The Mortar Method in the Wavelet Context*

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Abstract

This paper deals with the use of wavelets in the framework of the Mortar method.

We first review in an abstract framework the theory of the Mortar Method for non conforming domain decomposition, and point out some basic assumptions under which stability and convergence of such method can be proven. We study the application of the Mortar Method in the biorthogonal wavelet framework. In particular we define suitable multiplier spaces for imposing weak continuity. Unlike in the classical mortar method, such multiplier spaces are not a subset of the space of traces of interior functions, but rather of their duals.

For the resulting method, we provide with an error estimate, which is optimal in the geometrically conforming case. We also study an almost diagonal preconditioner based on using wavelet preconditioners as building blocks in a substructuring approach.

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Contents

1 Introduction

In the last few years there has been an increasing interest in the use of wavelet based methods for the numerical solution of Partial Differential Equations (see for instance [29, 37, 40]). The existence of diagonal preconditioners for elliptic operators of any given order ([44, 38]), as well as the possibility of designing efficient adaptive approximation schemes for different type of problems ([47, 18, 27, 30, 32, 34, 35, 16, 19]), are among the attractive features of such bases, which are raising the attention of the scientific computing community on the development in this field.

On the other hand, in order for such methods to be applicable in real life problems, several issues still need to be faced, among which the treatment of non trivial gemeotries. We recall that such bases were first introduced in $L^2(\mathbb{R})$ [49, 43]. Generalizations to $L^2(]0,1[)$ (and to $L^2(]0,1[^d))$ have been introduced in the early nineties ([31, 7]), and recently bases were constructed by a conforming domain decomposition approach, allowing to treat all domains which can be split as union of conformal images of rectangles or cubes ([25, 42]).

Rather than following such a philosophy, we prefer to consider here a *non conforming* domain decomposition approach. This has the great advantage of allowing to couple discretizations of different types. By this approach, wavelet bases can be coupled for instance, with Finite Elements. This would allow in principle to treat even very complicated geometries, by decomposing any given domain into subdomains, some of which are (conformal images of) squares or a cubes (and can then be discretized by wavelets) while the remaining ones can be discretized with Finite Elements.

In particular we consider here the *Mortar Method* ([14, 15]) which has been designed for such purpose. In the Mortar method the interface of the decomposition is itself decomposed into "mortars" (which in this paper will be called "multiplier sides"): each "mortar" is the whole edge of a given subdomain, and the "mortars" are disjoint from each other. Weak continuity is imposed by requiring that the jump on each mortar of a discrete function is orthogonal to a suitable "multiplier space". Such method has nowadays been applied to a wide variety of real-life problems in both two and three dimensions ([5, 51, 9, 10, 6]) and it is well suited for parallel implementation ([1]). We consider here the second version of such method ([8]) , in which strong continuity of the discrete functions at cross points is not required.

The aim of this paper is to introduce and analyze the application of such a method in the context of biorthogonal wavelet bases. We limit our analysis to the two dimensional case, and for simplicity we assume that all the subdomains are rectangular. For the sake of notational simplicity we chose to split the paper in two parts. In the first part we briefly review the theory of the Mortar Method. While doing that we underline the basic requirements on the discretization spaces needed in order to get stability and convergence. Moreover we point out the main differences between what happens in the framework of (for instance) finite elements and what will happen when dealing with wavelets.

In the second part we show that, for a suitable choice of the "multiplier space", wavelets fulfill such requirements. Due to the particular structure of the Mortar method we are able to do that by concentrating on one subdomain and one "mortar". In particular we provide with an error estimate, which, in the case of a geometrically conforming decomposition, is optimal.

In implementing the proposed method, one can take advantage of the properties of wavelets in several way. We concentrate here on the issue of preconditioning. Different preconditioners have been studied in the framework of non conforming domain decomposition (see for instance [2, 3, 4, 45, 20]). We study here the construction of a $O(N)$ operations preconditioner (N being the total number of degrees of freedom) for the resulting linear system, by a mixed substructuring/multilevel approach, for which we give an estimate on the resulting condition number.

2 The Mortar Method

Rather than introducing and analyzing the Mortar Wavelet Method directly, for the sake of notational simplicity we prefer to review at first the theory of the Mortar Method in an abstract framework. In doing that, we will point out some basic assumptions, which we will later check in the wavelet case. We follow essentially the guidelines of [15, 14]. Most of the proofs are the same in an abstract framework as they are in the finite element framework, and we will therefore omit them, or sometimes just briefly sketch them, for the sake of completeness. On the other hand, when dealing with wavelets, one is naturally confronted with two new issues:

- the natural choice of the multiplier space on an edge is not necessarily a subset of the space of traces of the interior functions, as it is in the classical Mortar Method; we will see that, under suitable assumptions which are easily verified in the wavelet case, this does not yield any major modification in the results that can be obtained, with no substantial difference with respect to the "classical" Mortar Method (in this respect see also [53]);
- since interpolation on general non dyadic (even coarse) grids in the wavelet context is still an open problem, we are led to analyze the approximation error also in the case in which the existence of an approximation operator that is interpolating at cross-points is not verified; we will see that this leads (in the geometrically non conforming case) to a loss of a factor of the order of $\log(H/h)$ in the error estimate.

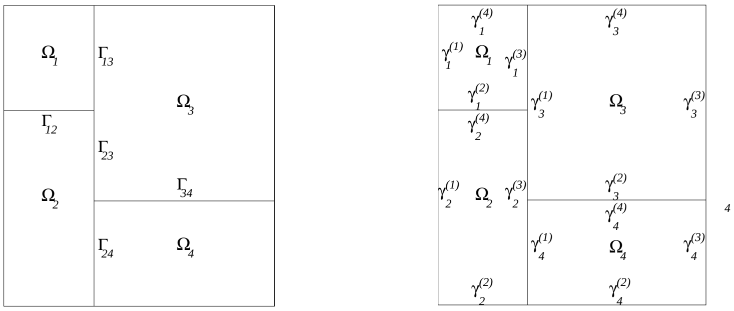

Figure 1: Geometrical decomposition of the domain Ω.

2.1 The Geometry

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. We will consider a decomposition of Ω as the union of L subdomains Ω_ℓ,

$$\Omega = \bigcup_{\ell=1,L} \Omega_\ell, \tag{1}$$

which, for the sake of simplicity we will assume to be rectangular. We set

$$\Gamma_{\ell n} = \partial\Omega_n \cap \partial\Omega_\ell, \tag{2}$$

$$S = \cup\Gamma_{\ell n}. \tag{3}$$

Throughout this section we will assume that the above decomposition is fixed. The constants in the bounds that we are going to prove will in general depend on the size of the subdomains, unless otherwise stated.

We denote by $\gamma_\ell^{(i)}$ $(i = 1, \ldots, 4)$ the i-th side of the ℓ-th domain:

$$\partial\Omega_\ell = \bigcup_{i=1}^{4} \gamma_\ell^{(i)}.$$

For $\ell = 1, \cdots, L$ let ν_ℓ denote the outer unit normal to the subdomain Ω_ℓ, and for $u_\ell \in H^1(\Omega_\ell)$ let $\frac{\partial u_\ell}{\partial \nu_\ell}$ denote the outer normal derivative.

Remark 2.1 *The decomposition is said to be* geometrically conforming *if each edge* $\gamma_\ell^{(i)}$ *coincides with* $\Gamma_{\ell n} (= \partial\Omega_\ell \cup \partial\Omega_n)$ *for some* n. *If the decomposition is not geometrically conforming, then each interior edge* $\gamma_\ell^{(i)}$ *will be in general split as the union of several segments* $\Gamma_{\ell n}$*:*

$$\gamma_\ell^{(i)} = \bigcup_{n \in I_\ell^{(i)}} \Gamma_{\ell n}, \tag{4}$$

where

$$I_\ell^{(i)} = \{n : \ |\partial\Omega_n \cap \gamma_\ell^{(i)}| \neq 0\}. \tag{5}$$

We assume that the subdomains are regular in shape, and from now on we will make the following grading assumption on the decomposition (which is automatically fulfilled in the case of a geometrically conforming decomposition).

(G1) The following bound holds

$$\max_{(\ell,i)} \left(\frac{|\gamma_\ell^{(i)}|}{\min_{n\in I_\ell^{(i)}} |\Gamma_{\ell n}|} \right) \leq \rho. \tag{6}$$

The constants appearing in the estimates of the following sections will in generally depend on the bound ρ.

For defining the Mortar Method we start by choosing a splitting of the skeleton S as the disjoint union of a certain number of subdomain sides $\gamma_\ell^{(i)}$, which we will call "multiplier sides" (in the usual terminology these are called "mortars" or "slave sides"). More precisely, we choose an index set $I \subset \{1,\ldots,L\}\times\{1,\ldots,4\}$ such that,

$$S = \bigcup_{(\ell,i)\in I} \gamma_\ell^{(i)}, \qquad \begin{array}{c} (\ell_1,i_1),(\ell_2,i_2) \in I, \\ (\ell_1,i_1) \neq (\ell_2,i_2) \end{array} \Rightarrow \gamma_{\ell_1}^{(i_1)} \cap \gamma_{\ell_2}^{(i_2)} = \emptyset. \tag{7}$$

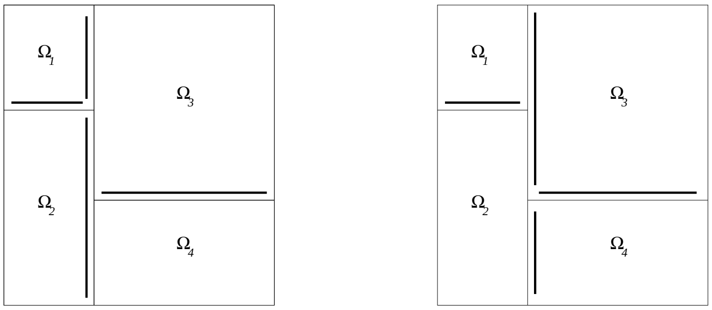

Figure 2: Two possible choices of "multiplier sides". For the one on the left we have (using the indexing of Figure 2.1) $I = \{(2,3),(1,2),(1,3),(3,2)\}$ and $I^* = \{(4,1),(4,4),(3,1),(2,4)\}$ while for the one on the right $I = \{(1,2),(3,1),(3,2),(4,1)\}$. Remark that in all cases the set $\{1,\ldots,L\} \times \{1,\ldots,4\} \setminus I \cup I^*$ individuates the edges which belong to the external boundary $\partial\Omega$.

Furthermore we will denote by $I^* \subset \{1,\cdots,L\}\times\{1,\cdots,4\}$ the index-set corresponding to "trace sides" ("non mortars" or "master sides" in the usual terminology), which is defined in such a way that $I^* \cap I = \emptyset$ and $S = \cup_{(\ell,i)\in I^*} \gamma_\ell^{(i)}$.

Corresponding to the splitting (7) we define a normal direction on the skeleton. More precisely for $(\ell,i) \in I$ ($\gamma_\ell^{(i)}$ multiplier side) we set

$$\nu = \nu_\ell, \qquad \text{on } \gamma_\ell^{(i)}. \tag{8}$$

Since $S = \cup_{(\ell,i)\in I}\gamma_\ell^{(i)}$, this defines ν on S. For any $u = (u_\ell)_{\ell=1,\cdots,L} \in \prod_{\ell=1}^L H^1(\Omega_\ell)$, $\dfrac{\partial u}{\partial \nu}$ will be defined accordingly: on $\gamma_\ell^{(i)}$ $(\ell,i) \in I$ ($\gamma_\ell^{(i)}$ multiplier side) we set $\dfrac{\partial u}{\partial \nu} = \dfrac{\partial u_\ell}{\partial \nu_\ell}$. Moreover, we let $[u]$ denote the jump of u along the skeleton S with the proper sign: on $\Gamma_{\ell n} \subset \gamma_\ell^{(i)}$, with $(\ell,i) \in I$ ($\gamma_\ell^{(i)}$ multiplier side),

$$[u]|_{\Gamma_{\ell n}} = u_\ell|_{\Gamma_{\ell n}} - u_n|_{\Gamma_{\ell n}}. \tag{9}$$

In order to simplify the notation in the following we will sometime make use of a multi-index $m = (\ell,i)$. We will for instance write $\gamma_m, m \in I$ (respectively $m \in I^*$) for $\gamma_\ell^{(i)}, (\ell,i) \in I$ (respectively $(\ell,i) \in I^*$).

2.2 The continuous problem

For simplicity we will consider the following problem. Given $f \in L^2(\Omega)$ find $u : \Omega \longrightarrow \mathbb{R}$ such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0. & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Given the splitting of the domain Ω introduced in the previous section, we will consider a non conforming domain decomposition method for the solution of such a problem. In order to do that let

$$X = \prod_\ell \{u_\ell \in H^1(\Omega_\ell) |\ \ u_\ell = 0 \text{ on } \partial\Omega \cap \partial\Omega_\ell\}, \tag{11}$$

$$T = \prod_\ell H_*^{1/2}(\partial\Omega_\ell), \tag{12}$$

where $H_*^{1/2}(\Omega_\ell)$ is defined by

$$H_*^{1/2}(\partial\Omega_\ell) = H^{1/2}(\partial\Omega_\ell) \qquad \text{if } \partial\Omega_\ell \cap \partial\Omega = \emptyset$$

and

$$H_*^{1/2}(\partial\Omega_\ell) = \{\eta \in H^{1/2}(\partial\Omega_\ell),\ \eta|_{\partial\Omega_\ell\cap\partial\Omega} \equiv 0\} \sim H_{00}^{1/2}(\partial\Omega_\ell \setminus \partial\Omega),$$

otherwise. We will denote by $\|\cdot\|_{1/2,\partial\Omega_\ell}$ the related norm, and by $\|\cdot\|_{-1/2,\ell}$ the norm of the corresponding dual space.

Remark 2.2 *From now on we will always consider functions satisfying (whenever this makes sense) an homogeneous boundary condition on $\partial\Omega$ even if not explicitely stated.*

On X we introduce the following broken norm and semi-norm:

$$\|u\|_X = \left(\sum_\ell \|u\|^2_{1,\Omega_\ell}\right)^{\frac{1}{2}}, \tag{13}$$

$$|u|_X = \left(\sum_\ell |u|^2_{1,\Omega_\ell}\right)^{\frac{1}{2}}. \tag{14}$$

In the following it will also be convenient to introduce the following norm on T:

$$\|\eta\|_T = \left(\sum_\ell \|\eta_\ell\|^2_{1/2,\partial\Omega_\ell}\right)^{1/2} \tag{15}$$

Moreover, for $\lambda \in L^2(S)$ we will use the notation:

$$\|\lambda\|_{-1/2,S} = \left(\sum_\ell \|\lambda\|_{-1/2,\ell}\right)^{1/2} \tag{16}$$

Let now a composite bilinear form $a_X : X \times X \longrightarrow \mathbb{R}$ be defined as follows:

$$a_X(u,v) = \sum_\ell \int_{\Omega_\ell} \nabla u_\ell \nabla v_\ell. \tag{17}$$

The bilinear form a_X is clearly not coercive on X. In order to obtain a well posed problem we will then consider proper subspaces of X, consisting of functions satisfying a suitable *weak continuity* constraint.

More precisely, for any subspace M of $L^2(S)$ let a *constrained space* $\mathcal{X}(M)$ be defined as follows:

$$\mathcal{X}(M) = \{u \in X : \int_S [u]\lambda = 0, \quad \forall \lambda \in M\} \tag{18}$$

We will consider the following problem (depending on the choice of the "*multiplier space*" M):

Problem 2.1 **(P_M)** *Find $u_M \in \mathcal{X}(M)$ such that for all $v \in \mathcal{X}(M)$*

$$a_X(u_M, v) = \int_\Omega f v. \tag{19}$$

In the following we will consider multiplier spaces M satisfying the following assumption.

