of a simplified construction which can be implemented by using only the Fast Wavelet Transform on $]0,1[$. For completeness we describe this construction in Section 4.

# 2 Stabilization of the Continuous Three Fields Formulation

## 2.1 Notation and Functional Setting

In the following, for any Hilbert space $H$, we will denote by $(\cdot,\cdot)_H$ its inner product, and by ${}_{H'}\langle\cdot,\cdot\rangle_H$ the action of its dual $H'$, though at times we may introduce a different notation. Moreover, we will frequently employ the notation $a \lesssim b$ (respectively $a \gtrsim b$) for saying that $a$ can be bounded from above (respectively, from below) by a constant multiple of $b$ uniformly in any parameters on which $a, b$ may depend. Furthermore, $a \sim b$ means $a \lesssim b$ and $b \lesssim a$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded polygonal domain with Lipschitz boundary $\partial\Omega$. Let $\Omega$ be decomposed into a finite number of non-overlapping polygonal subdomains $\Omega^k$, $k = 1,\dots,K$,

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}^k, \qquad \Omega^k \cap \Omega^m = \emptyset, \quad k \neq m, \tag{2.1.1}$$

where

$$\Sigma = \left(\bigcup_{k=1}^{K} \partial\Omega^k\right) \setminus \partial\Omega \tag{2.1.2}$$

is called the *skeleton* of the decomposition. We will also employ the notation

$$\Gamma^k := \partial\Omega^k \setminus \partial\Omega \tag{2.1.3}$$

so that

$$\Sigma = \bigcup_{k=1}^{K} \Gamma^k. \tag{2.1.4}$$

Let $H^{1/2}(\partial\Omega^k)$ and $H^{1/2}_{00}(\Gamma_k)$ be defined as the trace space of, respectively, $H^1(\Omega^k)$ and $H^1_{\partial\Omega}(\Omega^k) := \{v^k \in H^1(\Omega^k),\ v^k = 0 \text{ on } \partial\Omega \cap \partial\Omega^k\}$, with norms

$$\|\sigma\|_{H^{1/2}(\partial\Omega^k)} := \inf_{v^k \in H^1(\Omega^k):\ v^k|_{\partial\Omega^k} = \sigma} \|v^k\|_{H^1(\Omega^k)}, \tag{2.1.5}$$

$$\|\sigma\|_{H^{1/2}_{00}(\Gamma^k)} := \inf_{v^k \in H^1_{\partial\Omega}(\Omega^k):\ v^k|_{\Gamma^k} = \sigma} \|v^k\|_{H^1(\Omega^k)}.$$

**Remark 2.1** *Equivalent norms for $H^{1/2}(\partial\Omega^k)$ and for $H^{1/2}_{00}(\Gamma_k)$ can be defined through the norms of $H^{1/2}(]0,1[^{n-1})$ and $H^{1/2}_{00}(]0,1[^{n-1})$ by using an atlas and partitions of unity, the constants in the equivalence depending on the diameter of $\Omega^k$.*

For each $k$, we let $H^{1/2}_*(\Gamma^k)$ denote either $H^{1/2}(\Gamma^k)$ or $H^{1/2}_{00}(\Gamma^k)$, depending on whether $\Gamma^k$ is a closed or an open set, with norm

$$|\sigma|_{1/2,\Gamma^k} := \begin{cases} \|\sigma\|_{H^{1/2}_{00}(\Gamma^k)}, & \text{if } \Gamma^k \cap \partial\Omega \neq \emptyset, \\ \|\sigma\|_{H^{1/2}(\partial\Omega^k)}, & \text{otherwise, (i.e., if } \Gamma^k = \partial\Omega^k). \end{cases} \tag{2.1.6}$$

We will denote by $(\cdot,\cdot)_{1/2,\Gamma^k}$ the corresponding inner product. Moreover, let $H^{-1/2}_*(\Gamma^k)$ be the corresponding dual, whose norm and inner product will be denoted by $|\cdot|_{-1/2,\Gamma^k}$ and $(\cdot,\cdot)_{-1/2,\Gamma^k}$. Duality between $H^{1/2}_*(\Gamma^k)$ and $H^{-1/2}_*(\Gamma^k)$ will be written as $\langle\cdot,\cdot\rangle_k$.

We can now introduce the functional setting for the domain decomposition method we are going to consider. Let $V$ be the product space

$$V := \{(v^1,\ldots,v^K), v^k \in H^1(\Omega^k),\ v^k = 0 \text{ on } \partial\Omega\cap\partial\Omega^k\} \tag{2.1.7}$$

which is isomorphic to

$$\{v \in L^2(\Omega) :\ v^k = v|_{\Omega^k} \in H^1(\Omega^k),\ v^k = 0 \text{ on } \partial\Omega\cap\partial\Omega^k\},$$

endowed with the norm

$$\|v\|_V^2 := \sum_k \|v^k\|^2_{H^1(\Omega^k)}, \qquad v \in V, \tag{2.1.8}$$

induced by the inner product

$$(u,v)_V := \sum_k (u^k,v^k)_{H^1(\Omega^k)}. \tag{2.1.9}$$

Moreover, let $\Lambda$ and $\Phi$ be defined by

$$\Lambda := \prod_k H^{-1/2}_*(\Gamma^k) \tag{2.1.10}$$

and $\Phi = H^1_0(\Omega)\big|_\Sigma$, that is

$$\Phi := \{\sigma \in L_2(\Sigma) :\ \text{there exists } v \in H^1_0(\Omega) \text{ such that } \sigma = v|_\Sigma\}. \tag{2.1.11}$$

$\Lambda$ and $\Phi$ are endowed with the norms

$$\|\mu\|^2_\Lambda := \sum_k |\mu^k|^2_{-1/2,\Gamma^k}, \qquad \|\sigma\|_\Phi := \inf_{v\in H^1_0(\Omega):\ v|_\Sigma=\sigma} \|v\|_{H^1(\Omega)}, \tag{2.1.12}$$

respectively. We remark that $H^1_0(\Omega)$ can be identified with a subset of $V$:

$$H^1_0(\Omega) \cong \{v = (v^k) \in V :\ \ \exists\sigma\in\Phi,\ v^k = \sigma \text{ on } \Gamma^k\} \subset V.$$

In the following, when writing $v = (v^k) \in H^1_0(\Omega)$ for an element $v \in V$, we will refer to such an isomorphism.

An observation that will be important in the sequel is the following:

**Proposition 2.2** *It holds*

$$\Lambda' = \prod_k H_*^{1/2}(\Gamma^k), \qquad \|\sigma^k\|_{\Lambda'} \sim \left(\sum_k |\sigma|^2_{1/2,\Gamma^k}\right)^{1/2}. \tag{2.1.13}$$

*Moreover, one has that*

*(i) the space $\Phi$ can be identified with a proper subset of $\Lambda'$, by identifying $\sigma \in \Phi$ with $(\sigma^k) \in \Lambda'$, $\sigma^k = \sigma|_{\Gamma^k}$;*

*(ii) for $\sigma \in \Phi$ the equivalences*

$$\|\sigma\|_\Phi \sim \left(\sum_k |\sigma^k|_{1/2,\Gamma^k}\right)^{1/2} \sim \|\sigma\|_{\Lambda'}, \qquad (\sigma^k = \sigma|_{\Gamma^k}), \tag{2.1.14}$$

*hold.*

**Proof:** We start by proving that (2.1.13) holds. In fact, let $F \in \Lambda'$. Since $\Lambda$ is a separable Hilbert space, there exists an isomorphism between $\Lambda$ and $\Lambda'$. Thus, there is a (uniquely determined) element $\mu_F = (\mu_F^k) \in \Lambda$ such that

$${}_{\Lambda'}\langle F, \mu\rangle_\Lambda = \sum_k (\mu_F^k, \mu^k)_{-1/2,\Gamma^k} \text{ for all } \mu \in \Lambda, \qquad \|F\|^2_{\Lambda'} = \sum_k |\mu_F^k|^2_{-1/2,\Gamma^k}. \tag{2.1.15}$$

On the other hand, also $H_*^{-1/2}(\Gamma^k)$ is, for each $k$, a separable Hilbert space. Thus, there also exists an (uniquely determined) element $\sigma_F^k \in (H_*^{-1/2}(\Gamma^k))' = H_*^{1/2}(\Gamma^k)$ such that

$$(\mu_F^k, \mu^k)_{-1/2,\Gamma^k} = \langle \sigma_F^k, \mu^k\rangle_k \text{ for all } \mu^k \in H_*^{-1/2}(\Gamma^k) \tag{2.1.16}$$

and $|\mu_F^k|_{-1/2,\Gamma^k} = |\sigma_F^k|_{1/2,\Gamma^k}$. Comparing (2.1.15) and (2.1.16) yields (2.1.13).

As far as (ii) is concerned, let now $\sigma \in \Phi$ and let $v \in H_0^1(\Omega)$ be any function such that $u = \sigma$ on $\Sigma$. Then we have

$$\sum_k |\sigma|^2_{1/2,\Gamma^k} \leq \sum_k \|v\|^2_{H^1(\Omega^k)} = \|v\|^2_{H_0^1(\Omega)}.$$

The arbitrariness of $v$ yields

$$\sum_k |\sigma|^2_{1/2,\Gamma^k} \lesssim \inf_{v \in H_0^1(\Omega):\ v|_\Sigma = \sigma} \|v\|^2_{H^1(\Omega)} = \|\sigma\|^2_\Phi.$$

Let now $u$ in $H_0^1(\Omega)$ be defined such that $-\Delta u = 0$ in $\Omega^k$ and $u = \sigma$ on $\Gamma^k$ for all $k$. It holds

$$\|\sigma\|^2_\Phi \leq \|u\|^2_{H_0^1(\Omega)} = \sum_k \|u\|^2_{H^1(\Omega^k)} \lesssim \sum_k |\sigma|^2_{1/2,\Gamma^k}.$$

∎

Note that in principle the constants in (2.1.14) depend on the size of the subdomains.