(BP) There exists a constant $C_M > 0$ such that the following broken Poincaré inequality holds for all $u \in \mathcal{X}(M)$:

$$|u|_X \geq C_M \|u\|_X. \tag{20}$$

(20) is evidently equivalent to the coercivity of the bilinear form a_X over $\mathcal{X}(M)$. By simply observing that $M_1 \subset M_2$ implies $\mathcal{X}(M_2) \subset \mathcal{X}(M_1)$ one gets the following well known result (see for instance [14]).

Theorem 2.1 *Let $\bar{M}$ satisfy assumption (BP). Then for all M such that $\bar{M} \subset M$ we have the following:*

- *the solution u_M of problem P_M exists and is unique;*
- *tor u solution of (10) the following bound holds with constant $\bar{C} = \bar{C}(\bar{M})$:*

$$\|u - u_M\|_X \leq \bar{C} \inf_{\lambda \in M} \|\frac{\partial u}{\partial \nu} - \lambda\|_{-1/2,S}. \tag{21}$$

Proof: We first observe that for all M, the solution u of (10) verifies $u \in X_M$. For any $v \in H^1(\Omega_\ell)$ we have

$$\int_{\Omega_\ell} \nabla u \nabla v = \int_{\partial\Omega_\ell} \frac{\partial u}{\partial \nu_\ell} v - \int_{\Omega_\ell} \Delta u v = \int_{\partial\Omega_\ell} \frac{\partial u}{\partial \nu_\ell} v + \int_{\Omega_\ell} f v,$$

that is, for all $v \in \mathcal{X}(M)$ we have

$$a_X(u, v) = \int_\Omega f v + \sum_\ell \int_{\partial\Omega_\ell} \frac{\partial u}{\partial \nu_\ell} v = \int_\Omega f v + \int_S \frac{\partial u}{\partial \nu} [v]$$

As a result for all $\lambda \in M$ we have

$$a_X(u, v) = \int_\Omega f v + \int_S \left(\frac{\partial u}{\partial \nu} - \lambda \right) [v]$$

This implies:

$$\begin{aligned}
\bar{C}\|u - u_M\|_X &\leq a_X(u - u_M, u - u_M) = \left| \int_S \left(\frac{\partial u}{\partial \nu} - \lambda \right) [u - u_M] \right| = \\
&= \left| \sum_\ell \sum_n \int_{\Gamma_{\ell n}} \left(\frac{\partial u}{\partial \nu} - \lambda \right) [u - u_M] \right| = \\
&= \sum_\ell \left| \int_{\partial\Omega_\ell} \left(\frac{\partial u}{\partial \nu} - \lambda \right) (u - u_M) \right| \leq \\
&\leq \sum_\ell \left\| \frac{\partial u}{\partial \nu} - \lambda \right\|_{-1/2,\ell} \|u - u_M\|_{1,\Omega_\ell} \leq \\
&\leq C \left\| \frac{\partial u}{\partial \nu} - \lambda \right\|_{-1/2,S} \|u - u_M\|_X.
\end{aligned}$$

By dividing both sides by $\|u - u_M\|_X$, since λ is arbitrary, we get (21). $\square$

2.3 Mortar Discrete Problem - Approximation Spaces

For each ℓ let now $\mathcal{V}_\delta^\ell$ be a family of finite dimensional subspaces of $H^1(\Omega_\ell) \cap C^0(\bar{\Omega}_\ell)$, depending on a parameter $\delta = \delta_\ell > 0$ and satisfying an homogeneous boundary condition on $\partial\Omega \cap \partial\Omega_\ell$. Set

$$T_\delta^\ell = \mathcal{V}_\delta^\ell|_{\partial\Omega_\ell}, \tag{22}$$

and, for each edge $\gamma_\ell^{(i)}$ of the subdomain Ω_ℓ let

$$T_{\ell,i} = \{\eta : \quad \eta \text{ is the trace on } \gamma_\ell^{(i)} \text{ of some } u_\ell \in \mathcal{V}_\delta^\ell \ \} \tag{23}$$

$$T_{\ell,i}^0 = \{\eta \in T_{\ell,i} : \quad \eta = 0 \text{ at the vertices of } \gamma_\ell^{(i)}\}. \tag{24}$$

We set

$$X_\delta = \prod_{\ell=1}^{L} \mathcal{V}_\delta^\ell \ \subset X, \qquad T_\delta = \prod_{\ell=1}^{L} T_\delta^\ell \ \subset T. \tag{25}$$

For each $m = (\ell, i) \in I$ let the finite dimensional multiplier space M_δ^m (also depending on the parameter δ) on γ_m,

$$M_\delta^m \subset L^2(\gamma_m), \qquad\qquad dim(M_\delta^m) = dim(T_m^0), \tag{26}$$

be given. We set:

$$M_\delta = \{\eta \in H^{-1/2}(S), \ \forall m \in I \ \ \eta|_{\gamma_m} \in M_\delta^m\} \sim \prod_{m\in I} M_m. \tag{27}$$

The *constrained* approximation and trace spaces $\mathcal{X}_\delta$ and $\mathcal{T}_\delta$ are then defined as follows:

$$\mathcal{X}_\delta = \{v_\delta \in X_\delta, \quad \int_S [v_\delta]\lambda = 0, \ \forall\lambda \in M_\delta\} \subset \mathcal{X}(M_\delta), \tag{28}$$

$$\mathcal{T}_\delta = \{\eta \in T_\delta, \int_S [\eta]\lambda = 0, \ \forall\lambda \in M_\delta\}. \tag{29}$$

Moreover we will denote by $\mathcal{V}_\delta^{\ell,0} \subset \mathcal{V}_\delta^\ell$ and $\mathcal{X}_\delta^0 \subset \mathcal{X}_\delta$ the subspaces of functions vanishing on the skeleton:

$$\mathcal{V}_\delta^{\ell,0} = \mathcal{V}_\delta^\ell \cap H_0^1(\Omega_\ell), \qquad\qquad \mathcal{X}_\delta^0 = \prod \mathcal{V}_\delta^{\ell,0}. \tag{30}$$

Remark that the elements of $\mathcal{X}_\delta^0$ have null jump, hence they trivially satisfy the jump constraint ($\mathcal{X}_\delta^0 \subset \mathcal{X}_\delta$).

We can now introduce the following discrete problem:

Problem 2.2 (PD) *Find $u_\delta \in \mathcal{X}_\delta$ such that for all $v_\delta \in \mathcal{X}_\delta$*

$$a_X(u_\delta, v_\delta) = \int_\Omega f v_\delta. \tag{31}$$

The following result holds:

Theorem 2.2 *Assume that $\bar{M} = \cap_{\delta>0} M_\delta$ satisfies assumption (BP). Then for all $\delta >$, problem P_δ admits a unique solution u_δ which satisfy the following error estimate:*

$$\|u - u_\delta\|_X \leq \bar{C} \left(\inf_{v_\delta \in \mathcal{X}_\delta} \|u - v_\delta\|_X + \inf_{\lambda \in M_\delta} \left\| \frac{\partial u}{\partial \nu} - \lambda \right\|_{-1/2,S} \right), \tag{32}$$

with $\bar{C}$ constant depending on $\bar{M}$.

Proof: As we already observed, $\bar{M} \subset M_\delta$ implies $\mathcal{X}_\delta \subset \mathcal{X}(\bar{M})$. In view of (21), using the triangular inequality and applying Cea's Lemma to the solution in $\mathcal{X}_\delta$ of problem P_M (for $M = M_\delta$,) (which, thanks to the assumption on $\bar{M}$ is coercive), we get

$$\|u_\delta - u_M\|_X \leq \inf_{v_\delta \in \mathcal{X}_\delta} \|u_M - v_\delta\|_X \leq \|u - u_M\|_X + \inf_{v_\delta \in \mathcal{X}_\delta} \|v_\delta - u_M\|_X$$

and

$$\|u - u_\delta\|_X \leq \|u - u_M\|_X + \|u_M - u_\delta\|_X,$$

which imply (32). □

2.4 Stability

In order to apply the result of the previous sections, we need to choose the multiplier spaces M_δ^m in such a way that $\bar{M} = \cap_\delta M_\delta$ satisfies assumption (BP). If, as it is in our case, the space $\bar{M}$ is defined on each multiplier side γ_m independently, that is

$$\bar{M} = \oplus_{m \in I} \bar{M}_m, \qquad \bar{M}_m = \cap_\delta M_\delta^m,$$

it is well known ([14]) that a sufficient condition for $\bar{M}$ to satisfy assumption (BP) is the following condition holds far all $m = (\ell, i) \in I$ ($\gamma_\ell^{(i)}$ multiplier side): for any piecewise constant function g, (g constant on each $\Gamma_{\ell k}$, $k \in I_\ell^{(i)}$) we have

$$\int_{\gamma_m} g\lambda = 0, \forall \lambda \in \bar{M}_m \qquad \Longrightarrow \qquad g = 0. \tag{33}$$

(we recall that $\gamma_m = \bigcup_{n \in I_\ell^{(i)}} \Gamma_{\ell k}$).

The validity of assumption (33) has been studied for several types of discretizations. Several sufficient conditions for it to hold are therefore known. We recall for instance the following result [13].

Proposition 2.1 (**Sufficient condition I**) *If* $card\{I_\ell^{(i)}\} \leq \tilde{N}$ *($\tilde{N}$ being the number of polynomials exactly reproduced in $\bar{M}_m$), then (33) holds.*

It is also easy to prove that the following condition holds.

Proposition 2.2 (**Sufficient condition II**) *A sufficient condition for $\bar{M}_m$ to satisfy (33) is that $\forall i \in I_\ell^{(i)}$ there $\exists \beta_i \in M_\delta^m$ such that*

$$supp\, \beta_i \subset \Gamma_{\ell n}, \qquad \int_{\Gamma_{\ell n}} \beta_i = 1. \tag{34}$$

Remark 2.3 *In the case of a geometrically conforming decomposition (far all $m = (\ell, k)$, $\gamma_m = \Gamma_{\ell n}$, for some n) the above condition is also necessary, which is in general not true in the case of geometrically non conforming decompositions (see for instance the case of the mortar element method in the framework of a spectral decomposition, in which case M_δ^m is constructed out of polynomial functions).*

2.5 Approximation Error

In order to bound the right hand side of (32) we will make the following assumptions on the spaces considered.

(A1) $\forall m = (\ell, i) \in I$ ($\gamma_\ell^{(i)}$ multiplier side), there exists a bounded projection $\pi_m : L^2(\gamma_m) \longrightarrow T_m^0$, such that for all $\eta \in L^2(\gamma_m)$ and for all $\lambda \in M_\delta^m$

$$\int_{\gamma_m} (\eta - \pi_m \eta)\lambda = 0, \tag{35}$$

and for all $\eta \in H_{00}^{1/2}(\gamma_m)$

$$\|\pi_m \eta\|_{H_{00}^{1/2}(\gamma_m)} \lesssim \|\eta\|_{H_{00}^{1/2}(\gamma_m)}. \tag{36}$$

(A2) $\forall m = (\ell, i) \in I$ multiplier side, there exists a discrete lifting $R_m : T_m^0 \longrightarrow \mathcal{V}_\delta^\ell$ such that for all $\eta \in T_m^0$

$$R_m \eta = 0 \text{ on } \partial\Omega_\ell \backslash \gamma_m, \qquad R_m \eta = \eta \text{ on } \gamma_m, \tag{37}$$

and

$$\|R_m \eta\|_{1,\Omega_\ell} \lesssim \|\eta\|_{H_{00}^{1/2}(\gamma_m)}. \tag{38}$$

(A3) For all $m = (\ell, i) \in I$ ($\gamma_\ell^{(i)}$ multiplier side), the following inverse inequality holds: for all elements $\eta \in T_m^0$ and for all s, $0 \le s < 1/2$ it holds

$$\|\eta\|_{H_{00}^{1/2}(\gamma_m)} \lesssim C h_m^{s-1/2} \|\eta\|_{H^s(\gamma_m)}. \tag{39}$$

We remark that by space interpolation, assumption (A.1) implies that the projection operator π_m verifies for all s, $0 < s < 1/2$:

$$\|\pi_m \eta\|_{H^s(\gamma_m)} \lesssim \|\eta\|_{H^s(\gamma_m)}, \tag{40}$$

uniformly in s.

We also introduce the following assumption which will be needed in the next section in order to construct preconditioners for the linear system resulting from the mortar discretization.

(A4) The spaces $\mathcal{V}_\delta^\ell$ satisfy the following inverse inequality: for all $w \in \mathcal{V}_\delta^\ell$ it holds

$$\|\nabla w\|_{\infty,\Omega_\ell} \lesssim \delta_\ell^{-1} \|w\|_{\infty,\Omega_\ell}. \tag{41}$$

In assumptions (A3) and (A4), h_m and δ_ℓ act as "mesh sizes" on γ_m and Ω_ℓ respectively. We remark that, for each subdomain Ω_ℓ, not only the "mesh size" δ_ℓ will be relevant, but also the mesh size induced by the discretization of the adjacent subdomains, in particular on trace sides of Ω_ℓ. Therefore we will introduce the following notation: for γ_m, $m = (\ell, i) \in I$ multiplier side, we denote by h_m the "smallest mesh size" of T_m (defined as the parameter for which (A3) holds), and we introduce a mesh size parameter

$$h_\ell = \min_{m : |\gamma_m \cap \partial\Omega_\ell| > 0} \{\delta_\ell, h_m\}. \tag{42}$$

Moreover we set

$$H_\ell = \textit{diameter of } \Omega_\ell, \tag{43}$$

and we write conventionally

$$\frac{H}{h} = \min_\ell \left\{ \frac{H_\ell}{h_\ell} \right\}. \tag{44}$$

We observe that (A4) still holds if δ_ℓ is substituted by h_ℓ.

The following lemma holds:

Lemma 2.1 *If assumptions (A1–A4) hold, then for any $\eta = (\eta_\ell)_{\ell=1,\cdots,L} \in T$ it holds:*

$$\sum_{m \in I} \|\pi_m(\eta)\|^2_{H_{00}^{1/2}(\gamma_m)} \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \|\eta\|_T^2 \tag{45}$$

Proof: We start by recalling that the following inequalities hold. Let $U = \cup U_n$, $n = 1, \ldots, N$, U, U_n intervals. Assume that $|U|/|U_n| \leq \rho$. Then it holds for all ε, $0 < \varepsilon \leq 1/2$,

$$\|u\|^2_{H^{1/2-\varepsilon}(U)} \lesssim \sum_n \|u\|^2_{H_0^{1/2-\varepsilon}(U_n)} \lesssim \frac{|I|^{2\varepsilon}}{\varepsilon^2} \|u\|^2_{H^{1/2}(U)}, \tag{46}$$

with constants dependent on ρ, and independent of ε (we recall that $H^{1/2-\varepsilon}$ and $H_0^{1/2-\varepsilon}$ are the same space, the respective norms being indeed equivalent, but the constants in the equivalence depending on ε).