## 2.2 The Three Fields Formulation

We consider the second order elliptic boundary value problem

$$\begin{aligned} -\operatorname{div} a(x) \operatorname{grad} u(x) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.2.1}$$

with a sufficiently smooth and uniformly positive definite matrix $a(x)$ and $f \in L_2(\Omega)$. To solve this by a domain decomposition approach, for each $k \in \{1, \ldots, K\}$ let $a^k : H^1(\Omega^k) \times H^1(\Omega^k) \to I\!R$ be the bilinear form induced by the differential operator on the subdomain $\Omega^k$,

$$a^k(u, v) = \int_{\Omega^k} a(x) \ \operatorname{grad} u \cdot \operatorname{grad} v \, dx.$$

A composed bilinear form $a : V \times V \to I\!R$ can be defined by

$$a(\cdot, \cdot) := \sum_k a^k(\cdot, \cdot). \tag{2.2.2}$$

For all $v \in H_0^1(\Omega)$, $a(\cdot, \cdot)$ satisfies

$$a(v, v) = \sum_k \int_{\Omega^k} a(x) |\operatorname{grad} v|^2 \, dx \sim \|v\|^2_{H^1(\Omega)}. \tag{2.2.3}$$

In alternative to the standard weak formulation of the boundary value problem (2.2.1),

$$\begin{aligned} &\text{find } u^* \in H_0^1(\Omega) \text{ such that} \\ &a(u^*, v) = (f, v)_{L_2(\Omega)} \qquad \text{for all } v \in H_0^1(\Omega), \end{aligned} \tag{2.2.4}$$

we can then consider the *Three Fields Formulation*:

find $(u, \lambda, \tau) \in V \times \Lambda \times \Phi$ such that

$$\left\{ \begin{array}{rcll} a(u, v) - \sum_k \langle \lambda^k, v^k \rangle_k &=& (f, v)_{L_2(\Omega)} & \text{for all } v \in V, \\ \sum_k \langle \tau - u^k, \mu^k \rangle_k &=& 0 & \text{for all } \mu \in \Lambda, \\ \sum_k \langle \lambda^k, \sigma \rangle_k &=& 0 & \text{for all } \sigma \in \Phi. \end{array} \right. \tag{2.2.5}$$

Such formulation was introduced in [BM93], where it was shown that problem (2.2.5) has, for every $f \in L_2(\Omega)$, a unique solution $(u, \lambda, \tau)$, satisfying

$$\left\{ \begin{array}{rcll} u^k &=& u^* & \text{in } \Omega^k, \qquad k = 1, \ldots, K, \\ \lambda^k &=& a \dfrac{\partial u^*}{\partial n^k} & \text{on } \partial\Omega^k, \qquad k = 1, \ldots, K, \\ \tau &=& u^* & \text{on } \Sigma, \end{array} \right. \tag{2.2.6}$$

where $\partial u^*/\partial n^k$ is the outward normal derivative of the restriction of $u^*$ to $\Omega^k$.

We can write the system (2.2.5) more conveniently in operator form. Define $A : V \to V'$ by

$$\langle Av, v\rangle := a(v, v).$$

For each $k = 1, \dots, K$, let $B^k : H^1_{\partial\Omega}(\Omega^k) \to H^{1/2}_*(\Gamma^k)$ denote the *trace operator*. Let then $B : V \to \Lambda'$ be defined by $B := \mathrm{diag}(B^1, \dots, B^K)$. Moreover let $C^k : H^{-1/2}_*(\Gamma^k) \to \Phi'$ be defined as

$$\langle C^k \mu^k, \sigma\rangle := \langle \mu^k, \sigma\rangle_k,$$

and let $C : \Lambda \to \Phi'$ be assembled as $C := \mathrm{diag}(C^1, \dots, C^K)$. With these notations, the system (2.2.5) can be written as follows: find $(u, \lambda, \tau) \in V \times \Lambda \times \Phi$ solution of

$$\begin{pmatrix} A & -B^T & 0 \\ -B & 0 & C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \\ \tau \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}. \tag{2.2.7}$$

**Remark 2.3** *[BM93] One of the interests for the Three Fields Formulation lies in the observation that, for $\tau \in \Phi$ given, the computation of $u$ and $\lambda$ reduces to solving $K$ independent Dirichlet problems on the subdomains $\Omega^k$, each one of the form*

$$\begin{pmatrix} A^k & -(B^k)^T \\ -B^k & 0 \end{pmatrix} \begin{pmatrix} u^k \\ \lambda^k \end{pmatrix} = \begin{pmatrix} f^k \\ -(C^k)^T\tau \end{pmatrix}. \tag{2.2.8}$$

*$\tau$ can then be computed as solution of*

$$\mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T\tau = \mathbf{C}\mathbf{A}^{-1}\begin{pmatrix} f \\ 0 \end{pmatrix} \tag{2.2.9}$$

*with*

$$\mathbf{A} := \begin{pmatrix} A & -B^T \\ -B & 0 \end{pmatrix}, \qquad \mathbf{C} := (0 \quad C), \tag{2.2.10}$$

*Here the operator $\mathbf{S} := \mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T$ is just the Poincaré-Steklov operator on $\Sigma$. It has been shown in [BM93] that $\mathbf{S}$ is an isomorphism from $\Phi$ to $\Phi'$ that satisfies in addition ${}_{\Phi'}\langle \mathbf{S}\cdot, \cdot\rangle_\Phi \sim \|\cdot\|^2_\Phi$. Since $\mathbf{S}$ is an operator of order one, it needs to be preconditioned. On the other hand, each iteration implies solving (2.2.8), a task which, thanks to the previous observation, can be carried out in parallel.*

## 2.3 A New Stabilized Formulation

Let a bilinear form $(\cdot, \cdot)_{\Lambda'} : \Lambda' \times \Lambda' \to I\!R$ be defined by

$$(\sigma, \tau)_{\Lambda'} := \sum_k (\sigma, \tau)_{1/2,\Gamma^k} = \sum_k (\sigma|_{\Gamma^k}, \tau|_{\Gamma^k})_{1/2,\Gamma^k}. \tag{2.3.1}$$

Thanks to Remark 2.2 (ii), $(\cdot,\cdot)_{\Lambda'}$ is a scalar product inducing on $\Phi \subset \Lambda'$ a norm which is equivalent to $\|\cdot\|_\Phi$. Then for any $\delta > 0$, and for $(u, \lambda, \tau)$ solution of (2.2.5), it holds

$$\delta(\tau - u, \sigma)_{\Lambda'} + \sum_k \langle \lambda^k, \sigma \rangle_k = 0 \qquad \text{for all } \sigma \in \Phi, \tag{2.3.2}$$

since $(u, \tau, \lambda)$ satisfies $u = \tau$ on $\Sigma$. We can then introduce the following equivalent formulation of (2.2.5):

find $(u, \lambda, \tau) \in V \times \Lambda \times \Phi$ such that

$$\left\{ \begin{array}{rcll} a(u,v) - \sum_k \langle \lambda^k, v^k \rangle_k &=& (f,v)_{L_2(\Omega)} & \text{for all } v \in V, \\ \sum_k \langle \tau - u^k, \mu^k \rangle_k &=& 0 & \text{for all } \mu \in \Lambda, \\ \delta(\tau - u, \sigma)_{\Lambda'} + \sum_k \langle \lambda^k, \sigma \rangle_k &=& 0 & \text{for all } \sigma \in \Phi. \end{array} \right. \tag{2.3.3}$$

Also (2.3.3) can be rewritten in operator form, by introducing the operator $D : \Lambda' \to \Phi'$ defined by

$$\langle D\tau, \sigma \rangle = (\tau, \sigma)_{\Lambda'}.$$

Then problem (2.3.3) rewrites as follows: find the solution $(u, \lambda, \tau)$ of

$$\begin{pmatrix} A & -B^T & 0 \\ -B & 0 & C^T \\ \delta DB & C & \delta D \end{pmatrix} \begin{pmatrix} u \\ \lambda \\ \tau \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}. \tag{2.3.4}$$

In order to analyse the problem (2.3.4) as well as its discretization, it will be useful to reorder the equations and the unknowns. To this end, define $\mathcal{Z} = V \times \Phi$, endowed with the norm $\|v, \sigma\|^2_{\mathcal{Z}} := \|v\|^2_V + \|\sigma\|^2_\Phi$. Rewrite (2.3.3) as follows: find $(u, \tau; \lambda) \in \mathcal{Z} \times \Lambda$ such that

$$\begin{array}{lclcll} \mathcal{A}_\delta(u,\tau; v,\sigma) &+& \mathcal{B}(v,\sigma;\lambda) &=& (f,v)_{L_2(\Omega)} & \text{for all } (v,\sigma) \in \mathcal{Z}, \\ \mathcal{B}(u,\tau;\mu) & & &=& 0 & \text{for all } \mu \in \Lambda, \end{array} \tag{2.3.5}$$

with $\mathcal{A}_\delta : \mathcal{Z} \times \mathcal{Z} \to I\!R$ and $\mathcal{B} : \mathcal{Z} \times \Lambda \to I\!R$ defined respectively as the bilinear forms

$$\begin{array}{rcl} \mathcal{A}_\delta(u,\tau; v,\sigma) &:=& a(u,v) + \delta(\tau - u, \sigma)_{\Lambda'}, \\ \mathcal{B}(u,\tau;\mu) &:=& \sum_k \langle u^k - \tau, \mu^k \rangle_k. \end{array} \tag{2.3.6}$$

Existence and uniqueness of the solution of (2.3.3) can be proven by applying the classical theory for saddle point problems. In fact, we recall that the following theorem holds.