Let now $\eta = (\eta_\ell)_{\ell=1,\ldots,L}$ be any element of $T = \prod_{\ell=1,\cdots,L} H_*^{1/2}(\partial\Omega_\ell)$. Thanks to (40), for any ε, $0 < \varepsilon < 1/2$ it holds:

$$\begin{aligned}
&\sum_{m\in I} \|\pi_m([\eta])\|^2_{H_{00}^{1/2}(\gamma_m)} \lesssim \sum_{m\in I} h_m^{-2\varepsilon} \|\pi_m([\eta])\|^2_{H_0^{1/2-\varepsilon}(\gamma_m)} \lesssim \\
&\quad \lesssim \sum_{\substack{m\in I\\ m=(\ell,i)}} h_m^{-2\varepsilon} \sum_{n\in I_\ell^{(i)}} (\|\eta_\ell\|^2_{H_0^{1/2-\varepsilon}(\Gamma_{\ell n})} + \|\eta_n\|^2_{H_0^{1/2-\varepsilon}(\Gamma_{\ell n})}) = \\
&\quad = \sum_{\ell=1}^{L} h_\ell^{-2\varepsilon} \sum_{n:\Gamma_{\ell n}\neq\emptyset} \|\eta_\ell\|^2_{H_0^{1/2-\varepsilon}(\Gamma_{\ell n})} \lesssim \frac{h_\ell^{-2\varepsilon} H_\ell^{2\varepsilon}}{\varepsilon^2} \sum_{\ell=1}^{L} \|\eta_\ell\|^2_{1/2,\partial\Omega_\ell} \lesssim \\
&\quad \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \sum_{\ell=1}^{L} \|\eta_\ell\|^2_{1/2,\partial\Omega_\ell},
\end{aligned}$$

where the last bound derives by choosing $\varepsilon = \frac{1}{|\log_2(H/h)|}$. $\square$

We can now define a linear operator $\pi : \prod_{\ell=1}^L L^2(\partial\Omega_\ell) \longrightarrow \prod_{\ell=1}^L L^2(\partial\Omega_\ell)$ that we will need in the following: more precisely, for $\eta = (\eta_\ell)_{\ell=1,\cdots,L}$, $\pi(\eta) = (\eta_\ell^*)_{\ell=1,\cdots,L}$ is defined on multiplier sides as π_m applied to the jump of η, while it is set identically zero on trace sides and on the external boundary $\partial\Omega$:

$$\begin{aligned}
\eta_\ell^*|_{\gamma_m} &= \pi_m([\eta]|_{\gamma_m}), \quad \text{for } m = (\ell, i) \in I \qquad (47)\\
\eta_\ell^*|_{\gamma_m} &= 0, \quad \text{for } m = (\ell, i) \in I^*,\\
\eta_\ell^* &\equiv 0 \quad \text{on } \partial\Omega_\ell \cap \partial\Omega.
\end{aligned}$$

By observing that for all $\eta \in H^{1/2}(\partial\Omega_\ell)$ satisfying $\eta|_{\gamma_\ell^{(i)}} \in H_{00}^{1/2}(\gamma_\ell^{(i)})$, $(i = 1, \ldots, 4)$ it holds

$$\|\eta\|^2_{H^{1/2}(\partial\Omega_\ell)} \lesssim \sum_{i=1}^{4} \|\eta\|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})} \tag{48}$$

we obtain the following corollary of lemma 2.1.

Corollary 2.1 *If assumptions (A1–A4) hold, than for any $\eta = (\eta_\ell)_{\ell=1,\cdots,L}$ in the trace space T it holds:*

$$\|\pi(\eta)\|_T \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) \|\eta\|_T \tag{49}$$

We are then able to prove the following theorem.

Theorem 2.3 *Let assumptions (A1 – A3) hold. Then for any $u \in H_0^1(\Omega)$ we have:*

$$\inf_{v_\delta \in \mathcal{X}_\delta} \|u - v_\delta\|_X \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) \left(\sum_\ell \inf_{v_{\delta,\ell} \in \mathcal{V}_\delta^\ell} \|u - v_{\delta,\ell}\|^2_{1,\Omega_\ell}\right)^{1/2} \tag{50}$$

Proof: For each ℓ let $v_{\delta,\ell}$ be an arbitrary element of $\mathcal{V}_\delta^\ell$. Let $\bar{v}_\delta = (v_{\delta,\ell})_{\ell=1,\dots,L}$. Since $\bar{v}_\delta$ does not necessarily satisfy the jump condition, it may not belong to $\mathcal{X}_\delta$. We now define an element $v_\delta \in \mathcal{X}_\delta$ as follows:

$$v_\delta = \bar{v}_\delta - \sum_{m \in I} R_m \pi_m([\bar{v}_\delta]), \tag{51}$$

where, by abuse of notation for $\eta \in T_m^0$ $(m = (\ell, i))$ we indicate by $R_m(\eta)$ the element of $\mathcal{X}_\delta$ which coincides with $R_m(\eta)$ in Ω_ℓ and which is identically zero on the other subdomains. We easily check that v_δ belongs to $\mathcal{X}_\delta$. In fact for $\eta \in T_m^0$ we have,

$$[R_m \eta] = \eta, \text{ on } \gamma_m,$$

and hence

$$[v_\delta] = [\bar{v}_\delta] - \pi_m([\bar{v}_\delta]), \text{ on } \gamma_m,$$

which implies, for $\lambda_\delta \in M_\delta$, thanks to (35),

$$\int_S [v_\delta]\lambda_\delta = \sum_m \int_{\gamma_m} ([\bar{v}_\delta] - \pi_m([\bar{v}_\delta]))\lambda_\delta = 0$$

We can now bound

$$\begin{aligned}
\|\sum_{m \in I} R_m \pi_m([\bar{v}_\delta])\|^2_X &= \sum_\ell \| \sum_{\substack{m \in I \\ m=(\ell,i)}} R_m \pi_m([\bar{v}_\delta])\|^2_{1,\Omega_\ell} \\
&\lesssim \sum_\ell \sum_{\substack{m \in I \\ m=(\ell,i)}} \|R_m \pi_m([\bar{v}_\delta])\|^2_{1,\Omega_\ell} \\
&\lesssim \sum_{m \in I} \|\pi_m([\bar{v}_\delta])\|^2_{H^{1/2}_{00}(\gamma_m)} \\
&= \sum_{m \in I} \|\pi_m([\bar{v}_\delta - u])\|^2_{H^{1/2}_{00}(\gamma_m)},
\end{aligned}$$

where the last equality descends from the observation that for $u \in H^1(\Omega)$ it holds $[u] = 0$, and hence $[\bar{v}_\delta] = [\bar{v}_\delta - u]$.

Thus, by applying lemma 2.1, for $\eta = [\bar{v}_\delta - u]$, we obtain

$$\inf_{v_\delta \in \mathcal{X}_\delta} \|u - v_\delta\|_X^2 \lesssim \|u - \bar{v}_\delta\|_T^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \sum_{\ell=1}^{L} \|u - v_{\delta,\ell}\|_{1,\Omega_\ell}^2.$$

Since the $v_{\delta,\ell}$ are arbitrary, we get the thesis.

□

Corollary 2.2 *Let the family $\mathcal{V}_\delta^\ell$ be given and let M_δ be defined by (27) and $\mathcal{X}_\delta$ by (28). Assume that the following properties hold.*

(i) *(accuracy) Assumptions (A1 – A3) are satisfied for T_m^0 and M_δ^m defined by (24), (26),*

(ii) *(stability) $\bar{M} = \cap_\delta M_\delta$ satisfies assumption (BP).*

Then, for u solution of (10) and u_δ solution of (31) the following error estimate holds:

$$\|u - u_\delta\|_X \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) \left(\sum_\ell \inf_{v_{\delta,\ell} \in \mathcal{V}_\delta^\ell} \|u - v_{\delta,\ell}\|_{1,\Omega_\ell}^2\right)^{1/2} + \inf_{\lambda \in M_\delta} \|\frac{\partial u}{\partial \nu} - \lambda\|_{-1/2,S} \tag{52}$$

Remark 2.4 *We remark that assumption (A.1 – 3) deal* independently *with each subdomain and also with the coupling between the discretization on a subdomain and the multiplier space M_δ^m defined on* each one *of its sides. In the construction of suitable discretization spaces, it will then be sufficient to study the properties of the discretization on one subdomain and the multiplier space induced on one of its sides.*

Clearly, Theorem 2.3 yields only a sub-optimal error estimate, where, due to the constraint, a factor of the order $\log(H/h)$ is lost with respect to the optimal approximation rate.

Nevertheless, if the solution is sufficiently regular, an optimal error estimate can be retrieved, provided that a suitable projector exists, verifying an interpolation property at cross points. More precisely the following theorem holds.

Theorem 2.4 *For some $\bar{s}$, $\bar{t}$, $\bar{t} \geq \bar{s} > 1$ assume that $\mathcal{V}_\delta^\ell \subset H^{\bar{s}}(\Omega_\ell)$ and that an operator $\Pi_{\delta,\ell} : H^{\bar{s}}(\Omega_\ell) \to \mathcal{V}_\delta^\ell$ exists such that*

- *$\forall s, t$, $0 \leq s \leq \bar{s}$, $\bar{s} \leq t \leq \bar{t}$, for all $u \in H^t(\Omega_\ell)$ with $u = 0$ on $\partial\Omega_\ell \cap \partial\Omega$ we have*

$$\|u - \Pi_{\delta,\ell} u\|_{H^s(\Omega_\ell)} \lesssim \delta_\ell^{t-s} \|u\|_{H^t(\Omega_\ell)}, \tag{53}$$

δ_ℓ being the "mesh-size" of the discretization in the subdomain Ω_ℓ;

- *for all $A \in \partial\Omega_\ell$ such that A is a vertex of $\Gamma_{\ell n}$ for some n,*

$$\Pi_{\delta,\ell} u(A) = u(A), \qquad \forall u \in H^{\bar{s}}(\Omega_\ell). \tag{54}$$

Then, for all u, $u|_{\Omega_\ell} \in H^t(\Omega_\ell)$ with $u = 0$ on $\partial\Omega_\ell \cap \partial\Omega$ it holds

$$\inf_{v_\delta \in \mathcal{X}_\delta} \|u - v_\delta\|_X \lesssim \sum_\ell \delta_\ell^{t-1} \|u\|_{H^t(\Omega_\ell)} \tag{55}$$

Proof: Let $v_{\delta,\ell} = \Pi_{\delta,\ell} u$. Setting $\bar{v}_\delta = (v_{\delta,\ell})_{\ell=1,\dots,L}$ and $v_\delta = \bar{v}_\delta - \sum_{m\in I} R_m \pi_m([\bar{v}_\delta])$, as in Theorem 2.3 we have

$$\inf_{v_\delta \in \mathcal{X}_\delta} \|u - v_\delta\|_X \lesssim \|u - v_\delta\|_X \lesssim \sum_\ell \|u - \Pi_{\delta,\ell} u\|_{H^1(\Omega_\ell)} + \sum_{m\in I} \|\pi_m([\bar{v}_\delta - u])\|_{H^{1/2}_{00}(\gamma_m)}.$$

Now we observe that thanks to (54), $[v_\delta - u]$ belongs to $H^{1/2}_{00}(\gamma_m)$, hence we can write

$$\sum_{m\in I} \|\pi_m([\bar{v}_\delta - u])\|_{H^{1/2}_{00}(\gamma_m)} \lesssim \sum_{m\in I} \|[\bar{v}_\delta - u]\|_{H^{1/2}_{00}(\gamma_m)} \lesssim \sum_{\ell,n} \|\Pi_{\delta,\ell} u - u\|_{H^{1/2}_{00}(\Gamma_{\ell n})}$$

The conclusion follows by observing that

$$\|\Pi_{\delta,\ell} u - u\|_{H^{1/2}_{00}(\Gamma_{\ell n})} \lesssim \delta_\ell^{t-1} \|u\|_{H^t(\Omega_\ell)}.$$

This last bound can be proven by space interpolation. On one hand, we have

$$\|\Pi_{\delta,\ell} u - u\|_{H_0^{\bar{s}-1/2}(\Gamma_{\ell n})} \lesssim \|\Pi_{\delta,\ell} u - u\|_{H^{\bar{s}-1/2}(\partial\Omega_\ell \setminus \partial\Omega)} \lesssim \|\Pi_{\delta,\ell} u - u\|_{H^{\bar{s}}(\Omega_\ell)} \lesssim \delta_\ell^{t-\bar{s}} \|u\|_{H^t(\Omega_\ell)}.$$

On the other hand

$$\|\Pi_{\delta,\ell} u - u\|_{H_0^{1/4}(\Gamma_{\ell n})} \lesssim \|\Pi_{\delta,\ell} u - u\|_{H^{3/4}(\Omega_\ell)} \lesssim \delta_\ell^{t-3/4} \|u\|_{H^t(\Omega_\ell)}.$$

Since $H^{1/2}_{00}(\Gamma_{\ell n})$ is the interpolated of order $\theta = 1/[4(\bar{s}-1)+1]$ between $H^{1/4}(\Gamma_{\ell n})$ and $H_0^{\bar{s}-1/2}(\Gamma_{\ell n})$, we obtain that

$$\|\Pi_{\delta,\ell} u - u\|_{H^{1/2}_{00}(\Gamma_{\ell n})}\| \lesssim \delta_\ell^{(1-\theta)(t-3/4)} \delta_\ell^{\theta(t-\bar{s})} \|u\|_{H^t(\Omega_\ell)} = \delta_\ell^{t-1} \|u\|_{H^t(\Omega_\ell)}.$$

□

Corollary 2.3 *Under the assumptions of Theorem 2.4, if for all $\ell = 1,\dots,L$ the solution u of (10) satisfies $u|_{\Omega_\ell} \in H^t(\Omega_\ell)$ for some t (possibly depending on ℓ), $\bar{s} \le t \le \bar{t}$ then*

$$\|u - u_\delta\|_X \lesssim \inf_{\lambda \in M_\delta} \|\frac{\partial u}{\partial \nu} - \lambda\|_{-1/2,S} + \sum_{\ell=1}^{L} \delta_\ell^{t-1} \|u\|_{H^t(\Omega_\ell)} \tag{56}$$

Remark 2.5 *If we consider the Lagrange multiplier formulation of problem (31), the previous Corollary would imply that a discrete* inf-sup *condition of the form*

$$\inf_{\lambda_\delta \in M_\delta} \sup_{u_\delta \in X_\delta} \frac{\int_S [u_\delta] \lambda_\delta}{\|\lambda_\delta\|_{-1/2,S} \|u_\delta\|_X} \geq \alpha(H/h) > 0$$

is fulfilled with a stability constant $\alpha = \alpha(H/h)$*, decreasing as* $(\log(H/h))^{-1}$*. In fact, it is easy to see that Fortin's lemma ([23]) can be applied by setting the Fortin projector equals to* $(Id - \pi)$*. An alternative to such an approach is to substitute* $\|\cdot\|_{-1/2,S}$ *with a suitable* mesh dependent *norm. Then, in the finite element framework, a discrete* inf-sup *condition can be proven to hold uniformly in h ([21, 20]). However, working with mesh dependent norms usually yields (when, as in the present framework it is not possible to work "triangle by triangle"), a dependence of the constant in the estimates obtained on the ratio between the coarser and the finer "mesh sizes" of each subdomain. This is due to the concurrent use of direct and inverse inequalities. In the case of a very non uniform discretization, as one would have in an adaptive wavelet scheme, such a ratio can be much larger than the logarithmic factor in the bound (49) and in the resulting error estimate. Therefore, though for the sake of simplicity we will concentrate, later on, on a uniform wavelet discretization, we chose here to use the* natural norm $\|\cdot\|_{-1/2,S}$*. We remark that in the geometrically conforming case, if the solution is sufficiently regular, we sill get an optimal error estimate (see Theorem 2.4).*

2.6 Substructuring Preconditioners for the Mortar Element Methods

The aim of this section is to review the properties of a class of preconditioners for the linear system arising from the discretization described in the previous section. We will consider the "substructuring" approach first introduced in ([22]) and already studied in the case of the Mortar Finite Element method in [3]. The principle of these preconditioners consists in distinguishing three types of degrees of freedom: *interior* degrees of freedom (corresponding to basis functions vanishing on the skeleton and supported on one sub-domain), *edge* degrees of freedom, and *vertex* degrees of freedom. This corresponds to split the functions $u \in \mathcal{X}_\delta$ as the sum of three suitably defined components: $u = u^0 + u^E + u^V$. When expressed in a basis corresponding to such a splitting, substructuring preconditioners are of the block diagonal form

$$\hat{A} = \begin{pmatrix} \hat{A}^{00} & 0 & 0 \\ 0 & \hat{A}^{EE} & 0 \\ 0 & 0 & \hat{A}^{VV} \end{pmatrix}, \tag{57}$$

where the matrices A^{00} and A^{EE} are in turn block diagonal.

In analyzing such preconditioners we will also try to analyze the dependence of the condition number of the resulting preconditioned matrix with respect to the sizes H_ℓ of the subdomains Ω_ℓ. In order to do that we need to define more precisely the norms $\|\cdot\|_{H^{1/2}(\partial\Omega_\ell)}$ and $\|\cdot\|_{H^1(\Omega_\ell)}$ appearing in the definition of the X-norm and the T-norm. More precisely we set:

$$\begin{aligned}
\|u\|^2_{H^1(\Omega_\ell)} &= H_\ell^{-2}\int_{\Omega_\ell}|u|^2+\int_{\Omega_\ell}|\nabla u|^2,\\
|u|^2_{H^1(\Omega_\ell)} &= \int_{\Omega_\ell}|\nabla u|^2,\\
|\eta|^2_{1/2,\partial\Omega_\ell} &= \int_{\partial\Omega_\ell}\int_{\partial\Omega_\ell}\frac{|\eta(x)-\eta(y)|^2}{|x-y|^2}\,dx\,dy,\\
\|\eta\|^2_{1/2,\partial\Omega_\ell} &= |\eta|^2_{1/2,\partial\Omega_\ell}+H_\ell^{-1}\int_{\partial\Omega_\ell}|\eta|^2.
\end{aligned}$$

Remark that the above norms are defined in such a way that they are scaling invariant, that is they are preserved when Ω_ℓ is rescaled to the reference domain $]0,1[^2$. In order to construct the preconditioner, we assume that we can construct, for each subdomain Ω_ℓ

- an (easily computable) local discrete lifting operator $R_\delta^\ell : T_\delta^\ell \to \mathcal{V}_\delta^\ell$ such that for all $\eta \in T_\delta^\ell$ it holds

$$R_\delta^\ell\eta|_{\partial\Omega_\ell}=\eta \qquad \text{and} \qquad \begin{array}{lcl}\|R_\delta^\ell\eta\|_{H^1(\Omega_\ell)} & \lesssim & \|\eta\|_{1/2,\partial\Omega_\ell},\\ |R_\delta^\ell\eta|_{H^1(\Omega_\ell)} & \lesssim & |\eta|_{1/2,\partial\Omega_\ell};\end{array} \tag{58}$$

- a bilinear form $\tilde a_\ell^0 : \mathcal{V}_\delta^{\ell,0}\times\mathcal{V}_\delta^{\ell,0}\to\mathbb{R}$ such that for all $u,\ v\in\mathcal{V}_\delta^{\ell,0}$

$$\tilde a_\ell^0(u,v) \lesssim \|u\|_{H^1(\Omega_\ell)}\|v\|_{H^1(\Omega_\ell)}, \tag{59}$$

$$\tilde a_\ell^0(u,v) \gtrsim |u|^2_{H^1(\Omega_\ell)}. \tag{60}$$

We can then easily assemble a global lifting operator $R_\delta : T_\delta \to X_\delta$; more precisely for $\eta=(\eta_\ell)_{\ell=1,\cdots,L}$ we set

$$R_\delta(\eta)=(R_\delta^\ell\eta_\ell)_{\ell=1,\cdots,L}.$$

Remark 2.6 *Trivially, if η satisfies the jump constrain, so does $R_\delta(\eta)$. In other words, if $\eta\in\mathcal{T}_\delta$ then $R_\delta(\eta)\in\mathcal{X}_\delta$.*

In view of (71) and using the classical trace Theorem [46], it is easy to see that for all $\eta\in T_\delta$ it holds:

$$\|R_\delta(\eta)\|_X \simeq \|\eta\|_T. \tag{61}$$

Moreover, given any discrete function $w = (w_\ell)_{\ell=1,\cdots,L} \in X_\delta$ we can split it in a unique way as the sum of an *interior* function $w^0 \in \mathcal{X}_\delta^0$ and the discrete lifting of its trace $\eta(w) = (w_h^\ell|_{\Omega_\ell})_{\ell=1,\cdots,L}$ which by abuse of notation we will denote by $R_\delta(w)$ (rather than using the heavier notation $R_\delta(\eta(w_h))$):

$$w = w^0 + R_\delta(w), \qquad w^0 \in \mathcal{X}_\delta^0. \tag{62}$$

That is we can split respectively the spaces X_δ of unconstrained functions and $\mathcal{X}_\delta$ of constrained functions as direct sums of an interior and of a (respectively unconstrained or constrained) trace component:

$$X_\delta = \mathcal{X}_\delta^0 \oplus R_\delta(T_\delta), \tag{63}$$

$$\mathcal{X}_\delta = \mathcal{X}_\delta^0 \oplus R_\delta(\mathcal{T}_\delta). \tag{64}$$

We can then define a global bilinear operator $\tilde{a}_X : X_\delta \times X_\delta \to \mathbb{R}$ as follows:

$$\tilde{a}_X(v, w) = \tilde{a}_X^0(v^0, w^0) + \tilde{s}(\eta(v), \eta(w)), \tag{65}$$

where, for $v^0 = (v_\ell^0)_{\ell=1,\cdots,L}$ and $w^0 = (w_\ell^0)_{\ell=1,\cdots,L} \in \mathcal{X}_\delta^0$, and $\eta = (\eta_\ell)_{\ell=1,\cdots,L}$ and $\xi = (\xi_\ell)_{\ell=1,\cdots,L} \in T_\delta$ we set

$$\tilde{a}_X^0(v^0, w^0) = \sum_{\ell=1,\cdots,L} \tilde{a}_\ell^0(v_\ell^0, w_\ell^0), \tag{66}$$

$$\tilde{s}(\xi, \eta) = \sum_{\ell=1,\cdots,L} \int_{\Omega_\ell} \nabla R_\delta^\ell(\xi_\ell) \nabla R_\delta^\ell(\eta_\ell). \tag{67}$$

Remark 2.7 *One possible choice for R_δ^ℓ, though not necessarily the most efficient one, is the discrete harmonic extension:*

$$\int_{\Omega_\ell} \nabla R_\delta^\ell(\eta_\delta) \nabla v_\delta = 0, \quad \forall v_\delta \in \mathcal{V}_\delta^{\ell,0}, \qquad R_\delta^\ell(\eta_\delta) = \eta_\delta \quad \text{on } \partial\Omega_\ell.$$

In this case, $\tilde{s}$ is the bilinear form corresponding to the discrete Steklov-Poincaré operator*. We will see in the following sections that in the wavelet case another lifting exists, which is more easily computable. The corresonding bilinear form $\tilde{s}$ is still a first order operator, with the same spectral behavior of the discrete Steklov-Poincaré operator.*

Lemma 2.2 *The bilinear forms $\tilde{a}_X$ and a_X are spectrally equivalent on $\mathcal{X}_\delta$.*

Proof: We have

$$a_X(w_h, w_h) =\lesssim |w_h^0|_X^2 + |R_\delta(w)|_X^2 \lesssim \tilde{a}_X(w_h^0, w_h^0) + |R_\delta(w)|_X^2 = \tilde{a}_X(w_h, w_h),$$

and, since for $w_h^0 \in \mathcal{X}_\delta^0$ it holds $\|w_h^0\|_X \lesssim |w_h^0|_X$,

$$\tilde{a}_X(w_h, w_h) \lesssim |w_h^0|_X^2 + |R_\delta(w)|_X^2 \lesssim |w_h|_X^2 = a_X(w_h, w_h)$$

where we used the inequalities

$$|R_\delta(w)|_X \lesssim |w|_T \lesssim |w|_X \qquad \text{and} \qquad |w_h^0|_X \lesssim |w|_X$$

the second inequality descending from the first one which is a consequence of the assumptions on R_δ. □

The problem of preconditioning the matrix A corresponding to the discretization of a_X, reduces to finding good preconditioners for the matrices $\tilde{A}_0$ and $\tilde{S}$ corresponding respectively to the bilinear forms $\tilde{a}_X : \mathcal{X}_\delta^0 \times \mathcal{X}_\delta^0 \longrightarrow \mathbb{R}$ and $s : T_\delta \times T_\delta \longrightarrow \mathbb{R}$.

We now assume - and we will later check that in the wavelet case such an assumption holds - that the matrix $\tilde{A}_0$ is easily invertible. In other words we concentrate on the preconditioning of the discrete Steklov-Poincaré type operator $\tilde{S}$.

In order to construct a preconditioner for $\tilde{S}$, we start by observing that the space of constrained skeleton functions $\mathcal{T}_\delta$ can be further split as the sum of *vertex* and *edge* functions.

If we denote by $\mathfrak{L} \subset \prod_{\ell=1}^L H_*^{1/2}(\partial\Omega_\ell)$ the space

$$\mathfrak{L} = \{(\eta_\ell)_{\ell=1,\cdots,L},\ \eta_\ell \text{ is linear on each edge of } \Omega_\ell\}, \tag{68}$$

then the space of constrained *vertex* functions can be defined by

$$\mathcal{T}_\delta^V = (Id - \pi)\mathfrak{L}. \tag{69}$$

In the following we will make the (not restrictive) assumption $\mathfrak{L} \subset T_\delta$, which yields $\mathcal{T}_\delta^V \subset \mathcal{T}_\delta$.

At the same time we introduce the space of constrained *edge* functions $\mathcal{T}_\delta^E \subset \mathcal{T}_\delta$ defined by

$$\mathcal{T}_\delta^E = \{\eta = (\eta_\ell)_{\ell=1,\cdots,L} \in \mathcal{T}_\delta,\ \eta_\ell(A) = 0,\ \forall \text{ vertex } A \text{ of } \Omega_\ell\}. \tag{70}$$

It is quite easy to see that

$$\mathcal{T}_\delta = \mathcal{T}_\delta^V \oplus \mathcal{T}_\delta^E. \tag{71}$$

Moreover it is quite simple to check that a function in $\mathcal{T}_\delta^E$ is uniquely defined by its value on trace edges, the value on multiplier edges being forced by the constraint. For any trace side $\gamma_\ell^{(i)}$, $m = (\ell, i) \in I^*$, let now $b_{\ell,i} : T_{\ell,i}^0 \times T_{\ell,i}^0 \longrightarrow \mathbb{R}$ be a symmetric bilinear form satisfying for all $\eta \in T_{\ell,i}^0$:

$$b_{\ell,i}(\eta,\eta) \simeq \|\eta\|_{H_{00}^{1/2}(\gamma_\ell^{(i)})}. \tag{72}$$

We define a block diagonal global bilinear form $b : \mathcal{T}_\delta^E \times \mathcal{T}_\delta^E \longrightarrow \mathbb{R}$ by

$$b(\eta,\xi) = \sum_{(\ell,i)\in I^*} b_{\ell,i}(\eta_\ell, \xi_\ell), \tag{73}$$

and we let $\hat{s} : \mathcal{T}_\delta \times \mathcal{T}_\delta \longrightarrow \mathbb{R}$ be defined by

$$\hat{s}(\eta,\xi) = \tilde{s}(\eta^V, \xi^V) + b(\eta^E, \xi^E). \tag{74}$$

The following theorem holds:

Theorem 2.5 *Let $\eta \in \mathcal{T}_\delta$. We have:*

$$\left(1+\log\left(\frac{H}{h}\right)\right)^{-2} \tilde{s}(\eta,\eta) \lesssim \hat{s}(\eta,\eta) \lesssim \left(1+\log\left(\frac{H}{h}\right)\right)^{2} \tilde{s}(\eta,\eta). \tag{75}$$

The proof of theorem 2.5 follows essentially the guidelines of the proofs of the analogous results in [22, 3]. However, for the sake of completeness we will retrace in the next section the main steps in the framework of the abstract formulation here considered.

Corollary 2.4 *Let A and $\hat{A}$ be the matrices obtained by discretizing respectively the bilinear forms a and $\hat{a}$ with $\hat{a}(u,v) = \tilde{a}_X(u_0,v_0) + \hat{s}(\eta_u,\eta_v)$. Then it holds*

$$Cond(\hat{A}^{-1}A) \lesssim \left(1+\log\left(\frac{H}{h}\right)\right)^{4}. \tag{76}$$

Remark 2.8 *An alternative to the preconditioner presented here is take into account also the contribution of multiplier sides in the definition of b, that is to substitute*

$$\check{b}(\eta,\xi) = \sum_{(\ell,i)\in I\cup I^*} b_{\ell,i}(\eta_\ell,\xi_\ell),$$

to

$$b(\eta,\xi) = \sum_{(\ell,i)\in I^*} b_{\ell,i}(\eta_\ell,\xi_\ell)$$

in the definition of $\hat{s}$. From the properties of the bilinear forms $b_{\ell,i}$ it easily descends that in such case we would have

$$\tilde{s}(\eta,\eta) \lesssim \hat{s}(\eta,\eta) \lesssim \left(1+\log\left(\frac{H}{h}\right)\right)^{2} \tilde{s}(\eta,\eta),$$

whence, letting $\check{A}$ being the matrix discretizing the bilinear form thus obtained, we would have

$$Cond(\check{A}^{-1}A) \lesssim \left(1+\log\left(\frac{H}{h}\right)\right)^{2}.$$

Remark 2.9 *Following [3] a better condition number (increasing like $\left(1+\log\left(\frac{H}{h}\right)\right)^2$) can also be obtained in the case of a geometrically conforming decomposition.*

2.7 Proof of Theorem (2.5)

We start by remarking that, thanks to (61) and properties (i) and (ii) of the previous section we have for all $\xi \in T_\delta$

$$\tilde{s}(\xi,\xi) \lesssim \|\xi\|_T^2, \tag{77}$$

while for all constrained functions $\xi \in \mathcal{T}_\delta$ it holds

$$\tilde{s}(\xi,\xi) \simeq \|\xi\|_T^2. \tag{78}$$

Lemma 2.3 *For all $\eta = (\eta_\ell)_{\ell=1,\cdots,L} \in \mathcal{T}_\delta^E$ we have*

$$\|\eta\|_T^2 \lesssim \left(1+\log\left(\frac{H}{h}\right)\right)^2 \sum_{(\ell,i)\in I^*} \|\eta_\ell\|^2_{H^{1/2}_{00}(\gamma_\ell^{(i)})}. \tag{79}$$

Remark 2.10 *(79) does not descend directly from (48), since here we take the broken $H^{1/2}_{00}$ norm on the right hand side with only the contribution of trace sides.*

Proof: Let $\check{\eta} = (\check{\eta}_\ell)_{\ell=1,\cdots,L}$ be the function coinciding with η on master sides and vanishing on slave sides

$$\begin{aligned}\check{\eta}_\ell &= \eta_\ell \text{ on } \gamma_\ell^{(i)}, \quad (\ell,i)\in I^*,\\ \check{\eta}_\ell &= 0 \text{ on } \gamma_\ell^{(i)}, \quad (\ell,i)\in I.\end{aligned}$$

It is easy to check that we have $\eta = \check{\eta} - \pi(\check{\eta})$. By applying (45) we conclude that

$$\|\eta\|_T^2 \lesssim \left(1+\log\left(\frac{H}{h}\right)\right)^2 \|\check{\eta}\|_T^2 \lesssim \left(1+\log\left(\frac{H}{h}\right)\right)^2 \sum_{(\ell,i)\in I^*} \|\eta_\ell\|^2_{H^{1/2}_{00}(\gamma_\ell^{(i)})}.$$

□

Moreover we have

Lemma 2.4 *Let $\eta \in \mathfrak{L}$, then we have*

$$\|\pi(\eta)\|_T \lesssim \left(1+\log\left(\frac{H}{h}\right)\right)^{1/2} \|\eta\|_T. \tag{80}$$

The proof of (80) is essentially identical to the proof of Lemma 45, once we observed that property (46) can be enhanced as follows: let $U = \cup U_n$ and let η a function (not necessarily continuous) η *linear on each subinterval U_n*. Then by direct computation it is not difficult to check that:

$$\|\eta\|^2_{H^{1/2-\varepsilon}(U)} \lesssim \sum_n \|u\|^2_{H_0^{1/2-\varepsilon}(U_n)} \lesssim \frac{|U|^{2\varepsilon}}{\varepsilon}\|u\|^2_{H^{1/2}(U)}, \tag{81}$$

In order to prove theorem 2.5 we will need the following two results, whose proofs are already implicitly contained in ([22], Lemmas 3.2 and 3.4).