**Theorem 2.4** *[BF91] Let $\mathcal{A} : \mathcal{Z} \times \mathcal{Z} \to I\!R$, $\mathcal{B} : \mathcal{Z} \times \Lambda \to I\!R$ be continuous bilinear forms. Assume that $\mathcal{A}$ is coercive on $\ker \mathcal{B} := \{z \in \mathcal{Z} : \mathcal{B}(z;\lambda) = 0$ for all $\mu \in \Lambda\}$, that is, $\mathcal{A}(z;z) \geq \alpha \|z\|_{\mathcal{Z}}^2$ for all $z \in \ker \mathcal{B}$. Furthermore, assume that $\mathcal{B}$ satisfies the inf-sup condition*

$$\inf_{\mu \in \Lambda} \sup_{z \in \mathcal{Z}} \frac{\mathcal{B}(z;\mu)}{\|z\|_{\mathcal{Z}} \|\mu\|_{\Lambda}} \geq \beta > 0. \tag{2.3.7}$$

*Then, for any $f \in \mathcal{Z}'$, there exists a solution $(y, \lambda) \in \mathcal{Z} \times \Lambda$ of the saddle point problem*

$$\begin{aligned} \mathcal{A}(y;z) + \mathcal{B}(z;\lambda) &= \langle f, z \rangle \qquad \text{for all } z \in \mathcal{Z}, \\ \mathcal{B}(y;\mu) &= 0 \qquad \text{for all } \mu \in \Lambda. \end{aligned} \tag{2.3.8}$$

*Moreover, $y \in \mathcal{Z}$ is unique and $\lambda \in \Lambda$ is unique up to an element of $\ker \mathcal{B}^T := \{\mu \in \Lambda : \mathcal{B}(z;\mu) = 0$ for all $z \in \mathcal{Z}\}$.*

We will show that the assumptions of Theorem 2.4 hold in our case. We begin by proving the following lemma.

**Lemma 2.5** *For $0 < \delta < 2$ the bilinear form $\mathcal{A}_\delta$ defined in (2.3.6) is coercive on $H_0^1(\Omega) \times \Phi$ with respect to the norm $\|v, \sigma\|_{\mathcal{Z}}$, i.e., for all $(v, \sigma) \in H_0^1(\Omega) \times \Phi$ it holds*

$$\mathcal{A}_\delta(v, \sigma; v, \sigma) \gtrsim \alpha_1 \|v, \sigma\|_{\mathcal{Z}}^2, \tag{2.3.9}$$

*where*

$$\alpha_1 := \min\{\frac{\delta}{2}, 1 - \frac{\delta}{2}\}. \tag{2.3.10}$$

*Moreover, $\mathcal{A}_\delta$ is continuous on $\mathcal{Z}$, i.e.,*

$$\mathcal{A}_\delta(v, \sigma; v, \sigma) \leq M_1 \|v, \sigma\|_{\mathcal{Z}}^2, \tag{2.3.11}$$

*where*

$$M_1 := \max\{1 + \frac{\delta}{2}, \frac{3\delta}{2}\}. \tag{2.3.12}$$

**Proof:** We have by (2.2.3) and (2.1.14)

$$\begin{aligned} \mathcal{A}_\delta(v, \sigma; v, \sigma) &= a(v, v) + \delta(\sigma - v, \sigma)_{\Lambda'} \\ &\gtrsim \|v\|_V^2 + \delta \|\sigma\|_\Phi^2 - |\delta(v, \sigma)_{\Lambda'}| \\ &\geq \|v\|_V^2 + \delta \|\sigma\|_\Phi^2 - \frac{\delta}{2} \|v\|_{\Lambda'}^2 - \frac{\delta}{2} \|\sigma\|_\Phi^2, \end{aligned}$$

upon employing $2|(v, \sigma)_{\Lambda'}| \leq \|v\|_{\Lambda'}^2 + \|\sigma\|_\Phi^2$. Now it holds

$$\|v\|_{\Lambda'}^2 = \sum_k |v^k|_{1/2, \Gamma^k}^2 \leq \sum_k \|v^k\|_{H^1(\Omega^k)}^2 = \|v\|_{H^1(\Omega)}^2 \tag{2.3.13}$$

by using (2.1.13) and, hence,

$$
\begin{aligned}
\mathcal{A}_\delta(v,\sigma;v,\sigma) &\gtrsim \left(1-\frac{\delta}{2}\right)\|v\|^2_{H^1(\Omega)} + \frac{\delta}{2}\|\sigma\|^2_\Phi \\
&\gtrsim \alpha_1 \|v,\sigma\|^2_{\mathcal{Z}},
\end{aligned}
$$

where $\alpha_1$ is defined by (2.3.10).

As far as the continuity of $\mathcal{A}_\delta$ is concerned, one has by (2.3.13) for any $(v,\sigma) \in \mathcal{Z}$

$$
|(v,\sigma)_{\Lambda'}| \le \frac{1}{2}\|v\|^2_{\Lambda'} + \frac{1}{2}\|\sigma\|^2_\Phi \le \frac{1}{2}\|v\|^2_V + \frac{1}{2}\|\sigma\|^2_\Phi. \tag{2.3.14}
$$

Hence, we can estimate

$$
\begin{aligned}
\mathcal{A}_\delta(v,\sigma;v,\sigma) &= a(v,v) + \delta(\sigma - v,\sigma)_{\Lambda'} \\
&\le \|v\|^2_V + \delta\|\sigma\|^2_\Phi + |\delta(v,\sigma)_{\Lambda'}| \\
&\le \left(1+\frac{\delta}{2}\right)\|v\|^2_V + \frac{3\delta}{2}\|\sigma\|^2_\Phi \\
&\le M_1\|v,\sigma\|^2_{\mathcal{Z}}
\end{aligned}
$$

which proves the assertion. ∎

**Remark 2.6** *It is important to observe that $\mathcal{A}_\delta$ is coercive on a much larger set than simply* $\ker \mathcal{B}$*, namely on the set $H^1_0(\Omega) \times \Phi$, that is, $\sigma$ does not need to be the trace of $u$ in order for (2.3.9) to hold. The bilinear form $\mathcal{A}_0$ corresponding to the unstabilized problem is also coercive, but only on* $\ker \mathcal{B}$*, as easily descends by observing that $(v,\sigma) \in \ker \mathcal{B}$ implies $v = \sigma$ on $\Sigma$, and hence $\|v\|_{H^1(\Omega)} \gtrsim \|\sigma\|_\Phi$. The improved coercivity of $\mathcal{A}_\delta$ with respect to $\mathcal{A}_0$ will allow in the next section to choose a discretization on the skeleton completely uncoupled from the discretizations in the subdomains.*

**Remark 2.7** *The largest value for the coercivity constant $\alpha_1$ is attained when $\frac{\delta}{2} = 1 - \frac{\delta}{2}$ yielding $\delta = 1$ and thus $\alpha_1 = 1/2$ and $M_1 = 3/2$.*

**Lemma 2.8** *The bilinear form $\mathcal{B}$ defined in (2.3.6) is continuous,*

$$
\mathcal{B}(v,\sigma;\mu) \lesssim \|v,\sigma\|_{\mathcal{Z}}\|\mu\|_\Lambda \qquad \text{for all } (v,\sigma) \in \mathcal{Z} \text{ and } \mu \in \Lambda, \tag{2.3.15}
$$

*and satisfies the inf-sup condition*

$$
\inf_{\mu \in \Lambda} \sup_{(v,\sigma)\in\mathcal{Z}} \frac{\mathcal{B}(v,\sigma;\mu)}{\|v,\sigma\|_{\mathcal{Z}}\|\mu\|_\Lambda} \gtrsim 1, \tag{2.3.16}
$$

**Proof:** As far as the continuity is concerned, we simply observe that

$$\begin{aligned}\mathcal{B}(v,\sigma;\mu) &= \sum_k \langle v^k - \sigma, \mu^k\rangle_k \\ &\leq \sum_k |v^k - \sigma|_{1/2,\Gamma^k} |\mu^k|_{-1/2,\Gamma^k} \\ &\leq \left(\sum_k |v^k - \sigma|^2_{1/2,\Gamma^k}\right)^{1/2} \|\mu\|_\Lambda.\end{aligned}$$

Furthermore, we get

$$\begin{aligned}\sum_k |v^k - \sigma|^2_{1/2,\Gamma^k} &\lesssim \sum_k \left(|v^k|^2_{1/2,\Gamma^k} + |\sigma|^2_{1/2,\Gamma^k}\right) \\ &\lesssim \|v,\sigma\|^2_{\mathcal{Z}}.\end{aligned}$$

This together with the previous estimate proves (2.3.15).

For showing (2.3.16), note that for $\mu \in \Lambda$ and for all $\eta \in \Lambda'$

$$\frac{{}_{\Lambda'}\langle \eta, \mu\rangle_\Lambda}{\|\eta\|_{\Lambda'}} \lesssim \frac{{}_{\Lambda'}\langle B v_\eta, \mu\rangle_\Lambda}{\|v_\eta\|_V}$$

where $v_\eta = (v_\eta^k)_k$, $v_\eta^k$ being the harmonic extension of $\eta^k$ on $\Omega^k$. Hence, we can write

$$\|\mu\|_\Lambda = \sup_{\eta\in\Lambda'} \frac{{}_{\Lambda'}\langle \eta, \mu\rangle_\Lambda}{\|\eta\|_{\Lambda'}} \lesssim \sup_{v\in V, \Delta v^k = 0} \frac{{}_{\Lambda'}\langle B v, \mu\rangle_\Lambda}{\|v\|_V} \lesssim \sup_{v\in V} \frac{{}_{\Lambda'}\langle B v, \mu\rangle_\Lambda}{\|v\|_V},$$

which implies (2.3.16) since

$$\sup_{(v,\sigma)\in\mathcal{Z}} \frac{\mathcal{B}(v,\sigma;\mu)}{\|v,\sigma\|_{\mathcal{Z}}} \geq \sup_{v\in V} \frac{\mathcal{B}(v,0;\mu)}{\|v\|_V}.$$

■

We now observe that $(v,\sigma) \in \ker \mathcal{B}$ implies that $v \in H_0^1(\Omega)$ since it is continuous across the interfaces and it belongs to $H^1$ on each subdomain. Then Lemma 2.5 and Lemma 2.8 combined with Theorem 2.4 yield, after observing that

$$\ker \mathcal{B}^T = \{\mu \in \Lambda : \sum_k \langle \sigma - v^k, \mu^k\rangle_k = 0 \quad \text{for all } (v,\sigma) \in \mathcal{Z}\} = \{0\}, \tag{2.3.17}$$

the following corollary [BF91].