Lemma 2.5 *Let $\eta \in T_\delta^\ell$ and let $\check{\eta} \in \mathfrak{L}$ be the function linear on each edge of Ω_ℓ and satisfying $\check{\eta}(A) = \eta(A)$ at all vertex A of Ω_ℓ. Then it holds*

$$\|\check{\eta}^V\|_T^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) \|\eta\|_T^2. \tag{82}$$

Proof: A direct computation using the linearity of $\check{\eta}$ shows

$$\|\check{\eta}^\ell\|^2_{H^{1/2}(\partial\Omega_\ell)} \lesssim \sum_{i=1}^4 (\eta(a_i) - \eta(b_i))^2$$

a_i and b_i being the two vertices of $\gamma_\ell^{(i)}$. Now it is known ([22], Lemma 3.3) that assumption (A4) yields for all $w \in \mathcal{V}_\delta^\ell$

$$\|w\|^2_{L^\infty(\Omega_\ell)} \lesssim (H_\ell^{-2}\|w\|^2_{0,\Omega_\ell} + \log(H_\ell/h_\ell)|w|^2_{H^1(\Omega_\ell)}).$$

The prove of lemma 2.6 is then identical to the proof of ([22], Lemma 3.4). More precisely, for any function $w \in H^1(\Omega_\ell)$, denoting by $< w >$ the average value of w, we have

$$H_\ell^{-2}\|w - \alpha\|^2_{0,\Omega_\ell} \lesssim |w|^2_{H^1(\Omega_\ell)}.$$

Let now $w_\eta \in \mathcal{V}_\delta^\ell$ denote the discrete harmonic extension of η. The function $w_\eta - \eta(b_i)$ vanishes at b_i. This yields

$$|< w_\eta - \eta(b_i) >| \lesssim \|w_\eta - \eta(b_i) - < w_\eta - \eta(b_i) > \|_{L^\infty(\Omega_\ell)}.$$

By triangle inequality

$$\|w_\eta - \eta(b_i)\|^2_{L^\infty(\Omega_\ell)} \lesssim \|w_\eta - \eta(b_i) - < w_\eta - \eta(b_i) > \|^2_{L^\infty(\Omega_\ell)} \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) |w_\eta|_{H^1(\Omega_\ell)}.$$

Since $|w_\eta|_{H^1(\Omega_\ell)} \lesssim \|\eta\|_{1/2,\partial\Omega_\ell}$, and since $(\eta(a_i) - \eta(b_i))^2 \lesssim \|w_\eta - \eta(b_i)\|^2_{L^\infty(\Omega_\ell)}$ we get the thesis. □

Lemma 2.6 *Let assumption (A4) hold, and let $\xi \in T_\delta^\ell$, $\xi(A) = 0$ for all A vertex of Ω_ℓ. Let $\zeta_L \in H^{1/2}(\partial\Omega_\ell)$, ζ_L linear on each edge of Ω_ℓ. Then it holds*

$$\sum_{k=1}^4 \|\xi\|^2_{H^{1/2}_{00}(\gamma_\ell^{(i)})} \lesssim \left(1 + \log\left(\frac{H_\ell}{h_\ell}\right)\right)^2 \|\xi + \zeta_L\|^2_{H^{1/2}(\partial\Omega_\ell)}. \tag{83}$$

Proof: We start by observing that, under assumption (A4), for all $\xi \in T_\delta^\ell$ such that $\xi(P) = 0$ for some $P \in \gamma_\ell^{(i)}$ it holds (see [22], p. 115, Lemma 3.3 and 3.4):

$$\|\xi\|^2_{L^\infty(\gamma_\ell^{(i)})} \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) |\xi|^2_{1/2,\gamma_\ell^{(i)}}. \tag{84}$$

With the same argument to the one used in the proof of Lemma 2.1, since $\xi \in H^{1/2}_{00}(\gamma_\ell^{(i)})$, $i = 1, \dots, 4$ it is also easy to see that it holds

$$\sum_{k=1}^4 \|\xi\|^2_{H^{1/2}_{00}(\gamma_\ell^{(i)})} \lesssim \left(1 + \log\left(\frac{H_\ell}{h_\ell}\right)\right)^2 \|\xi\|^2_{H^{1/2}(\partial\Omega_\ell)}.$$

In order to prove (83), let $\zeta_0 \in T_\delta^\ell$ be the unique element of T_δ^ℓ satisfying $\zeta_0(A) = 0$ for all A vertex of Ω_ℓ and

$$(\zeta_0, \tau)_{1/2,\partial\Omega_\ell} = (\zeta_L, \tau)_{1/2,\partial\Omega_\ell}$$

for all $\tau \in T_\delta^\ell$ with $\tau(A) = 0$ for all A vertex of Ω_ℓ, where

$$(\zeta, \xi)_{1/2,\partial\Omega_\ell} = \int_{\partial\Omega_\ell} \int_{\partial\Omega_\ell} \frac{(\zeta(x) - \zeta(y))(\xi(x) - \xi(y))}{|x - y|^2} \, dx \, dy$$

is the bilinear form inducing on $H^{1/2}(\partial\Omega_\ell)$ the seminorm $|\cdot|_{1/2,\partial\Omega_\ell}$. It is not difficult to see that $|\cdot|_{1/2,\partial\Omega_\ell}$ is a norm on the subspace of functions in T_δ^ℓ vanishing at the vertices of Ω_ℓ and then, by standard arguments we get that ζ_0 is well defined and

$$|\zeta_0|_{1/2,\partial\Omega_\ell} \lesssim |\zeta_L|_{1/2,\partial\Omega_\ell}$$

Now we can write:

$$\sum_{i=1}^4 \|\xi\|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})} \lesssim \sum_{i=1}^4 \|\xi + \zeta_0\|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})} + \sum_{i=1}^4 \|\zeta_0\|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})} \tag{85}$$

The first sum on the right hand side of (85) can be bound by using the previous lemma as

$$\sum_{i=1}^4 \|\xi + \zeta_0\|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})} \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 |\xi + \zeta_0|^2_{1/2,\partial\Omega_\ell} \tag{86}$$

$$\lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 |\xi + \zeta_L|^2_{1/2,\partial\Omega_\ell} \tag{87}$$

where on one hand we used Poincaré inequality to bound the $H^{1/2}$ norm of $\xi + \zeta_0$ (which vanishes at the vertices of Ω_ℓ) by the corresponding seminorm, while the last inequality derives by observing that, by the definition of ζ_0 we have that $\xi + \zeta_0 \in T_\delta^\ell$ is null at the vertices of Ω_ℓ and it verifies $(\zeta_L - \zeta_0, \xi + \zeta_0)_{1/2,\partial\Omega_\ell} = 0$, hence we have

$$|\xi + \zeta_L|^2_{1/2,\partial\Omega_\ell} = |\xi + \zeta_0|^2_{1/2,\partial\Omega_\ell} + |\zeta_L - \zeta_0|^2_{1/2,\partial\Omega_\ell} \geq |\xi + \zeta_0|^2_{1/2,\partial\Omega_\ell}.$$

Let us now bound the second sum on the right hand side of (85): we first observe that, denoting by A_i and B_i the two vertices of the edge $\gamma_\ell^{(i)}$, we have

$$\|\zeta_0\|^2_{H^{1/2}(\partial\Omega_\ell)} = |\zeta_0|^2_{H^{1/2}(\gamma_\ell^{(i)})} + I_1(\zeta_0) + I_2(\zeta_0)$$

with

$$I_1(\zeta_0) = \int_{A_i}^{B_i} \frac{|\zeta_0(x)|^2}{|x - A_i|} \, dx, \qquad I_2(\zeta_0) = \int_{A_i}^{B_i} \frac{|\zeta_0(x)|^2}{|x - B_i|} \, dx.$$

Now we can write

$$\sum_{i=1}^4 |\zeta_0|^2_{H^{1/2}(\gamma_\ell^{(i)})} \lesssim |\zeta_0|^2_{H^{1/2}(\partial\Omega_\ell)} \lesssim |\zeta_L|^2_{H^{1/2}(\partial\Omega_\ell)} \lesssim \sum_{i=1}^4 (\zeta_L(A_i) - \zeta_L(B_i))^2 \lesssim \left(1 + \log\frac{H}{h}\right) \; |\xi + \zeta_L|^2_{H^{1/2}(\partial\Omega_\ell)},$$

where the inequality $|\zeta_L|^2_{H^{1/2}(\partial\Omega_\ell)} \lesssim \sum_{i=1}^4 (\zeta_L(A_i) - \zeta_L(B_i))^2$ is proven in [22] by direct computation, and the last inequality descends by applying the bound (84) to the function $(\xi + \zeta_L)(x) - (\xi + \zeta_L)(B_i)$. Let us now bound I_1. I_2 can be bound by the same argument. For notational simplicity let us identify $A_i = 0$ and $B_i = H$. We have

$$I_1(\zeta_0) = \int_0^H \frac{|\zeta_0|^2}{|x|} \, dx = \int_0^H \frac{|\zeta_0(x) + \zeta_L(x) - \zeta_L(x) + \zeta_L(0) - \zeta_L(0)|^2}{|x|} \, dx$$

$$\lesssim \int_0^H \frac{|\zeta_0(x) - \zeta_L(x) + \zeta_L(0)|^2}{|x|} \, dx + \int_0^H \frac{|\zeta_L(x) - \zeta_L(0)|^2}{|x|} \, dx.$$

Let us bound the first integral. Let us set $\zeta_\perp = \zeta_0 - \zeta_L$. We have

$$\int_0^H \frac{|\zeta_\perp(x) - \zeta_\perp(0)|^2}{|x|}\,dx = \int_0^h \frac{|\zeta_\perp(x) - \zeta_\perp(0)|^2}{|x|}\,dx + \int_h^H \frac{|\zeta_\perp(x) - \zeta_\perp(0)|^2}{|x|}\,dx.$$

The first term is bound by

$$\begin{aligned}\int_0^h \frac{|\zeta_\perp(x) - \zeta_\perp(0)|^2}{|x|}\,dx &= \int_0^h \frac{|\int_0^x (\zeta_\perp)_x(\tau)\,d\tau|^2}{|x|}\,dx \lesssim \int_0^h \frac{\int_0^x |(\zeta_\perp)_x(\tau)|^2\,d\tau}{}dx \\ &\lesssim h|\zeta_\perp|^2_{H^1(\gamma_\ell^{(i)})} \lesssim |\zeta_\perp|^2_{H^{1/2}(\gamma_\ell^{(i)})},\end{aligned}$$

while we bound the second by

$$\int_h^H \frac{|\zeta_\perp(x) - \zeta_\perp(0)|^2}{|x|}\,dx \lesssim \|\zeta_\perp - \zeta_\perp(0)\|^2_{L^\infty(\gamma_\ell^{(i)})} \log\frac{H}{h} \lesssim \left(\log\frac{H}{h}\right)^2 |\zeta_\perp|^2_{H^{1/2}(\gamma_\ell^{(i)})}.$$

On the other hand, by direct calculation, using the linearity of ζ_L we have

$$\int_0^H \frac{|\zeta_L(x) - \zeta_L(0)|^2}{|x|}\,dx \lesssim (\zeta_L(b_i) - \zeta_L(a_i))^2 \lesssim \log\frac{H}{h}|\zeta_L + \xi|^2_{H^{1/2}(\gamma_\ell^{(i)})}.$$

hence we conclude that

$$I_1(\zeta_0) \lesssim \left(1 + \left(\log\frac{H}{h}\right)^2\right)|\zeta_0 - \zeta_L|^2_{H^{1/2}(\gamma_\ell^{(i)})} + \log\frac{H}{h}|\zeta_L + \xi|^2_{H^{1/2}(\gamma_\ell^{(i)})}.$$

□

We are now able to prove Theorem 2.5. Let $\eta \in \mathcal{T}_\delta$, $\eta = \eta^V + \eta^E$. By applying (77),(72) and (79) we get

$$\begin{aligned}\tilde{s}(\eta,\eta) \lesssim \|\eta^E\|_T^2 + \tilde{s}(\eta^V,\eta^V) &\lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 b(\eta^E,\eta^E) + \tilde{s}(\eta^V,\eta^V) \\ &\lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \hat{s}(\eta,\eta).\end{aligned}$$

On the other hand we have

$$\hat{s}(\eta,\eta) = \tilde{s}(\eta^V,\eta^V) + b(\eta^E,\eta^E). \tag{88}$$

In order to bound the first term of the sum on the righthandside of (88) we let $\check{\eta}^V = (\check{\eta}^V_\ell)_{\ell=1,\cdots,L} \in \mathfrak{L}$ be unique element satisfying

$$\check{\eta}^V_\ell(A) = \eta_\ell(A) \text{ for all A vertex of } \Omega_\ell.$$

We easily check that $\eta^V = \check{\eta}^V - \pi(\check{\eta}^V)$. Then we can apply lemma 2.6 with $\xi = \eta - \check{\eta}^V$ and $\zeta_L = \check{\eta}^V$: thanks to lemma 2.5 it holds

$$\|\eta^V\|_T^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)\|\check{\eta}^V\|_T^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \|\eta\|_T^2. \tag{89}$$

In view of (78), the bound (89) implies

$$\tilde{s}(\eta^V, \eta^V) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \tilde{s}(\eta, \eta). \tag{90}$$

Let us now consider the second term in the sum at the right hand side of (88). We can write

$$b(\eta^E, \eta^E) \lesssim \sum_{m \in I^*} \|\eta^E\|^2_{H^{1/2}_{00}(\gamma_m)} \lesssim \sum_{m \in I^*} \|\eta^E\|^2_{H^{1/2}_{00}(\gamma_m)} + \sum_{m \in I} \|\eta - \check{\eta}\|^2_{H^{1/2}_{00}(\gamma_m)}$$

We now observe that on "trace sides" ($m \in I^*$) we have $\eta^E = \eta - \check{\eta}$. We can then apply lemma 2.6 again and we get

$$b(\eta^E, \eta^E) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \tilde{s}(\eta, \eta). \tag{91}$$

3 The Mortar Wavelet Method

In this section we construct mortar approximation spaces in a general wavelet context. We focus here on one (rectangular) subdomain Ω_ℓ which for simplicity we assume to be the unit square. The approximation spaces on $]0,1[^2$ will be obtained from tensor-product of one-dimensional spaces. Starting from a (now classical) multiresolution analysis on the interval [31, 7, 29], we construct a wavelet family (adapted to the "multiplier" sides) which will allow to define a suitable multiplier space verifying by construction the basic assumptions (A.1-3) needed to apply the abstract result of the previous section. This will allow us to derive the error estimates.

3.1 Multiresolution Analyses on the interval and approximation properties

3.1.1 Scaling functions on the Interval

The construction of multiresolution analyses and associated wavelet bases on the interval, which preserve the approximation properties of the wavelet bases on $\mathbb{R}$ has nowadays already been discussed in a number of papers (*see* [31, 7] for the first constructions, but also [48, 39, 28, 50, 52]). To be as general as possible, we will consider the case of biorthogonal wavelet bases, that includes the orthonormal bases. Let us point out which are the properties of such bases, which will be needed for the design and the analysis of the *Mortar Wavelet Method* and which we will assume to be verified by the chosen basis. We recall that a wide class of bases exists, which satisfy by construction such assumptions.