**Corollary 2.9** *The solution $(u,\tau;\lambda) \in \mathcal{Z}\times\Lambda$ of the system (2.3.5) is uniquely determined, and it holds*

$$\|u\|_V + \|\tau\|_\Phi + \|\lambda\|_\Lambda \lesssim \|f\|_{V'}. \tag{2.3.18}$$

# 3 Discretization

## 3.1 The Discrete Stabilized System

The solution of (2.3.3) can be approximated by a Galerkin scheme. One chooses finite-dimensional subspaces $V_h = \prod_k V_h^k \subset V$, $\Phi_h \subset \Phi$ and $\Lambda_h = \prod_k \Lambda_h^k \subset \Lambda$. Moreover, it is not restrictive to assume that $\Lambda_h^k \subset L_2(\Gamma^k)$. The discretized system reads as follows:

find $(u_h, \tau_h; \lambda_h) \in \mathcal{Z}_h \times \Lambda_h := (V_h \times \Phi_h) \times \Lambda_h$ such that

$$\left\{ \begin{array}{rcll} a(u_h, v_h) - \sum_k \langle \lambda_h^k, v_h^k \rangle_k &=& (f, v_h)_{L_2(\Omega)} & \text{for all } v_h \in V_h, \\ \sum_k \langle \tau_h - u_h^k, \mu_h^k \rangle_k &=& 0 & \text{for all } \mu_h \in \Lambda_h, \\ \delta(\tau_h - u_h, \sigma_h)_{\Lambda'} + \sum_k \langle \lambda_h^k, \sigma_h \rangle_k &=& 0 & \text{for all } \sigma_h \in \Phi_h. \end{array} \right. \tag{3.1.1}$$

In order to guarantee that this system has a unique solution, we will make the following assumptions on the discretization spaces $V_h$, $\Lambda_h$.

(A1) For all $k$, the following inf-sup condition holds:

$$\inf_{\mu_h^k \in \Lambda_h^k} \sup_{v_h^k \in V_h^k} \frac{\langle v_h^k, \mu_h^k \rangle_k}{|v_h^k|_{1/2,\Gamma^k} |\mu_h^k|_{-1/2,\Gamma^k}} \geq \beta^k > 0. \tag{3.1.2}$$

(A2) Letting $\Lambda_h^\cap \subset L^2(\Sigma)$ be defined by

$$\Lambda_h^\cap = \{\mu \in L^2(\Sigma) : \ \mu|_{\Gamma^k} \in \Lambda_h^k \ \text{ for all } k\},$$

for all $v_h \in V_h$ such that

$$\int_\Sigma [v_h] \mu_h = 0 \qquad \text{for all } \mu_h \in \Lambda_h^\cap \tag{3.1.3}$$

it holds

$$a(v_h, v_h) \geq C_0 \|v_h\|_V^2,$$

($[v]$ denoting the jump of $v$ at the corresponding edge).

**Remark 3.1** *Assumption (A1) is just the inf-sup condition for the discrete spaces on each of the subdomains. There are number of results in this direction, starting from the one by [Bab73]. A general criterion to ensure (A1) in the multilevel context is derived in [DK98]. We remark that it is possible to apply a stabilization techniques also at this level (see, for instance, [BH92, Ste95, BFMR97, Ber98b, Buf98]).*
*Assumption (A2) is also not restrictive: if, for instance, for all k $\Lambda_h^k$ contains the piecewise constants on the (very coarse) grid composed by the "elements" $\partial\Omega^k \cap \partial\Omega^\ell$, then it is possible to prove that (A2) holds (see for instance [BMP94]).*

Under assumptions (A1) and (A2), we can again apply Theorem 2.4, now in the discrete setting, yielding existence and uniqueness of the solution of (3.1.1), as well as an error estimate.

**Lemma 3.2** *Assume that (A2) holds. Then there exists a $\delta_0$ such that for $0 < \delta \leq \delta_0$, the bilinear form $\mathcal{A}_\delta$ defined in (2.3.6) is coercive on*

$$\ker \mathcal{B}_h = \{(v_h, \sigma_h) \in \mathcal{Z}_h : \sum_k \langle v_h^k - \sigma_h, \mu_h^k \rangle_k = 0 \quad \text{for all } \mu_h \in \Lambda_h\} \tag{3.1.4}$$

*with respect to the norm of $\mathcal{Z}$, i.e., for all $(v_h, \sigma_h) \in \ker \mathcal{B}_h$ the estimate*

$$\mathcal{A}_\delta(v_h, \sigma_h; v_h, \sigma_h) \geq \alpha_2 \, \|v_h, \sigma_h\|_{\mathcal{Z}}^2 \tag{3.1.5}$$

*holds, where $\alpha_2$ depends on $\delta$ and $C_0$.*

**Proof:** Assume that $(v_h, \sigma_h) \in \ker \mathcal{B}_h$. We will show that (3.1.3) holds. In fact, let $\mu_h \in \Lambda_h^\cap$. Then there exists an element $\breve{\mu}_h \in \Lambda_h$ such that $\mu_h|_{\Gamma^k} = \breve{\mu}_h^k|_{\Gamma^k}$. Let us denote $\mu_h^{k,\ell} = \mu_h|_{\Gamma^k \cap \Gamma^\ell}$. If $(v_h, \sigma_h) \in \ker \mathcal{B}_h$, then we have, since $\Gamma^k = \cup_\ell \Gamma^k \cap \Gamma^\ell$,

$$\mathcal{B}_h(v_h, \sigma_h; \breve{\mu}_h) = \sum_k \sum_\ell \int_{\Gamma^k \cap \Gamma^\ell} (v_h^k - \sigma_h) \mu_h^{k,\ell} = 0.$$

Since on $\Gamma^k \cap \Gamma^\ell$ we have that $\breve{\mu}_h^k = \mu_h^{k,\ell} = \breve{\mu}_h^\ell$, we also have

$$\mathcal{B}_h(v_h, \sigma_h; \breve{\mu}_h) = \sum_\ell \sum_k \int_{\Gamma^k \cap \Gamma^\ell} (v_h^\ell - \sigma_h) \mu_h^{k,\ell} = 0.$$

Subtracting the two identities, we obtain

$$2 \int_\Sigma [v_h] \mu_h = \sum_{k,\ell} \int_{\Gamma^k \cap \Gamma^\ell} (v_h^k - v_h^\ell) \mu_h^{k,\ell} = 0,$$

that is, (3.1.3) holds. Then, by assumption (A2), it holds

$$a(v_h, v_h) \geq C_0 \|v_h\|_V^2.$$

This yields

$$\begin{aligned} \mathcal{A}_\delta(v_h, \sigma_h; v_h, \sigma_h) &\geq C_0 \|v_h\|_V^2 + \delta \|\sigma_h\|_\Phi^2 - \delta \|v_h\|_{\Lambda'} \|\sigma_h\|_\Phi \\ &\geq (C_0 - \frac{\delta}{2}) \|v_h\|_V^2 + \frac{\delta}{2} \|\sigma_h\|_\Phi^2. \end{aligned} \tag{3.1.6}$$

By choosing $\delta$ such that $0 < \delta < 2C_0$, we get that (3.1.5) holds for $\alpha_2 = \min\{C_0 - \delta/2, \delta/2\}$.
■

Let us now prove that a discrete inf-sup condition on $\mathcal{Z}_h \times \Lambda_h$ holds.

**Lemma 3.3** *Let assumption (A1) hold. Then the bilinear form $\mathcal{B}$ defined in (2.3.6) satisfies*

$$\inf_{\mu_h \in \Lambda_h} \sup_{(v_h,\sigma_h)\in \mathcal{Z}_h} \frac{\mathcal{B}(v_h,\sigma_h;\mu_h)}{\|v_h,\sigma_h\|_{\mathcal{Z}}\|\mu_h\|_\Lambda} \geq \beta_2 > 0, \tag{3.1.7}$$

*where*

$$\beta_2 := \min_{k=1,\dots,K} \beta^k. \tag{3.1.8}$$

**Proof:** Let $(V_h^k)'$ be the dual of $V_h^k$, endowed with the norm

$$\|F\|_{(V_h^k)'} = \sup_{v_h \in V_h} \frac{F(v_h)}{\|v_h\|_{H^1(\Omega^k)}}.$$

The bound (3.1.2) implies that the operator $(B_h^k)^T : \Lambda_h^k \to (V_h^k)'$ defined by

$${}_{(V_h^k)'}\langle (B_h^k)^T \mu_h^k, v_h^k\rangle_{V_h^k} = \langle \mu_h^k, v_h^k\rangle_k$$

satisfies, for all $\mu \in \Lambda_h^k$,

$$\|(B_h^k)^T\mu_h^k\|_{(V_h^k)'} \geq \beta^k |\mu_h^k|_{-1/2,\Gamma^k}.$$

On the other hand, since $V_h^k$ is isomorphic to its dual, there exists a (uniquely determined) element $w_h^k(\mu_h^k) \in V_h^k$ such that

$${}_{(V_h^k)'}\langle (B_h^k)^T \mu_h^k, v_h^k\rangle_{V_h^k} = (w_h^k(\mu_h^k), v_h^k)_{V_h^k} \qquad \text{for all } v_h^k \in V_h^k$$

and

$$\|(B_h^k)^T\mu_h^k\|_{(V_h^k)'} = \|w_h^k(\mu_h^k)\|_{V_h^k}.$$

We can then write

$$\begin{aligned}
\sup_{v_h \in V_h} \frac{\sum_k \langle \mu_h^k, v_h^k\rangle_k}{\|v_h\|_V} &= \sup_{v_h\in V_h} \frac{\sum_k {}_{(V_h^k)'}\langle (B_h^k)^T\mu_h^k, v_h^k\rangle_{V_h^k}}{\|v_h\|_V} \\
&= \sup_{v_h\in V_h} \frac{\sum_k (w_h^k(\mu_h^k), v_h^k)_{V_h^k}}{\|v_h\|_V} \\
&= \sup_{v_h\in V_h} \frac{(w_h(\mu_h), v_h)_V}{\|v_h\|_V} \\
&\geq \|w_h(\mu_h)\|_V = \left(\sum_k \|w_h^k(\mu_h^k)\|^2_{H^1(\Omega^k)}\right)^{1/2} \\
&= \left(\sum_k \|(B_h^k)^T\mu_h^k\|_{(V_h^k)'}\right)^{1/2} \\
&\geq (\min_k \beta^k) \left(\sum_k |\mu_h^k|^2_{-1/2,\Gamma^k}\right)^{1/2} \\
&= \beta_2 \|\mu_h\|_\Lambda.
\end{aligned} \tag{3.1.9}$$

$\blacksquare$

A standard argument [BF91] then yields an optimal error estimate:

**Corollary 3.4** *Under assumptions (A1) and (A2), the solution $(u_h, \lambda_h, \sigma_h) \in V_h \times \Lambda_h \times \Phi_h$ of problem (3.1.1) exists and is unique. Moreover the error estimate*

$$\begin{aligned} \|u - u_h\|_V \; + \; \|\lambda - \lambda_h\|_\Lambda + \|\sigma - \sigma_h\|_\Phi \\ \lesssim \inf_{v_h \in V_h} \|u - u_h\|_V + \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_\Lambda + \inf_{\tau_h \in \Phi_h} \|\sigma - \tau_h\|_\Phi \end{aligned} \tag{3.1.10}$$

*holds.*

## 3.2 Solving the Discrete Problem

We now consider the linear system stemming from the discrete problem (3.1.1). Again, the solution of such system can be reduced to the inversion of the discrete Steklov-Poincaré operator. In fact, for $\tau_h \in \Phi_h$ given, the problem of finding $(u_h, \lambda_h) \in V_h \times \Lambda_h$ satisfying

$$\left\{ \begin{aligned} a(u_h, v_h) - \sum_k \langle \lambda_h^k, v_h^k \rangle_k \; &= \; (f, v_h)_{L_2(\Omega)} \qquad \text{for all } v_h \in V_h, \\ \sum_k \langle u_h^k, \mu_h^k \rangle_k \; &= \; \sum_k \langle \tau_h, \mu_h^k \rangle_k \qquad \text{for all } \mu_h \in \Lambda_h, \end{aligned} \right. \tag{3.2.1}$$

decouples into $K$ independent discrete problems on the $K$ subdomains. We can then consider the formal solution $(u_h(f, \tau_h), \lambda_h(f, \tau_h))$ of problem (3.2.1). We remark that it holds

$$(u_h(f, \tau_h), \lambda_h(f, \tau_h)) = (u_h(f) + u_h(\tau_h), \lambda_h(f) + \lambda_h(\tau_h))$$

$(u_h(\tau_h), \lambda_h(\tau_h))$ (resp.$(u_h(f), \lambda_h(f))$) denoting the (formal) solution of problem (3.2.1) for $f = 0$ (resp. $\tau_h = 0$). By substituting $(u_h(f, \tau_h), \lambda_h(f, \tau_h))$ into the last equation of (3.1.1), we obtain a discrete problem in the unknown $\tau_h$:

find $\tau_h \in \Phi_h$ such that

$$\begin{aligned} s(\tau_h, \sigma_h) \; &:= \; \delta(\tau_h - u_h(\tau_h), \sigma_h)_{\Lambda'} + \sum_k \langle \lambda_h^k(\tau_h), \sigma_h \rangle_k \\ &= \; \delta(u_h(f), \sigma_h)_{\Lambda'} + \sum_k \langle \lambda_h^k(f), \sigma_h \rangle_k \qquad \text{for all } \sigma_h \in \Phi_h \end{aligned} \tag{3.2.2}$$

holds.

Again, the evaluation of the bilinear form $s(\cdot, \cdot)$ reduces to solving problem (3.2.1). This is reflected by the terms $u_h(\tau_h)$ and $\lambda_h(\tau_h)$ in (3.2.2).

**Lemma 3.5** *If assumptions (A1) and (A2) hold, there exists some $\delta_1$ such that for all $\delta$, $0 < \delta < \delta_1$, the bilinear form $s : \Phi_h \times \Phi_h \to I\!R$ defined by (3.2.2) is continuous and coercive uniformly in $h$, that is, one has*

$$s(\tau_h, \sigma_h) \;\; \lesssim \;\; \|\tau_h\|_\Phi \|\sigma_h\|_\Phi, \tag{3.2.3}$$

$$s(\sigma_h, \sigma_h) \;\; \gtrsim \;\; \|\sigma_h\|_\Phi^2. \tag{3.2.4}$$

**Proof:** We first observe that, thanks to assumption (A1), the solution $(u_h(\tau_h), \lambda_h(\tau_h))$ of problem (3.2.1) for $f = 0$ satisfies the bound

$$\|u_h(\tau_h)\|_V + \|\lambda_h(\tau_h)\|_\Lambda \lesssim \|\tau_h\|_\Phi.$$

As far as continuity is concerned, we have then

$$s(\tau_h, \sigma_h) \lesssim \|\tau_h\|_{\Lambda'}\|\sigma_h\|_{\Lambda'} + \|u_h(\tau_h)\|_{\Lambda'}\|\sigma_h\|_{\Lambda'} + \|\lambda_h(\tau_h)\|_\Lambda\|\sigma_h\|_{\Lambda'}.$$

Observing that $\|u_h(\tau_h)\|_{\Lambda'} \leq \|u_h(\tau_h)\|_V$, by combining the two previous inequalities, we get (3.2.3).

In order to prove coercivity, we have, according to the definition of $u_h(\tau_h), \lambda_h(\tau_h)$, (that is using (3.2.1) for $f = 0$)

$$\sum_k \langle \lambda_h^k(\tau_h), \tau_h\rangle_k = \sum_k \langle u_h^k(\tau_h), \lambda_h^k(\tau_h)\rangle_k = a_h(u_h(\tau_h), u_h(\tau_h)) \geq C_0\|u_h(\tau_h)\|_V^2,$$

where the inequality holds because of assumption (A2), exactly as in Lemma 3.2. Hence, using again $\|u_h(\tau_h)\|_{\Lambda'} \leq \|u_h(\tau_h)\|_V$, we obtain

$$\begin{aligned} s(\tau_h, \tau_h) &\geq \delta\|\tau_h\|_{\Lambda'}^2 - \delta c_2\|u_h(\tau_h)\|_V\|\tau_h\|_{\Lambda'} + C_0\|u_h(\tau_h)\|_V^2 \\ &\geq \delta\left(1 - \frac{c_2}{2\kappa}\right)\|\tau_h\|_{\Lambda'}^2 + \left(C_0 - \frac{\delta c_2 \kappa}{2}\right)\|u_h(\tau_h)\|_V^2. \end{aligned}$$

By choosing $\kappa$ in such a way that $(1-c_2/2\kappa) > 0$ and $\delta$ in such a way that $(C_0-\delta c_2\kappa/2) > 0$, we thus get coercivity. ∎

## 3.3 Preconditioning the Schur Complement

The fast and efficient solution of the discrete problem (3.2.2) relies on the performance of a good preconditioner. Based on a wavelet basis for $\Phi$, one can construct discretization spaces $\Phi_h$ and asymptotically optimal diagonal preconditioners for the discrete linear operator $\mathbf{S}_h$ corresponding to the bilinear form $s(\cdot, \cdot)$ defined by

$$\langle \mathbf{S}_h \tau_h, \sigma_h\rangle := s(\tau_h, \sigma_h), \quad \sigma_h, \tau_h \in \Phi_h. \tag{3.3.1}$$

In fact, assume that we have a couple of biorthogonal wavelet bases for $L_2(\Sigma)$

$$\begin{aligned} \boldsymbol{\psi} &:= \{\psi_{j,m},\ (j,m) \in \nabla := \cup_{j \geq j_0} \nabla_j\}, \\ \tilde{\boldsymbol{\psi}} &:= \{\tilde{\psi}_{j,m},\ (j,m) \in \nabla := \cup_{j \geq j_0} \nabla_j\}, \end{aligned} \tag{3.3.2}$$

with the following properties:

(P1) any function $\sigma \in L_2(\Sigma)$ can be expanded in terms of either $\boldsymbol{\psi}$ or $\tilde{\boldsymbol{\psi}}$,

$$\sigma = \sum_{j \geq j_0} \sum_{m \in \nabla_j} \langle \sigma, \tilde{\psi}_{j,m}\rangle \psi_{j,m} = \sum_{j \geq j_0} \sum_{m \in \nabla_j} \langle \sigma, \psi_{j,m}\rangle_k \tilde{\psi}_{j,m}, \tag{3.3.3}$$

($\langle \cdot, \cdot\rangle$ denoting here the $L_2(\Sigma)$ scalar product);

(P2) one has $\psi_{j,m} \in \Phi$, and the following *norm equivalence* holds

$$\|\sigma\|_\Phi^2 \sim \sum_{j \geq j_0} 2^j \sum_{m \in \nabla_j} |\langle \sigma, \tilde{\psi}_{j,m} \rangle|^2, \qquad \sigma \in \Phi; \tag{3.3.4}$$

(P3) $\boldsymbol{\psi}$ and $\tilde{\boldsymbol{\psi}}$ have local support, i.e.,

$$\operatorname{diam}(\operatorname{supp} \psi_{j,m}) \sim \operatorname{diam}(\operatorname{supp} \tilde{\psi}_{j,m}) \sim 2^{-j}. \tag{3.3.5}$$

There are by now a number of constructions of such biorthogonal wavelets [CM98, CTU99, DS97, DS98] that can be applied to the present setting. For completeness we will briefly describe in Section 4 a particularly simple construction which, in the two-dimensional case, is sufficient for the present purpose.