We assume that we are given a couple of biorthogonal Multi-Resolution Analyses (MRA) of $L^2(0,1)$, that is a couple of increasing sequences of finite dimensional subspaces $(V_j = V_j(]0,1[))_{j\geq j_0}$ and $\left(\tilde{V}_j = \tilde{V}_j(]0,1[)\right)_{j\geq j_0}$, whose respective union is dense in $L^2(0,1)$. Without loss of generality, we can assume that $dim V_j = dim \tilde{V}_j = 2^j + 2$ (*see* [7, 50, 33] for example).

V_j and $\tilde{V}_j$ are respectively spanned by *scaling function* Riesz bases $(\varphi_{j,k})_{k=0,2^j+1}$ and $(\tilde{\varphi}_{j,k})_{k=0,2^j+1}$ which are biorthogonal, i.e. they verify:

$$\int_0^1 \varphi_{j,k}\tilde{\varphi}_{j,k} \;=\; \delta_{kk'}, \;\; \forall k,k' = 0,\ldots,2^j+1.$$

The scaling function bases are usually constructed as a modification of the corresponding (compactly supported) scaling functions in $L^2(\mathbb{R})$ (*see* [43]). In this process, one naturally distinguishes between *edge* (*left* and *right*) functions and *interior functions*: interior functions coincide with scaling functions on the line whose support is included into $]0,1[$ while edge function are linear combinations of scaling functions on the line (restricted to $]0,1[$), whose support overlap the left (resp. right) edge. Consequently, for $j \geq j_0$, the scaling basis of V_j is usually indexed as follows, with N and $\tilde{N}$ two given integers:

- $\varphi_{j,k}^{left}$, $\;k = 0,\ldots,N-1$, the N scaling functions at the left edge,
- $\varphi_{j,k}$, $\;k = N,\ldots,2^j-N+1$, the interior scaling functions,
- $\varphi_{j,k}^{right}$, $\;k = 2^j-N+2,\ldots,2^j+1$, the N scaling functions at the right edge.

Similarly for the scaling basis of $\tilde{V}_j$:

- $\tilde{\varphi}_{j,k}^{left}$, $\;k = 0,\ldots,\tilde{N}-1$, the $\tilde{N}$ scaling functions at the left edge,
- $\tilde{\varphi}_{j,k}$, $\;k = \tilde{N},\ldots,2^j-\tilde{N}+1$, the interior scaling functions,
- $\tilde{\varphi}_{j,k}^{right}$, $\;k = 2^j-\tilde{N}+2,\ldots,2^j+1$, the $\tilde{N}$ scaling functions at the right edge.

For notational simplicity, we will omit in the following the suffixes "$left$" and "$right$". We will then note $(\varphi_{j,k})_{k=0,2^j+1}$ and $(\tilde{\varphi}_{j,k})_{k=0,2^j+1}$ the above bases. We recall that these functions have compact support and are scale invariant, *i.e.* $\forall j \geq j_0$, $\forall x \in [0,1]$, and $forall k = 0,\ldots,N-1$ (resp. $forall k = 0,\ldots,\tilde{N}-1$) on the left boundary it holds

$$\varphi_{j,k}(x) = 2^{\frac{j-j_0}{2}}\varphi_{j_0,k}(2^{j-j_0}x), \qquad \tilde{\varphi}_{j,k}(x) = 2^{\frac{j-j_0}{2}}\tilde{\varphi}_{j_0,k}(2^{j-j_0}x),$$

while on the right boundary we have

$$\begin{aligned}
\varphi_{j,2^j+1-k}(x) &= 2^{\frac{j-j_0}{2}} \varphi_{j_0,2_0^j+1-k}(1-2^{j-j_0}(1-x)), \\
\tilde{\varphi}_{j,2^j+1-k}(x) &= 2^{\frac{j-j_0}{2}} \varphi_{j_0,2_0^j+1-k}(1-2^{j-j_0}(1-x)).
\end{aligned}$$

Moreover, the *interior* scaling functions coincide with the original scaling functions on the real line and $\forall j \geq j_0$, $\forall k = N, \ldots, 2^j - k + N$, they take the form:

$$\varphi_{j,k}(x) = \varphi_{j,N}(x - 2^{-j}(k-N)),$$

(and similarly for the duals $\tilde{\varphi}_{j,k}$). Finally we can also suppose, that all scaling functions of V_j vanish at the edges 0 and 1, except one function at each edge. For example we will assume that only the functions $\varphi_{j,0}$ and $\varphi_{j,2^j+1}$ verify a nonvanishing boundary condition:

$$\begin{array}{ll}
\varphi_{j,0}(0) \neq 0 \quad \text{and} & \varphi_{j,2^j+1}(1) \neq 0 \;, \\
\varphi_{j,k}(0) = 0, & \forall k = 1, \ldots, 2^j + 1, \\
\varphi_{j,k}(1) = 0, & \forall k = 0, \ldots, 2^j.
\end{array} \tag{92}$$

We will assume that the two MRA are respectively R-regular and $\tilde{R}$-regular, that is $\varphi_{j,k} \in H^R(0,1)$ and $\tilde{\varphi}_{j,k} \in H^{\tilde{R}}(0,1)$, with $R \geq 1$, $\tilde{R} > 0$.

Following [31] such scaling functions are constructed in such a way that they satisfy Strang-Fix conditions, that is they allow to reconstruct polynomials up to degree $N-1$ in the space V_j and up to degree $\tilde{N}-1$ in the space $\tilde{V}_j$. More precisely, we will have for all $j \geq j_0$ and for $n = 0, \ldots, N-1$

$$2^{j/2}(2^j x)^n = \sum_{k=0}^{2^j+1} a_k^n \; \varphi_{j,k}(x), \tag{93}$$

and for $n = 0, \ldots, \tilde{N}-1$

$$2^{j/2}(2^j x)^n = \sum_{k=0}^{2^j+1} \tilde{a}_k^n \; \tilde{\varphi}_{j,k}(x), \tag{94}$$

with a_k^n and $\tilde{a}_k^n$ reals independent of j. Remark that the hypothesis (92) implies that $a_0^0 \neq 0$, $a_{2^j+1}^0 \neq 0$ and $a_0^n = 0$ for all $1 \leq n \leq N-1$ in equation (93).

Let now $V_j^0 \subset H_0^1(0,1)$ be the space of functions of V_j vanishing at the boundaries of the interval:

$$V_j^0 = V_j \cap H_0^1(0,1) = span < \varphi_{j,k} \ , \ k = 1, \ldots, 2^j, >,$$

$$V_j = V_j^0 \oplus span < \varphi_{j,0} \ , \ \varphi_{j,2^j+1}. > .$$

When designing the Mortar Wavelet Method, the space V_j^0 will play the role of the space which, in section 2.3, was denoted T_m^0, that is the space of traces on a "multiplier side" of discrete functions, vanishing at the extrema. The corresponding multiplier space (which will play the the role of the M_δ^m of section 2.3) is most naturally defined with the aid of a suitable dual space, which will have to satisfy the assumptions of section 2.5. The construction of such dual space for V_j^0 is the object of the following theorem.

Theorem 3.1 *Let $j \geq j_0$. Let $(\alpha_k)_{k=0,\tilde{N}-1}$ and $(\beta_k)_{k=2^j-\tilde{N}+1,2^j+1}$ be the solutions of the following linear systems*

$$\sum_{k=0}^{\tilde{N}-1} \tilde{a}_k^n \alpha_k = \tilde{a}_{\tilde{N}}^n, \quad and \quad \sum_{k=2^j-\tilde{N}+2}^{2^j+1} \tilde{a}_k^n \beta_k = \tilde{a}_{2^j-\tilde{N}+1}^n, \quad \forall n = 0, \tilde{N}-1. \tag{95}$$

Then the family:

$$\begin{array}{lcll} \tilde{\varphi}_{j,k}^* & = & \tilde{\varphi}_{j,k} - \frac{\alpha_k}{\alpha_0} \ \tilde{\varphi}_{j,0} \ , & for \ k = 1, \tilde{N}-1 \\ \tilde{\varphi}_{j,\tilde{N}}^* & = & \tilde{\varphi}_{j,\tilde{N}} + \frac{1}{\alpha_0} \ \tilde{\varphi}_{j,0} \ , & \\ \tilde{\varphi}_{j,k}^* & = & \tilde{\varphi}_{j,k} \ , & for \ k = \tilde{N}+1, 2^j - \tilde{N} \\ \tilde{\varphi}_{j,2^j-\tilde{N}+1}^* & = & \tilde{\varphi}_{j,2^j-\tilde{N}+1} + \frac{1}{\beta_{2^j+1}} \ \tilde{\varphi}_{j,2^j+1} \ , & \\ \tilde{\varphi}_{j,k}^* & = & \tilde{\varphi}_{j,k} - \frac{\beta_k}{\beta_{2^j+1}} \ \tilde{\varphi}_{j,2^j+1} \ , & for \ k = 2^j - \tilde{N}+2, 2^j \end{array} \tag{96}$$

is biorthogonal to the basis $\{\varphi_{j,k} \ , \ k = 1, \ldots, 2^j\}$ of V_j^0.

Moreover the space $\tilde{V}_j^*$ defined by:

$$\tilde{V}_j^* = span < \tilde{\varphi}_{j,k}^* \ , \ k = 1, \ldots, 2^j > \tag{97}$$

contains all polynomials of degree $\tilde{N}-1$.

Remark that $\tilde{V}_j^*$ does not verify homogeneous boundary conditions.

Proof: The biorthogonality between families $(\tilde{\varphi}^*_{j,k})$ and $(\varphi_{j,k})$ is a trivial consequence of the biorthogonality between $(\tilde{\varphi}_{j,k})$ and $(\varphi_{j,k})$. By construction, the $(\tilde{\varphi}_{j,k})$ reproduce polynomials up to order $\tilde{N}-1$. Indeed, by equation (94) we have, $\forall n = 0, \tilde{N}-1$:

$$
\begin{aligned}
2^{j/2}(2^j x)^n &= \sum_{k=0}^{2^j+1} \tilde{a}^n_k \; \tilde{\varphi}_{j,k}(x) \\
&= \left(\tilde{a}^n_0 + \sum_{k=1}^{\tilde{N}-1} \tilde{a}^n_k \frac{\alpha_k}{\alpha_0} - \frac{\tilde{a}^n_{\tilde{N}}}{\alpha_0} \right) \tilde{\varphi}_{j,0}(x) + \sum_{k=1}^{2^j} \tilde{a}^n_k \; \tilde{\varphi}^*_{j,k}(x) \\
&\quad + \left(\tilde{a}^n_{2^j-\tilde{N}+1} + \sum_{k=2^j-N+2}^{2^j} \tilde{a}^n_k \frac{\beta_k}{\beta_{2^j+1}} - \frac{\tilde{a}^n_{2^j-\tilde{N}+1}}{\beta_{2^j+1}} \right) \tilde{\varphi}_{j,2^j+1}(x)
\end{aligned}
$$

where we applied the definition (96).
The result holds then if $\alpha_0 \neq 0$, $\beta_{2^j+1} \neq 0$, and if the α_k, β_k satisfy the following equations:

$$
\begin{aligned}
\tilde{a}^n_N - \sum_{k=0}^{\tilde{N}-1} \tilde{a}^n_k \alpha_k = 0, \quad \forall n = 0, \tilde{N}-1 \\
\tilde{a}^n_{2^j-\tilde{N}+1} - \sum_{k=2^j-N+2}^{2^j+1} \tilde{a}^n_k \beta_k = 0, \quad \forall n = 0, \tilde{N}-1
\end{aligned}
$$

We will assume here that the matrix $[\tilde{a}^n_k]_{0\leq k,n \leq \tilde{N}-1}$ is invertible and that the coefficients α_0 and β_0 are non vanishing. This can be proven in particular cases, such as for biorthogonal spline wavelets [39, 32] or orthogonal wavelets. In this last example $\varphi_{j,k} = \tilde{\varphi}_{j,k}$ and the coefficients $\tilde{a}^n_k = \int_0^1 x^n \tilde{\varphi}_{j,k}$ can be written as $\tilde{a}^n_k = p_n(k)$, where p_n is a polynomial of degree n [50]. $[\tilde{a}^n_k]_{0\leq k,n\leq \tilde{N}-1}$ is then a nonsingular Vandermonde type matrix. The nonvanishing values of the coefficients α_0 and β_{2^j+1} follows from the same argument, by considering the matrix $[\tilde{a}^n_k]_{0\leq n\leq \tilde{N}-1, 1\leq k\leq \tilde{N}}$. In general we will have to verify case by case that such an assumption holds. Remark that, thanks to the scale invariance property of the scaling functions it will be enough to verify it once and for all for $j = j_0$. $\square$

3.1.2 Projectors on MRA spaces and approximation properties

Let P_j and $\tilde{P}_j$ be the biorthogonal projectors associated to V_j and $\tilde{V}_j$ defined, as usual, for all η and λ in $L^2(0,1)$ by:

$$\begin{aligned} P_j(\eta) &= \sum_{k=0}^{2^j+1} < \eta|\tilde{\varphi}_{j,k} > \varphi_{j,k} \\ \tilde{P}_j(\lambda) &= \sum_{k=0}^{2^j+1} < \lambda|\varphi_{j,k} > \tilde{\varphi}_{j,k} \end{aligned} \tag{98}$$

where $< \cdot|\cdot >$ denotes the $L^2(0,1)$ scalar product.

It is well known that under the assumptions of the previous section, the following Jackson and Bernstein inequalities hold [29]:

Theorem 3.2 *Let $0 \le r \le s \le N$, $r \le R$, then for all $\eta \in H^s(0,1)$, we have:*

$$\|\eta - P_j(\eta)\|_{r,]0,1[} \lesssim 2^{-j(s-r)}\|\eta\|_{s,]0,1[}, \tag{99}$$

while, for $0 \le r \le s \le \tilde{N}$, $r \le \tilde{R}$ and for all $\lambda \in H^s(0,1)$ it holds:

$$\|\lambda - \tilde{P}_j(\lambda)\|_{r,]0,1[} \lesssim 2^{-j(s-r)}\|\lambda\|_{s,]0,1[}. \tag{100}$$

Moreover, a scaling argument yields the following inverse inequalities

Theorem 3.3 *For all r, s, $0 \le r \le s \le R$ one has:*

$$\|\eta\|_{s,]0,1[} \lesssim 2^{j(s-r)}\|\eta\|_{r,]0,1[}, \quad if \;\; \eta \in V_j, \tag{101}$$

and for all r, s, $0 \le r \le s \le \tilde{R}$ one has:

$$\|\lambda\|_{r,]0,1[} \lesssim 2^{jr}\|\lambda\|_{0,]0,1[}, \quad if \;\; \lambda \in \tilde{V}_j. \tag{102}$$

In the following it will also be useful to consider the biorthogonal projector on V_j^0 induced by its dual $\tilde{V}_j^*$

Definition. *Let $\pi_j \; : \; L^2(0,1) \to V_j^0$ be defined, for all $\eta \in L^2(0,1)$ by:*

$$\pi_j(\eta) = \sum_{k=1}^{2^j} < \eta|\tilde{\varphi}^*_{j,k} > \varphi_{j,k} \tag{103}$$

We will also consider the adjoint projection of π_j, $\tilde{\pi}_j : \ L^2(0,1) \to \tilde{V}_j^0$, *defined for all* $\lambda \in L^2(0,1)$ *by:*

$$\tilde{\pi}_j(\lambda) = \sum_{k=1}^{2^j} < \lambda | \varphi_{j,k} > \tilde{\varphi}^*_{j,k} \tag{104}$$

The natural environment for analyzing the projectors π_j consists in the spaces with null zero-th order trace. More precisely, for $s \geq 1$ we then denote by $\mathcal{H}_0^s(0,1)$ the space

$$\mathcal{H}_0^s(0,1) = H^s(0,1) \cap H_0^1(0,1),$$

endowed with the H^s norm. For $s = 0$ we set $\mathcal{H}_0^0(0,1) = L^2(0,1)$, and for $0 < s < 1$ we define $\mathcal{H}_0^s$ by space interpolation. We remark that $\mathcal{H}_0^{1/2}(0,1) = H_{00}^{1/2}(0,1)$.

Since $V_j^0 \subset V_j$ and $\tilde{V}_j^* \subset \tilde{V}_j$ and since they reproduce respectively polynomials of degree $N-1$ satisfying homogeneous boundary condition at the edge 0 and 1 and polynomials of degree $\tilde{N}-1$, the following bounds also hold.

Theorem 3.4 *For all* $r \leq R$, *for all* s, $0 \leq r \leq s \leq N$, π_j *is continuous from* $\mathcal{H}_0^s(0,1)$ *to* $\mathcal{H}_0^r(0,1)$, *and for all* $\eta \in \mathcal{H}_0^s(0,1)$ *we have:*

$$\|\eta - \pi_j(\eta)\|_{\mathcal{H}_0^r(0,1)} \lesssim 2^{-j(s-r)} \|\eta\|_{\mathcal{H}_0^s(0,1)}. \tag{105}$$

For all $r \leq \tilde{R}$, *for all* s, $0 \leq r \leq s \leq \tilde{N}$, *for all* $\lambda \in H^s(0,1)$, *we have:*

$$\|\lambda - \tilde{\pi}_j(\lambda)\|_{r,]0,1[} \lesssim 2^{-j(s-r)} \|\lambda\|_{s,]0,1[} \tag{106}$$

Moreover, for all $0 \leq r \leq s \leq R$ *it holds*

$$\|\eta\|_{\mathcal{H}_0^s(0,1)} \lesssim 2^{j(s-r)} \|\eta\|_{\mathcal{H}_0^r(0,1)}. \tag{107}$$

In particular it holds

Corollary 3.1 π_j *is continuous from* $H_{00}^{1/2}(0,1)$ *to* $H_{00}^{1/2}(0,1)$, *that is for all* $\eta \in H_{00}^{1/2}(0,1)$:

$$\|\pi_j(\eta)\|_{H_{00}^{1/2}(0,1)} \lesssim \|\eta\|_{H_{00}^{1/2}(0,1)}. \tag{108}$$

Moreover, for all $\eta \in L^2(0,1)$ *and for all* $\lambda \in \tilde{V}_j^0$,

$$\int_0^1 (\eta - \pi_j \eta)\lambda = 0. \tag{109}$$

3.2 Wavelets

We can now introduce two couples of biorthogonal wavelet bases, which will both be needed for the analysis and/or for the implementation of the Mortar Wavelet Method. On one hand we will need the complement spaces W_j and $\tilde{W}_j$ defined as follows.

Definition. *The complement spaces W_j and its dual $\tilde{W}_j$ are defined by:*

$$\begin{aligned} W_j &= (P_{j+1} - P_j)V_{j+1}, \\ \tilde{W}_j &= (\tilde{P}_{j+1} - \tilde{P}_j)\tilde{V}_{j+1}. \end{aligned} \tag{110}$$

Let us now introduce the complement space W_j^0 of V_j^0 in V_{j+1}^0 and its dual. To do this, we will use the biorthogonal projector $\tilde{\pi}_j$ defined in the previous section.

Definition. *The complement spaces W_j^0 and its dual $\tilde{W}_j^*$ are defined by:*

$$\begin{aligned} W_j^0 &= (\pi_{j+1} - \pi_j)V_{j+1}^0, \\ \tilde{W}_j^* &= (\tilde{\pi}_{j+1} - \tilde{\pi}_j)\tilde{V}_{j+1}^*. \end{aligned} \tag{111}$$

Following [32] it is possible to construct wavelet bases for W_j, $\tilde{W}_j$, W_j^0 and $\tilde{W}_j^*$. We will then have biorthogonal wavelet Riesz1 bases $\{\psi_{jk},\ \ k = 1, \cdots, 2^j\}$, $\{\tilde{\psi}_{jk},\ \ k = 1, \cdots, 2^j\}$, and $\{\psi_{jk}^0,\ \ k = 1, \cdots, 2^j\}$, $\{\tilde{\psi}_{j,k}^*,\ \ k = 1, \cdots, 2^j\}$ such that the following identities hold for all $f \in L^2(0,1)$:

$$(P_{j+1} - P_j)f = \sum_{k=1}^{2^j} < f|\tilde{\psi}_{jk} > \psi_{jk}, \tag{112}$$

$$(\tilde{P}_{j+1} - \tilde{P}_j)f = \sum_{k=1}^{2^j} < f|\psi_{jk} > \tilde{\psi}_{jk}, \tag{113}$$

$$(\pi_{j+1} - \pi_j)f = \sum_{k=1}^{2^j} < f|\tilde{\psi}_{j,k}^* > \psi_{jk}^0, \tag{114}$$

$$(\tilde{\pi}_{j+1} - \tilde{\pi}_j)f = \sum_{k=1}^{2^j} < f|\psi_{jk}^0 > \tilde{\psi}_{j,k}^*. \tag{115}$$

Remark 3.1 *The functions in the complement spaces $\tilde{W}_j^*$ do not satisfy an homogeneous boundary condition.*

As usual in the wavelet framework, norm equivalences hold for Sobolev spaces of negative and/or fractional smoothness (either of the type $H^s(0,1)$ or $\mathcal{H}_0^s(0,1)$

as defined above), in terms the expansions in either one of the bases $\{\psi_{jk}\}$, $\{\tilde{\psi}_{jk}\}$, $\{\psi^0_{jk}\}$ and $\{\tilde{\psi}^*_{jk}\}$. It is beyond the goals of this paper to precisely state such norm equivalences. We will therefore limit ourselves to state the following theorem, which will play a key role in both the analysis and the implementation of the Mortar Wavelet Method and which can be proven by applying the results of [36, 24].

Theorem 3.5 *Let $f \in L^2(0,1)$. The following two norm equivalences hold:*

$$\|f\|^2_{H^{1/2}(0,1)} \sim \sum_{k=0}^{2^j+1} |<f|\tilde{\varphi}_{j_0,k}>|^2 + \sum_{j\geq j_0}\sum_{k=1}^{2^j} 2^j \; |<f|\tilde{\psi}_{jk}>|^2, \tag{116}$$

$$\|f\|^2_{H^{1/2}_{00}} \sim \sum_{k=1}^{2^j} |<f|\tilde{\varphi}_{j_0,k}>|^2 + \sum_{j\geq j_0}\sum_{k=1}^{2^j} 2^j \; |<f|\tilde{\psi}^*_{jk}>|^2. \tag{117}$$

Moreover $f \in H^{1/2}(0,1)$ (resp. $f \in H^{1/2}_{00}(0,1)$) if and only if the sum on the right hand side of (116) (resp. (117)).

3.3 Mortar approximation spaces in the Wavelet framework

3.3.1 2D Multiresolution analyses

As usual in the unit $]0,1[^2$ we will consider as approximation spaces a two-dimensional MRA $\mathcal{V}_j$ defined by tensor products of one-dimensional MRA:

$$\begin{aligned}\mathcal{V}_j &= V_j \otimes V_j \sim span < \Phi_{j,\mathbf{k}} \;,\; \mathbf{k} \in \mathbb{K}_j >,\\ \tilde{\mathcal{V}}_j &= \tilde{V}_j \otimes \tilde{V}_j \sim span < \tilde{\Phi}_{j,\mathbf{k}} \;,\; \mathbf{k} \in \mathbb{K}_j >,\end{aligned}$$

with, for $\mathbf{k} = (k_1, k_2) \in \mathbb{K}_j = \{0, \ldots, 2^j+1\}^2$

$$\Phi_{j,\mathbf{k}} = \varphi_{j,k_1} \otimes \varphi_{j,k_2} \qquad \text{and} \qquad \tilde{\Phi}_{j,\mathbf{k}} = \tilde{\varphi}_{j,k_1} \otimes \tilde{\varphi}_{j,k_2}.$$

In the same way, $\mathcal{V}^0_j$, the subspaces of $\mathcal{V}_j$ verifying homogeneous boundary conditions and its dual $\tilde{\mathcal{V}}^*_j$ are, for $\mathbb{K}^*_j = \{1, \ldots, 2^j\}^2$

$$\begin{aligned}\mathcal{V}^0_j &= V^0_j \otimes V^0_j \sim span < \Phi_{j,\mathbf{k}} \;,\; \mathbf{k} \in \mathbb{K}^*_j >,\\ \tilde{\mathcal{V}}^*_j &= \tilde{V}^*_j \otimes \tilde{V}^*_j \sim span < \tilde{\Phi}^*_{j,\mathbf{k}} \;,\; \mathbf{k} \in \mathbb{K}^*_j > .\end{aligned}$$

It is well known that the family $(\mathcal{V}_j)$ constitutes a MRA of $L^2(]0,1[^2)$, and $(\mathcal{V}^0_j)$ a MRA of $H^1_0(]0,1[^2)$. The two-dimensional biorthogonal projections on respectively

$\mathcal{V}_j$ and $\mathcal{V}_j^0$ will be denoted by $\mathcal{P}_j$ and Π_j and their adjoints by $\tilde{\mathcal{P}}_j$ and $\tilde{\Pi}_j$:

$$
\begin{aligned}
\mathcal{P}_j f &= \sum_{\mathbf{k}\in\mathbb{K}_j} < f|\tilde{\Phi}_{j,\mathbf{k}} > \Phi_{j,\mathbf{k}}, & \Pi_j f &= \sum_{\mathbf{k}\in\mathbb{K}_j^*} < f|\tilde{\Phi}^*_{j,\mathbf{k}} > \Phi_{j,\mathbf{k}}, \\
\tilde{\mathcal{P}}_j f &= \sum_{\mathbf{k}\in\mathbb{K}_j} < f|\Phi_{j,\mathbf{k}} > \tilde{\Phi}_{j,\mathbf{k}}, & \tilde{\Pi}_j f &= \sum_{\mathbf{k}\in\mathbb{K}_j^*} < f|\Phi_{j,\mathbf{k}} > \tilde{\Phi}^*_{j,\mathbf{k}}.
\end{aligned}
$$

Two-dimensional wavelets are constructed (as usual) by tensor products of univariate bases. We will then have two couples of biorthogonal wavelets, which we will denote by $\Psi_{j,\kappa}$, $\tilde{\Psi}_{j,\kappa}$, and $\Psi^0_{j,\kappa}$, $\tilde{\Psi}^*_{j,\kappa}$ such that:

$$
(\mathcal{P}_{j+1} - \mathcal{P}_j)f = \sum_{\kappa\in\mathbb{I}_j} < f|\tilde{\Psi}_{j,\kappa} > \Psi_{j,\kappa} \tag{118}
$$

$$
(\tilde{\mathcal{P}}_{j+1} - \tilde{\mathcal{P}}_j)f = \sum_{\kappa\in\mathbb{I}_j} < f|\Psi_{j,\kappa} > \tilde{\Psi}_{j,\kappa} \tag{119}
$$

$$
(\Pi_{j+1} - \Pi_j)f = \sum_{\kappa\in\mathbb{I}_j^*} < f|\tilde{\Psi}^*_{j,\kappa} > \Psi^0_{j,\kappa} \tag{120}
$$

$$
(\tilde{\Pi}_{j+1} - \tilde{\Pi}_j)f = \sum_{\kappa\in\mathbb{I}_j^*} < f|\Psi^0_{j,\kappa} > \tilde{\Psi}^*_{j,\kappa}, \tag{121}
$$

where $\mathbb{I}_j$ and $\mathbb{I}_j^*$ will denote suitable multi-index sets. The corresponding complement spaces will then be denoted by

$$
\begin{array}{lllll}
\mathcal{W}_j & = & (\mathcal{P}_{j+1} - \mathcal{P}_j)\mathcal{V}_j & = & span < \Psi_{j,\kappa}, \quad \kappa \in \mathbb{I}_j > \\
\tilde{\mathcal{W}}_j & = & (\tilde{\mathcal{P}}_{j+1} - \tilde{\mathcal{P}}_j)\mathcal{V}_j & = & span < \tilde{\Psi}_{j,\kappa}, \quad \kappa \in \mathbb{I}_j > \\
\mathcal{W}^0_j & = & (\Pi_{j+1} - \Pi_j)\mathcal{V}_j & = & span < \Psi^0_{j,\kappa}, \quad \kappa \in \mathbb{I}^*_j > \\
\tilde{\mathcal{W}}^*_j & = & (\tilde{\Pi}_{j+1} - \tilde{\Pi}_j)\mathcal{V}_j & = & span < \tilde{\Psi}^*_{j,\kappa}, \quad \kappa \in \mathbb{I}^*_j >
\end{array} \tag{122}
$$

Remark 3.2 *There are several ways of building wavelet bases for the spaces $\mathcal{W}_j$, $\tilde{\mathcal{W}}_j$, $\mathcal{W}_j^0$ and $\tilde{\mathcal{W}}_j^*$. For instance, one can define a basis for $\mathcal{W}_j$ composed by functions of the form $\psi_{j,k}(x)\psi_{j',k'}(y)$ and $\psi_{j',k'}(x)\psi_{j,k}(y)$ (with $j' \leq j$), or, as it is usually done in the wavelet context, by functions of the form $\phi_{j,k}(x)\psi_{j,k'}(y)$, $\psi_{j,k}(x)\psi_{j,k'}(y)$ and $\psi_{j,k}(x)\psi_{j,k'}(y)$. The same alternative holds for the three other spaces. Depending on the choice made the index sets $\mathbb{I}_j$ and $\mathbb{I}_j^*$ will have different forms. We remark however that Theorem 3.7 in the following holds for both choices.*

Again, Jackson and Bernstein inequalities and norm equivalences analogous to (99-105) and (116-117) are still valid in dimension 2. In particular, the following two Theorem hold.

Theorem 3.6 *For all r, $0 \leq r \leq R$, for all s, $r \leq s \leq N$ and for all $u \in H^s(]0,1[^2)$ it holds*

$$\|u - \mathcal{P}_j(u)\|_{r,]0,1[^2} \lesssim 2^{-j(s-r)} \|u\|_{s,]0,1[^2}. \tag{123}$$

For all r, s, $0 \leq r \leq s \leq R$

$$u \in \mathcal{V}_j \qquad \Longrightarrow \qquad \|u\|_{s,]0,1[^2} \lesssim 2^{j(s-r)} \|u\|_{r,]0,1[^2}. \tag{124}$$

Theorem 3.7 *The following norm equivalences hold: for all $u \in H^1(]0,1[^2)$*

$$\|u\|^2_{H^1(]0,1[^2)} \sim \sum_{\mathbf{k} \in \mathbb{K}_{j_0}} | < u | \tilde{\Phi}_{j_0,\mathbf{k}} > |^2 + \sum_{j \geq j_0} \sum_{\kappa \in \mathbb{I}_j} 2^{2j} | < u | \tilde{\Psi}_{j,\kappa} > |^2, \tag{125}$$

and for all $u \in H^1_0(]0,1[^2)$

$$\|u\|^2_{H^1(]0,1[^2)} \sim \sum_{\mathbf{k} \in \mathbb{K}^*_{j_0}} | < u | \tilde{\Phi}_{j_0,\mathbf{k}} > |^2 + \sum_{j \geq 0} \sum_{\kappa \in \mathbb{I}^*_j} 2^{2j} | < u | \tilde{\Psi}^*_{j,\kappa} > |^2. \tag{126}$$

Conversely, $u \in H^1(]0,1[^2)$ (resp. $u \in H^1_0(]0,1[^2)$) if and only if the sum on the right hand side of (125) (resp. of (126)) is bounded.