Let then $\Phi_h = \operatorname{span}_\Phi(\boldsymbol{\psi}_J)$ with $\boldsymbol{\psi}_J = \{\psi_{j,m},\ (j,m) \in \nabla_J := \cup_{j \geq j_0}^J \nabla_j\}$, and let $\mathbf{S}_{\boldsymbol{\psi}_J}$ be the matrix corresponding to the Galerkin discretization of $\mathbf{S}_h$ relative to the wavelet basis $\boldsymbol{\psi}_J$,

$$(\mathbf{S}_{\boldsymbol{\psi}_J})_{(j,m),(j',m')} = s(\psi_{j,m}, \psi_{j',m'}).$$

Furthermore, let $\mathbf{D}_J$ be the diagonal matrix defined by

$$(\mathbf{D}_J)_{(j,m),(j',m')} = 2^{j/2}\, \delta_{j,j'}\, \delta_{m,m'}, \tag{3.3.6}$$

for $j_0 \leq j, j' \leq J$, $m \in \nabla_j$, $m' \in \nabla_{j'}$. Then Lemma 3.5 together with (3.3.1) and (3.3.4) yields asymptotically optimal condition numbers [DK92].

**Theorem 3.6** *One has*

$$\operatorname{cond}_2(\mathbf{D}_J^{-1} \mathbf{S}_{\boldsymbol{\psi}_J} \mathbf{D}_J^{-1}) = \mathcal{O}(1), \quad J \to \infty. \tag{3.3.7}$$

### 3.4 Evaluating the Stabilizing Term

In order to set up (3.2.2), one needs to evaluate the stabilizing term

$$\delta(\tau_h, \sigma_h)_{\Lambda'} \tag{3.4.1}$$

in an effective way. The problem of computing (equivalent) scalar products for Sobolev spaces of negative or fractional order has recently been considered in several papers ([BPV98],[Ber98a]).

One can do so in terms of biorthogonal wavelet bases for $\Lambda'$ as follows.

Assume that there is a pair of biorthogonal bases (which, without fear of confusion, we will also call $\boldsymbol{\psi}$ and $\tilde{\boldsymbol{\psi}}$)

$$\begin{aligned} \boldsymbol{\psi} &:= \{\psi_{j,m}^k,\ k = 1, \ldots, K,\ (j,m) \in \nabla^k := \cup_{j \geq j_0} \nabla_j^k\}, \\ \tilde{\boldsymbol{\psi}} &:= \{\tilde{\psi}_{j,m}^k,\ \ k = 1, \ldots, K,\ (j,m) \in \nabla^k := \cup_{j \geq j_0} \nabla_j^k\}, \end{aligned} \tag{3.4.2}$$

for $\prod_k L_2(\Omega^k)$, satisfying properties analogous to (P1)–(P3). In particular we assume that (P2)' holds:

(P2)' one has the *norm equivalence*

$$\|f\|_{\Lambda'}^2 \sim \sum_k \sum_{j\geq j_0} 2^j \sum_{m\in\nabla_j^k} |\langle f^k, \tilde{\psi}_{j,m}^k\rangle_k|^2, \qquad f\in\Lambda'. \tag{3.4.3}$$

Wavelet bases with the properties required are again available in the literature, see the references in the previous subsection.

Like in [Ber98b], we can then define on $\Lambda'\times\Lambda'$ the bilinear form

$$(\tau,\sigma)_{\Lambda',\psi} := \sum_k \sum_{j\geq j_0} 2^j \sum_{m\in\nabla_j^k} \langle \tau, \psi_{j,m}^k\rangle_k \langle \sigma, \psi_{j,m}^k\rangle_k. \tag{3.4.4}$$

The following proposition is an immediate consequence of the norm equivalence (3.4.3).

**Proposition 3.7** *The bilinear form $(\cdot,\cdot)_{\Lambda',\psi}$ defines an inner product on $\Lambda'$ which in turn induces a norm $\|\cdot\|_{\Lambda',\psi}$ satisfying*

$$\|\sigma\|_{\Lambda',\psi} \sim \|\sigma\|_{\Lambda'} \qquad \textit{for all } \sigma\in\Lambda'. \tag{3.4.5}$$

Then the scalar product $(\cdot,\cdot)_{\Lambda'}$ in (2.3.3) can be substituted by $(\cdot,\cdot)_{\Lambda',\psi}$, and it is easy to check that all the subsequent arguments hold unchanged, yielding the same results (with possibly different constants depending on the constants in (3.4.3)).

Furthermore, on the discrete problem, the bilinear form $(\cdot,\cdot)_{\Lambda',\psi}$, which consists in an infinite sum, can be approximated by a finite sum while retaining the stability and accuracy bounds. If fact, defining $H_*^r(\Gamma^k)$ for $r>1/2$ in analogy with $H_*^{1/2}(\Gamma^k)$, the following lemma has been proven in [Ber98b].

**Lemma 3.8** *If the trial spaces $\Phi_h$ satisfy a Bernstein estimate: for some $r>1/2$, $\Phi_h^k\subset H^r(\Sigma)$ and for each $k$,*

$$\|\sigma_h\|_{H_*^r(\Gamma^k)} \lesssim h^{-(r-1/2)}|\sigma|_{1/2,\Gamma^k} \quad \sigma\in\Phi_h,$$

*then there exists a $J=J(h)$ such that the bilinear form $(\cdot,\cdot)_{\Lambda',\psi,J}:\Lambda'\times\Lambda'\to\mathbb{R}$, defined by*

$$(\tau,\sigma)_{\Lambda',\psi,J} := \sum_k \sum_{j=j_0}^{J} 2^j \sum_{m\in\nabla_j^k} \langle \tau, \psi_{j,m}^k\rangle_k \langle \sigma, \psi_{j,m}^k\rangle_k, \tag{3.4.6}$$

*verifies:*

*(i) for all $\tau$, $\sigma$ in $\Lambda'$ it holds*

$$(\tau,\sigma)_{\Lambda',\psi,J} \lesssim \|\tau\|_{\Lambda'}\|\sigma\|_{\Lambda'};$$

*(ii) for all $\sigma_h$ in $\Phi_h$ it holds*

$$(\sigma_h,\sigma_h)_{\Lambda',\psi,J} \gtrsim \|\sigma\|_{\Lambda'}^2.$$

It is then easy to see that the arguments of Section 3 carry over to the case where the bilinear form $(\cdot,\cdot)_{\Lambda'}$ in (3.1.1) is substituted by $(\cdot,\cdot)_{\Lambda',\psi,J}$. In particular, for the "totally discrete" problem thus obtained, all the results of Section 3 (existence and uniqueness of the solution, error estimate, preconditioning for the Schur complement) hold.

# 4 Wavelets

Let now $\Omega \subset I\!R^2$. $\Sigma$ will then be the union of one dimensional segments. The aim of this section is to describe a simple construction of biorthogonal wavelets $\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}$ satisfying (P1)–(P3). The idea like in [CTU99, DS97] is to start from a couple of sufficiently regular biorthogonal multiresolution analyses (MRA's) for $L_2(]0,1[)$ and to assemble them in a suitable way in order to obtain a MRA for $L_2(\Sigma)$ ($\sim L_2(\cup_n I_n)$, $I_n \sim ]0,1[$). In all of the above mentioned papers great care is taken in order to obtain continuous functions on both the primal and the dual side, and the constructions given there are perfectly well suited to be used in our framework. However, continuity on the dual side is not strictly necessary for our purpose, so that a simplified version of the above mentioned constructions can be used. For completeness we report now the details of such a construction. We follow here the approach of [DS97], which we simplify by giving up any continuity requirement on the dual side. Fpr the sake of simplicity, we start by considering the interval $]0,2[$ which we decompose as the closure of $]0,1[\cup]1,2[$, and then we will briefly sketch the generalization to $\Sigma = \cup_n I_n$.

As it is already done in the previously mentioned papers, we start with taking any couple $\{T_j^1\}$, $\{\tilde{T}_j^1\}$ of biorthogonal multiresolutions for $L_2(]0,1[)$ with corresponding (biorthogonal) *generator bases* [DKU96] which we assume to satisfy the following properties:

(G1) the increasing sequences

$$\begin{array}{l} T_{j_0}^1 \subset \ldots \subset T_j^1 \subset T_{j+1}^1 \subset \ldots \subset \quad H^1(]0,1[), \\ \tilde{T}_{j_0}^1 \subset \ldots \subset \tilde{T}_j^1 \subset \tilde{T}_{j+1}^1 \subset \ldots \subset \quad L_2(]0,1[), \end{array} \tag{4.7}$$

are such that

$$\operatorname{clos}_{H^1}(\cup_{j\geq j_0} T_j^1) = H^1(]0,1[) \tag{4.8}$$

and

$$\operatorname{clos}_{L_2}(\cup_{j\geq j_0} \tilde{T}_j^1) = L_2(]0,1[); \tag{4.9}$$

(G2) the generator bases for $T_j^1$, $\tilde{T}_j^1$:

$$T_j^1 = \operatorname{span} \boldsymbol{\varphi}_j^1, \quad \tilde{T}_j^1 = \operatorname{span} \tilde{\boldsymbol{\varphi}}_j^1, \tag{4.10}$$

are, for some index set $\Delta_j := \{0, \ldots, N_j\}$, $\#\Delta_j \sim 2^j$, of the following form:

$$\boldsymbol{\varphi}_j^1 = \{\varphi_{j,m}^1 : m \in \Delta_j\}, \quad \tilde{\boldsymbol{\varphi}}_j^1 = \{\tilde{\varphi}_{j,m}^1 : m \in \Delta_j\}; \tag{4.11}$$

they consist of functions with local support,

$$\operatorname{diam}(\operatorname{supp} \varphi_{j,m}^1) \sim \operatorname{diam}(\operatorname{supp} \tilde{\varphi}_{j,m}^1) \sim 2^{-j}, \tag{4.12}$$

which are $L_2$-stable,

$$\| \sum_{m\in\Delta_j} c_m \varphi_{j,m}^1 \|_{L_2(]0,1[)} \sim \| \sum_{m\in\Delta_j} c_m \tilde{\varphi}_{j,m}^1 \|_{L_2(]0,1[)} \sim \|\mathbf{c}\|_{\ell_2(\Delta_j)}, \tag{4.13}$$

and biorthogonal,

$$(\varphi^1_{j,m}, \tilde{\varphi}^1_{j,m'})_{L_2(]0,1[)} = \delta_{m,m'}; \tag{4.14}$$

moreover, without loss of generality, we assume

$$\begin{aligned} &\varphi^1_{j,0}(0) = 2^{j/2}; \qquad &&\varphi^1_{j,m}(0) = 0, \qquad &&m > 0; \\ &\varphi^1_{j,N_j}(1) = 2^{j/2}; \qquad &&\varphi^1_{j,m}(1) = 0, \qquad &&m < N_j; \end{aligned} \tag{4.15}$$

(G3) projectors $P^1_j : L_2(]0,1[) \longrightarrow T^1_j$ can be defined by

$$P^1_j f = \sum_{m \in \Delta_j} \langle f, \tilde{\varphi}^1_{j,m} \rangle \varphi^1_{j,m}, \tag{4.16}$$

which are uniformly bounded on $L_2(]0,1[)$;

(G4) polynomials up to order $M \geq 1$ (resp. $\tilde{M} \geq 0$) are contained in $T^1_j$ (resp. in $\tilde{T}^1_j$);

(G5) a Bernstein (or inverse) inequality for $T^1_j$ holds:

$$\|f_j\|_{H^1(]0,1[)} \lesssim 2^j \|f_j\|_{L_2(]0,1[)}, \qquad \text{for all } f_j \in T^1_j. \tag{4.17}$$

By a simple translation we can obtain a couple of MRA's for $L_2(]1,2[)$, which we will denote by $\{T^2_j\}, \{\tilde{T}^2_j\}$, with generators $\boldsymbol{\varphi}^2_j$, $\tilde{\boldsymbol{\varphi}}^2_j$ respectively, which do satisfy properties (G1)–(G5) on $]1,2[$. We can then define $T_j \subset H^1(]0,2[)$ as

$$T_j = \{u \in C^0(]0,2[) : \; u|_{]0,1[} \in T^1_j, \;\; u|_{]1,2[} \in T^2_j\}.$$

Since all functions in $\boldsymbol{\varphi}^i_j, \tilde{\boldsymbol{\varphi}}^i_j$ $(i = 1,2)$ except one vanish at $x = 1$, it is easy to define generators for $T_j$ by

$$\boldsymbol{\varphi}_j = \{\varphi_{j,m}, \; m = 0, \ldots, 2N_j\}, \tag{4.18}$$

with, for $m = 0, \ldots, N_j - 1$ and for $\ell = 1, \ldots, N_j$

$$\varphi_{j,m} = \begin{cases} \varphi^1_{j,m}, & \text{in } [0,1) \\ 0, & \text{in } [1,2] \end{cases}, \qquad \varphi_{j,\ell+N_j} = \begin{cases} 0, & \text{in } [0,1) \\ \varphi^2_{j,\ell}, & \text{in } [1,2] \end{cases},$$

while the function $\varphi_{j,N_j}$ at the *crosspoint* $x = 1$ is defined by

$$\varphi_{j,N_j}(x) := \begin{cases} \varphi^1_{j,N_j}, & \text{in } [0,1), \\ \varphi^2_{j,0}, & \text{in } [1,2]. \end{cases} \tag{4.19}$$

In order to define wavelet functions for the above MRA, we will construct projectors $P_j : L_2(]0,2[) \to T_j$. The dual MRA will then be defined as $\tilde{T}_j := \mathcal{I}m(P^*_j)$ and complement spaces $W_j$ as $W_j = (P_{j+1} - P_j)L_2(]0,2[)$.

We start by defining an extension operator $E_j : T_j^1 \to T_j$ by

$$(E_j f)(x) \;\; := \;\; \begin{cases} f(x) = \sum_{m=0}^{N_j} f_m \varphi_{j,m}^1(x), & x \in [0,1), \\ f_{N_j} \varphi_{j,0}^2(x), & x \in [1,2]. \end{cases} \tag{4.20}$$

$$= \sum_{m=0}^{N_j-1} f_m \varphi_{j,m}^1(x) + f_{N_j} \varphi_{j,N_j}(x).$$

By construction, $E_j f$ is continuous across $x = 1$. Furthermore, $E_j$ is $L_2$ bounded. In fact, by the $L_2$-stability (4.13) of $\boldsymbol{\varphi}_j^1, \boldsymbol{\varphi}_j^2$, one has

$$\|E_j f\|_{L_2(]0,2[)} \leq \|f\|_{L_2(]0,1[)} + |f_{N_j}| \, \|\varphi_{j,0}^2\|_{L_2(]1,2[)} \;\lesssim\; \|f\|_{L_2(]0,1[)}. \tag{4.21}$$

We next define the sets

$$T_{j,0}^2 := \text{span}\,\{\varphi_{j,m}^2, \;\, m = 1, \ldots, N_j\}, \tag{4.22}$$

which contain only those functions in $T_j^2$ vanishing at $x = 1$, and corresponding projectors $P_{j,0}^2 : L_2(]1,2[) \to T_{j,0}^2$ which are given by

$$P_{j,0}^2 f = \sum_{m=1}^{N_j} \langle f, \tilde{\varphi}_{j,m}^2 \rangle \varphi_{j,m}^2. \tag{4.23}$$

The two projectors $P_j^1$ and $P_{j,0}^2$ are respectively defined on $L_2(]0,1[)$ and on $L_2(]1,2[)$. In the following, for the sake of simplicity, for any $f \in L_2(]0,2[)$ we will write $P_j^1 f$ and $P_{j,0}^2 f$ for $P_j^1(f|_{]0,1[})$ and $P_{j,0}^2(f|_{]1,2[})$. At the same time, for any function $f$ in $L_2(]0,1[)$ or in $L_2(]1,2[)$ we will also denote by $f \in L_2(]0,2[)$ its trivial extension by zero.

We can now define the projectors $P_j : L_2(]0,2[) \to T_j$ by

$$P_j f := E_j P_j^1 f + P_{j,0}^2 f. \tag{4.24}$$

It is easy to see that $P_j$ can be computed as

$$P_j f = \sum_{m=0}^{N_j-1} \langle f, \tilde{\varphi}_{j,m}^1 \rangle \varphi_{j,m}^1 + \langle f, \tilde{\varphi}_{j,N_j}^1 \rangle \varphi_{j,N_j} + \sum_{m=1}^{N_j} \langle f, \tilde{\varphi}_{j,m}^2 \rangle \varphi_{j,m}^2 \tag{4.25}$$

(where $\langle \cdot, \cdot \rangle$ is the $L_2(]0,2[)$ scalar product). Note that the only primal function not vanishing at $x = 1$ is continuous there while the corresponding dual one is not.

**Remark 4.1** *By construction, the extension operator $E_j$ and the projector $P_{j,0}^2$ satisfy $P_{j,0}^2 E_j = 0$. By taking suitable linear combinations of the dual generators $\boldsymbol{\varphi}_j^2$, other projectors could be defined with better properties, which can substitute $P_{j,0}^2$ as defined in (4.23). Then, the projector $P_j$ should be defined as*

$$P_j f := E_j P_j^1 f + P_{j,0}^2 (f - E_j P_j^1 f).$$

*However, the projector defined by (4.24) is sufficient for our purpose, and the above relation between $E_j$ and $P_{j,0}^2$ simplifies the following arguments.*

**Remark 4.2** *The operator $P_j$ in (4.24) has the following properties:*

*(i) $P_j v = v$ for all $v \in T_j$;*

*(ii) $P_j$ is bounded on $L_2(]0,2[)$.*

Property (i) follows by construction while (ii) is a consequence of the $L_2$-boundedness of $P_j^1$, $P_{j,0}^2$ and $E_j$. Remark 4.2 yields

**Proposition 4.3** *For all $s$, $0 \le s \le M+1$, the space $T_j = \operatorname{span} \boldsymbol{\varphi}_j$ with $\boldsymbol{\varphi}_j$ defined in (4.18) satisfies the Jackson estimate*

$$\|f - P_j f\|_{L_2(]0,2[)} \lesssim 2^{-js}\|f\|_{H^1(]0,2[)}, \quad f \in H^s(]0,2[). \tag{4.26}$$

In fact, it is simple to check that polynomials up to order $M$ are locally contained in $T_j$, which, by a standard argument, yields (4.26). By construction, the spaces $T_j$ also satisfy a Bernstein estimate of the form (4.17) on $]0,2[$,

$$\|f_j\|_{H^1(]0,2[)} \lesssim 2^j\|f_j\|_{L_2(]0,2[)}, \quad f_j \in T_j. \tag{4.27}$$

**Lemma 4.4** *The projectors $P_j$ satisfy*

$$P_\ell P_j = P_j P_\ell = P_j \quad \textit{for all } \ell > j. \tag{4.28}$$

**Proof:** We first observe that we have $P_j^1 P_\ell^1 = P_j^1$ and $P_{j,0}^2 P_{\ell,0}^2 = P_{j,0}^2$. The first relation $P_\ell P_j = P_j$ for $\ell > j$ follows obviously from Remark 4.2(ii). As for showing $P_j P_\ell v = P_j v$ for $\ell > j$ and any $v \in L_2(\Sigma)$, we first observe that $P_\ell v|_{]0,1[} = P_\ell^1 v$. This yields

$$E_j P_j^1(P_\ell v) = E_j P_j^1(P_\ell^1 v) = E_j P_j^1 v.$$

Moreover it holds

$$P_{j,0}^2 E_\ell P_\ell^1 v = P_{j,0}^2 P_{\ell,0}^2 E_\ell P_\ell^1 v = 0,$$

where the last equality derives from the identity $P_{\ell,0}^2 E_\ell = 0$ which is a consequence of biorthogonality. Then we have

$$P_j(P_\ell v) = E_j P_j^1(P_\ell v) + P_{j,0}^2(E_\ell P_\ell^1 v + P_{\ell,0}^2 v) = E_j P_j^1 v + P_{j,0}^2 v = P_\ell v.$$

∎

Lemma 4.4 implies that the spaces $\tilde{T}_j$ spanned by the dual functions $\tilde{\boldsymbol{\varphi}}_j$ are refinable [Dah96] which in turn yields that the $\tilde{T}_j$ satisfy a Bernstein inequality

$$\|\tilde{f}_j\|_{H^1(]0,2[)} \lesssim 2^j\|\tilde{f}_j\|_{L_2(]0,2[)} \tag{4.29}$$

for any $\tilde{f}_j \in \tilde{T}_j$ and any $s < 1/2$ [Dah97]. Thus, Lemma 4.3, (4.27) on the primal side and (4.29) together with (G4) for the dual functions yield the following result [Dah96].

**Theorem 4.5** *For any $f \in H^s(]0,2[)$ and $s \in (-1/2, 1)$, we have the norm equivalence*

$$\|f\|^2_{H^s(]0,2[)} \sim \sum_{j \geq j_0} 2^{2js} \|(P_j - P_{j-1})v\|^2_{L_2(]0,2[)}. \tag{4.30}$$

The difference projector

$$Q_j := P_{j+1} - P_j$$

can be characterized in terms of the two difference projectors

$$Q^1_j := P^1_{j+1} - P^1_j, \qquad Q^2_{j,0} := P^2_{j+1,0} - P^2_{j,0}$$

as

$$Q_j v = E_{j+1} Q_j + Q^2_{j,0} \left( f - E_j P^1_j f \right). \tag{4.31}$$

If we then introduce the two couples of biorthogonal wavelets $\psi^1_j, \tilde{\psi}^1_j$ and $\psi^2_j, \tilde{\psi}^2_j$

$$\begin{aligned} \psi^i_j &:= \{\psi^i_{j,m},\ m \in \nabla^i_j := \{1, \ldots, M_j\}\}, \qquad i = 1,2, \\ \tilde{\psi}^i_j &:= \{\tilde{\psi}^i_{j,m},\ m \in \nabla^i_j := \{1, \ldots, M_j\}\}, \qquad i = 1,2, \end{aligned} \tag{4.32}$$

which are defined such that

$$Q^1_j v = \sum_{m \in \nabla^i_j} \langle f, \tilde{\psi}^1_{j,m} \rangle \psi^1_{j,m}, \qquad Q^2_{j,0} v = \sum_{m \in \nabla^i_j} \langle f, \tilde{\psi}^2_{j,m} \rangle \psi^2_{j,m}$$

and which we assume to be $L_2$-stable, it is then easy to check that the projector $Q_j$ then writes

$$Q_j v = \sum_{m=1}^{2M_j} \langle v, \tilde{\psi}_{j,m} \rangle \psi_{j,m} \tag{4.33}$$

with, for $m = 1, \ldots, M_j$

$$\psi_{j,m} = \begin{cases} \psi^1_{j,m}, & \text{in } [0,1), \\ 2^{-j}\, \psi^1_{j,m}(1) \varphi^2_{j,0}, & \text{in } [1,2], \end{cases} \qquad \tilde{\psi}_{j,m} = \begin{cases} \tilde{\psi}^1_{j,m}, & \text{in } [0,1), \\ 0, & \text{in } [1,2], \end{cases}$$

while, for $c^m_j = \langle \varphi^2_{j,0}, \tilde{\psi}^2_{j,m} \rangle$,

$$\psi_{j,M_j+m} = \begin{cases} 0, & \text{in } [0,1), \\ \psi^2_{j,m}, & \text{in } [1,2], \end{cases} \qquad \tilde{\psi}_{j,M_j+m} = \begin{cases} c^m_j \tilde{\varphi}^1_{j,N_j}, & \text{in } [0,1), \\ \tilde{\psi}^2_{j,m}, & \text{in } [1,2]. \end{cases}$$

The $L_2$ stability of the new wavelet system $\boldsymbol{\psi}$, $\tilde{\boldsymbol{\psi}}$

$$\begin{aligned} \boldsymbol{\psi} &:= \{\psi_{j,m}, (j,m) \in \nabla = \cup_{j \geq j_0} \nabla_j\}, \\ \tilde{\boldsymbol{\psi}} &:= \{\tilde{\psi}_{j,m}, (j,m) \in \nabla = \cup_{j \geq j_0} \nabla_j\}, \end{aligned}$$

with $\nabla_j := \{1, \cdots, 2M_j\}$ is proven by standard techniques.

We remark that the construction proposed is local. The basis functions which are actually modified with respect to the ones originally defined on $]0,1[$ or on $]1,2[$ are only the ones which interact with the cross point $x = 1$, and these do not interact with the other extrema of the two intervals. Boundary conditions can then be easily imposed at 0 and/or 2. By iterating the same technique, a construction for e.g. $L_2(]0,3[)$ is immediately obtained, consisting in splitting $]0,3[$ as $]0,3[=]0,1[\cup]1,2[\cup]2,3[$. If we then consider the skeleton $\Sigma$ of our domain decomposition method, which we regard as the union of segments $\Sigma = \cup_n I_n$, we can build a multiscale analysis by choosing for all cross point $x$ one of the edges having $x$ as a vertex, which will be considered as the conformal image of $]0,1[$ — $x$ being the image of 1 — while all the other segments having $x$ as a vertex will be treated as the image of $]1,2[$ (again $x$ being the image of 1), as illustrated in Figure 1. We will thus obtain a couple $T_j(\Sigma), \tilde{T}_j(\Sigma)$ of biorthogonal multiresolutions of $L_2(\Sigma)$ (with

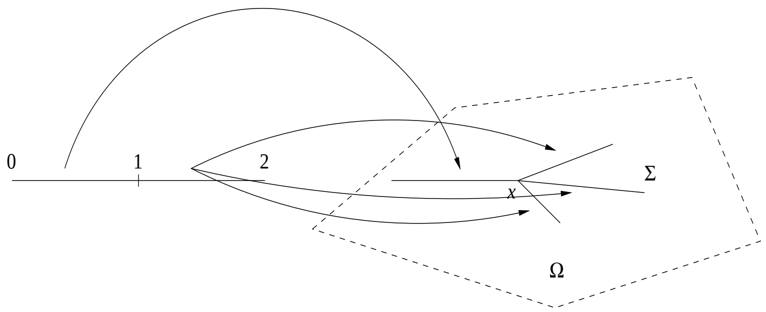


Figure 1: Decomposition of $\Sigma$

corresponding generators $\boldsymbol{\varphi}_j(\Sigma), \tilde{\boldsymbol{\varphi}}_j(\Sigma)$, projectors $P_j^\Sigma, \tilde{P}_j^\Sigma$, and wavelets $\boldsymbol{\psi}(\Sigma), \tilde{\boldsymbol{\psi}}(\Sigma)$), satisfying homogeneous boundary conditions on $\partial\Sigma = \bar{\Sigma} \cap \partial\Omega$. We want to point out that the implementation of the Fast Wavelet Transform on $L_2(\Sigma)$ can be easily carried out by using the FWT's corresponding to the MRA's $T_j^i, \tilde{T}_j^i$ and $T_{j,0}^i, \tilde{T}_{j,0}^i$, coupled with the extension operator $E_j$ which has a very simple structure. For $T_j(\Sigma), \tilde{T}_j(\Sigma)$ Bernstein and Jackson inequalities will hold — the Sobolev spaces $H^s$ on $\Sigma$ being defined according to [JW84] — with the usual modifications due to the homogeneous boundary conditions on $\partial\Sigma$. Then, by using the arguments in [DK], we can prove the norm equivalence

$$\|\sigma\|_\Phi^2 \sim \sum_j 2^j \|(P_{j+1}^\Sigma - P_j^\Sigma)\sigma\|_{L_2(\Sigma)}^2. \tag{4.34}$$

Thanks to the $L_2(\Sigma)$ stability of the wavelet basis $\psi_j$, we then obtain a norm equivalence of the form (3.3.4).

# 5 Conclusion

We presented a new stabilized formulation for the Three Fields domain decomposition method, introduced in [BM93]. Thanks to the stabilization, the discretization space for the interface variable can be chosen completely independently from the discretizations in the subdomains, provided the latter satisfy some mild compatibility condition.

This allows to discretize the interface by means of wavelet bases, without need of using wavelet functions in the subdomains. The discretization in the subdomains is regarded here as a "black box" which is *not* affected by the stabilization term.

Wavelet preconditioners can then be applied for the Steklov-Poincaré operator, for which we show asymptotically optimal (in term of the mesh size on the interface) bounds on the condition number.

The results are complemented by the briefly described construction of a wavelet basis on the skeleton which can be viewed as a simplification of the constructions in [CTU99, DS97]. The above results apply to such wavelet basis, which is particularly easy to implement.

The stabilization proposed is based on the computation of an equivalent scalar product for the space $H^{1/2}(\partial\Omega^k)$. Also this task can be performed by means of wavelets. It is our opinion that a suitable coupling of the wavelets used for stabilizing with those used for discretizing the interface variable can result in a particularly simple form for the stabilizing term. This will be exploited in a future work in order to design efficient numerical schemes.

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Silvia Bertoluzza
CNR–Istituto di Analisi Numerica
Via Abbiategrasso, 209
27100 Pavia, Italy
e–mail: `aivlis@dragon.ian.pv.cnr.it`
WWW: `http://www.ian.pv.cnr.it/~aivlis`

Angela Kunoth
Institut für Geometrie und Praktische Mathematik
RWTH Aachen
52056 Aachen, Germany
Fax: +49-241-8888 317
e–mail: `kunoth@igpm.rwth-aachen.de`
WWW: `http://www.igpm.rwth-aachen.de/kunoth`