3.3.2 Trace and multiplier spaces

Let γ be one edge of $]0,1[^2$. Thanks to the tensor product structure of the space $\mathcal{V}_j$, the trace space $T_j(\gamma)$ of functions of $\mathcal{V}_j$ can be identified to the space spanned by the basis of scaling functions on the interval:

$$\begin{aligned} T_j(\gamma) &= \{\text{Traces of } \mathcal{V}_j \text{ on } \gamma\} \\ &\sim V_j = span < \varphi_{j,k} \ , \ k = 0, \ldots, 2^j + 1 > . \end{aligned}$$

Then, the space $T^0_j(\gamma) \subset H^{1/2}_{00}(\gamma)$ of functions of $T_j(\gamma)$ vanishing at the edges of γ verifies:

$$T^0_j(\gamma) \sim V^0_j = span < \varphi_{j,k} \ , \ k = 1, \ldots, 2^j > . \tag{127}$$

Definition. *If γ is a* multiplier side *a natural choice for the* multiplier space *$M_j(\gamma)$ on γ will be:*

$$M_j(\gamma) \sim \tilde{V}^*_j = span < \tilde{\phi}^*_{j,k} \ , \ k = 1, \ldots, 2^j > . \tag{128}$$

Remark 3.3 - *$M_j \subset \tilde{T}_j$ (and not T_j as usual in the mortar methods, see [15]) is exactly the dual space of T_j^0, then the projector π_j is diagonal. This projector is diagonal for spectral approximation, and tridiagonal for finite elements P1 [12, 11]. If orthonormal wavelets are considered, then M_j is a subspace of codimension 2 of T_j, as in the classical mortar method.*

Theorem 3.8 *The spaces $T_j^0(\gamma)$ and $M_j(\gamma)$ verify assumptions (A.1–3).*

Proof:

- **(A1)** The projection: $\pi_j \ : \ L^2(\gamma) \to T_j^0(\gamma)$ defined by (103) verifies assumption (A1). This result is given by Corollary 3.1.
- **(A2)** Lifting: For simplicity let us assume that $\gamma = \{(0,y),\ y \in (0,1)\}$. The following map: $R_j : T_j^0(\gamma) \to \mathcal{V}_j$ defined, for all $\eta \in T_j^0(\gamma)$ by:

$$\begin{aligned} R_j(\eta) \ &= \ \sum_{j'=j_0}^{j-1} \sum_{k=1}^{2^{j'}} 2^{-j'/2} < \eta | \tilde{\psi}^*_{j',k} > \psi^0_{j',k} \otimes \varphi_{j',0} \\ &\quad + \sum_{k=1}^{2^{j_0}} 2^{-j_0/2} < \eta | \tilde{\varphi}^*_{j_0,k} > \varphi_{j_0,k} \otimes \varphi_{j,0} \end{aligned}$$

  is a lifting from $H^{1/2}_{00}(\gamma)$ to $H^1(]0,1[^2)$ which verifies assumption (A2).
  In fact, let us consider the MRA $\check{\mathcal{V}}_j = V_j^0 \otimes V_j$ and its dual $\hat{\mathcal{V}}_j = \tilde{V}_j^* \otimes \tilde{V}_j$. If we denote by $\{\check{\Psi}_{j,\kappa}, \kappa \in \check{\mathbb{I}}_j\}$ and $\{\hat{\Psi}_{j,\kappa}, \kappa \in \check{\mathbb{I}}_j\}$, the corresponding couple of biorthogonal wavelet bases, (with an analogous notation for scaling functions) the following norm equivalence holds: for all $u \in H^1_0(]0,1[^2)$, satisfying $u(0,y) \equiv u(1,y) \equiv 0$ for all $y \in (0,1)$:

$$\|u\|^2_{1,]0,1[^2} \sim \sum_{\mathbf{k} \in \check{\mathbb{K}}_{j_0}} | < u | \tilde{\Phi}_{j,\mathbf{k}} > |^2 + \sum_{j \geq j_0} 2^{2j} \sum_{\kappa \in \check{\mathbb{I}}_j} | < f | \hat{\Psi}_{j,\kappa} > |^2,$$

  where $\check{\mathbb{K}}_{j_0} = \{1, \ldots, 2^{j_0}\} \times \{0, \ldots, 2^{j_0}+1\}$. Now we observe that for suitable $\kappa = \kappa(k)$ we have

$$\check{\Psi}_{j,\kappa}(x,y) = \psi^0_{j,k}(x) \varphi_{j,0}(y), \qquad \hat{\Psi}_{j,\kappa}(x,y) = \tilde{\psi}^*_{j,k}(x) \tilde{\varphi}_{j,0}(y).$$

  Therefore, applying the previous norm equivalence to $R_j(\eta)$, we get

$$\|R_j(\eta)\|_{1,]0,1[^2} \sim \sum_{k=1}^{2^{j_0}} 2^{-j_0} | < \eta | \tilde{\varphi}^*_{j_0,k} > |^2 + \sum_{j'=j_0}^{j-1} \sum_{k=1}^{2^{j'}} 2^{j'} | < \eta | \tilde{\psi}^*_{j',k} > |^2, \qquad (129)$$

  and the right hand side of (129) is equivalent to $\|\eta\|_{H^{1/2}_{00}(\gamma)}$, thanks to norm equivalence (117).

- **(A3)** Let γ be a multiplier side of $]0,1[^2$. Then, for all $\eta \in T_j^0$ we have by (107):

$$\|\eta\|_{H^{1/2}_{00}(\gamma)} \lesssim 2^{-js/2}\|\eta\|_{H^s(\gamma)},$$

and the assumption (A.3) is fullfilled with $h = 2^{-j}$.
- **(A4)** By applying a scaling argument it is not difficult to prove that the space $\mathcal{V}_j$ satisfies also the following inverse inequality: for all $u \in \mathcal{V}_j$ it holds

$$\|\nabla u\|_{\infty,\Omega_\ell} \lesssim 2^j \|u\|_{\infty,\Omega_\ell}.$$

□

3.4 Error estimates

We can now use the function spaces thus defined in the framework of the mortar method described in section 2. For simplicity we will just consider the case in which all the subdomains are discretized by wavelets. More precisely, in each subdomain Ω_ℓ let the approximation space $\mathcal{V}_\delta^\ell$ be given by:

$$\mathcal{V}_\delta^\ell = \mathcal{V}_{j(\ell)}$$

with some fixed $j(\ell) \geq j_0$. To each edge $\gamma_\ell^{(i)}$ we associate the trace space of $\mathcal{V}_\delta^\ell$:

$$T_{\ell,i} \sim T_{j(\ell)}(\gamma_\ell^{(i)}).$$

For $m = (\ell, i) \in I$ ($\gamma_m = \gamma_\ell^{(i)}$ multiplier side), the multiplier space is

$$M_\delta^m \sim M_{j(\ell)}(\gamma_m).$$

The constrained approximation space $\mathcal{X}_\delta$ and its multiplier space M_δ are then defined according to (28, 27).

If the decomposition is geometrically conforming, the resulting multiplier space M_δ will automatically satisfy assumption (33) (ensuring stability), while in the case of a geometrically non conforming decomposition we will have to choose $j(\ell)$ in such a way that either one of the sufficient conditions **I** and **II** of section 2.4 are satisfied. This is true, provided one of the following two conditions is satisfied (we recall that $I_\ell^{(i)}$ is defined as the set of those subdomains whose boundary intersect $\gamma_\ell^{(i)}$ (see (5)).

(i) $card\{I_\ell^{(i)}\} \leq \tilde{N}$ ($\tilde{N}$ being the number of polynomials exactly reproduced in $\tilde{V}_J$).

(ii) If (i) does not hold, then $j(\ell) \geq \max\{J_{\ell,i},\ i = 1, \cdots, K\}$, where for each $i \in I_\ell^{(i)}$

$$J_{\ell,i} = \inf\{j : \exists \tilde{\varphi}_{jn} \in M_j(\gamma_m),\ supp \tilde{\varphi}_{jn} \subset \Gamma_{\ell n}\}. \tag{130}$$

With such a choice of the spaces, we will consider problem (31). Since assumptions (A.1-A.3) are fullfilled (as we checked in the previous section), the solution u_δ of such problem will satisfy the following error estimate.

Corollary 3.2 *Let u, solution of (10), and let u_δ be the solution of problem (PD-31), with the above choice of approximation and multiplier spaces. If $u|_{\Omega_\ell} \in H^{s_\ell}(\Omega_\ell)$, $s_\ell > 3/2$, then it holds*

$$\|u - u_\delta\|_X \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) \left(\sum_{\ell=1}^{L} 2^{-2j(\ell)(s_\ell - 1)} \|u\|^2_{s_\ell,\Omega_\ell}\right)^{1/2} + \tag{131}$$

$$+ \left(\sum_{m=(\ell,i)\in I} 2^{-2j(\ell)(s_\ell-1)} \left\| \frac{\partial u}{\partial \nu} \right\|^2_{s_\ell - 3/2, \gamma_m} \right)^{1/2} \tag{132}$$

Proof: Either one of the conditions (i) and (ii) is sufficient to ensure that assumption (BP) is satisfied with a constant $\bar{C}$ ultimately depending on the geometric decomposition of Ω in subdomains and on the particular wavelet space chosen. Since the approximation and multiplier spaces satisfy assumption (A.1–3), we can then apply corollary 2.2, which, together with the Jackson type inequalities (123) and (106) yields (131). □

Remark 3.4 *In the case of a geometrically non conforming decomposition, condition (i) might be quite restrictive, forcing the discretization to be very fine ($2^{-j(\ell)} \leq N^{-1} \inf |\Gamma_{\ell n}|$). It is opinion of the authors that such condition can in practice be relaxed by only asking $2^{-j(\ell)} \leq \inf |\Gamma_{\ell n}|$.*

In the geometrically conforming case we have instead the following (optimal) error estimate:

Theorem 3.9 *If the decomposition is geometrically conforming, under the assumptions of corollary 3.2, it holds*

$$\|u - u_\delta\|_X \lesssim \left(\sum_{\ell=1}^{L} 2^{-2j(\ell)(s_\ell - 1)} \|u\|^2_{s_\ell,\Omega_\ell}\right)^{1/2} + \tag{133}$$

$$+ \left(\sum_{m=(\ell,i)\in I} 2^{-2j(\ell)(s_\ell-1)} \left\| \frac{\partial u}{\partial \nu} \right\|^2_{s_\ell - 3/2, \gamma_m} \right)^{1/2} \tag{134}$$

Proof: The result follows from corollary 2.3, provided it exits a projector satisfying the assumptions of theorem 2.4. Indeed, observing that for a geometrically conforming decomposition all cross points at vertices of all the subdomains to which they belong, in the wavelet case, this projector $\Pi_j : H^s(]0,1[^2) \to \mathcal{V}_j$ can be simply defined for all $u \in H^s(]0,1[^2)$ by:

$$\Pi_j(u) = \mathcal{P}_j(u) + \sum_{i=1}^{4} \left(\frac{u(A_i)}{\Phi_j^{A_i}(A_i)} - < u|\tilde{\Phi}_j^{A_i} > \right) \Phi_j^{A_i}$$

where for each vertex A_i of $[0,1]^2$ vertex scaling functions $\Phi_j^{A_i}$ is the only scaling function of $\mathcal{V}_j$ that does not vanish at A_i,, in other words, for instance:

$$\begin{array}{lll} \text{If } A_0 = (0,0), & \text{then} & \Phi_j^{A_0}(x,y) = \varphi_{j,0}(x)\varphi_{j,0}(y), \\ \text{If } A_1 = (1,0), & \text{then} & \Phi_j^{A_1}(x,y) = \varphi_{j,2^j+1}(x)\varphi_{j,0}(y), \\ & \dots & \end{array}$$

It is easily verified that Π_j is indeed a projector and that it verifies $\Pi_j(u)(A_i) = u(A_i)$ for any vertex A_i. Moreover it is not difficult to check that the Jackson inequality of the form (53) valid for $\bar{s} = R$ and $\bar{t} = N$, provided Π_j is bounded from $H^t(\Omega_\ell)$ to $H^s(\Omega_\ell)$. This is the case, if $1 < t$. □

4 Some Remarks on the Implementation

The aim of this section is to stress out how some of the features of wavelet bases can be exploited for an efficient implementation of the Mortar Wavelet Method.

4.1 Multiplication by the Stiffness Matrix

It is well known that in the implementation of the mortar method, the multiplication by the stiffness matrix can be performed by applying subdomainwise the local stiffness matrix (which does not take into account the constraints), after multiplication by a "transfer matrix" which gives the values of the constrained degrees of freedom (the ones living on the interior of multiplier sides) in terms of the remaining (free) degrees of freedom. The transfer matrix is a discrete realization of the projector π (section 2.5).

In the case that a geometrically conforming decomposition is consider and that all subdomains are discretized by wavelets, applying such projector (π_j, see (103)) reduces to performing either a *Fast Wavelet Transform* or an *Inverse Fast Wavelet*

Transform depending on which of the two discretizations on the "trace" and on the "multiplier" side, is finer.

In general, when a non geometrically conforming decomposition is considered and/or wavelets are coupled with some other method, in the case in which a multiplier side γ_m is chosen in a subdomain discretized by wavelets, then, thanks to (103), applying the transfer matrix reduces to computing the scalar products of the functions on the corresponding "trace sides" with the scaling functions $\tilde{\varphi}^*_{j,k}$, $k = 1, \ldots, 2^j$. We recall that several techniques are available for the numerical computation of the integral of a given function times a scaling function, a task that can be performed in $O(N)$ operations, and we refer to [41, 17], for more details in such a direction. We also refer to [26] for an analysis of the effect that using numerical quadrature in computing the projector π has on the method.

4.2 Preconditioning Operators in the Wavelet case

Let us now come to the construction of the preconditioner described in section 2.6. For simplicity we assume here that all subdomains are discretized by *wavelets*. We recall that the considered preconditioner (expressed through the corresponding bilinear form rather than in matrix form) takes the block diagonal form

$$\begin{aligned} \hat{a}_X(u,v) &= \sum_{\ell=1,\cdots,L} \tilde{a}^0_\ell(u^0_\ell, v^0_\ell) + \sum_{(\ell,i)\in I^*} b_{\ell,i}(u^E_\ell, v^E_\ell) \\ &\quad + \sum_{\ell=1,\cdots,L} \int_{\Omega_\ell} \nabla R^\ell_\delta(u^V_\ell) \nabla R^\ell_\delta(v^V_\ell) \end{aligned} \tag{135}$$

with R^ℓ_δ suitable lifting operators, and $\tilde{a}^0_\ell$, $b_{\ell,i}$ suitable bilinear forms. The terms $u^0 = (u^0_\ell)_{\ell=1,\cdots,L}$, $u^E = (u^E_\ell)_{\ell=1,\cdots,L}$ and $u^V = (u^V_\ell)_{\ell=1,\cdots,L}$ correspond to the splitting

$$u = u^0 + \mathcal{R}_j(u^E + u^V)$$

of u in interior, edge and vertex components, and analogously for the test function v. We need then to construct a lifting operator $\mathcal{R}_j$, and bilinear forms $\tilde{a}_\ell$ and $b_{\ell,i}$ satisfying assumptions (58), (59) and (72) respectively. Again, we can take advantage of the features of wavelets for gaining efficiency.

Thanks to the scaling invariance of the norms involved in the analysis of the preconditioner we can work on the reference square $\hat{\Omega} =]0,1[^2$. We will denote by $\gamma_1 = \{(0,y), y \in]0,1[\}$, $\gamma_2 = \{(x,1), x \in]0,1[\}$, $\gamma_3 = \{(1,y), y \in]0,1[\}$ and $\gamma_4 = \{(x,0), x \in]0,1[\}$ the four edges of $\hat{\Omega}$, and $\hat{\Gamma} = \delta\hat{\Omega} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ the boundary of $\hat{\Omega}$.

Let us begin by constructing the *lifting operator* $\mathcal{R}_j$. We first define an operator $\check{R}_j : H^{1/2}(\hat{\Gamma}) \to \mathcal{V}_j$ as follows

$$\check{R}_j(\eta) = R_j^1(\eta) + R_j^2(\eta - R_j^1(\eta)),$$

where R_j^1 is a lifting from $\bar{\gamma}_1 \cup \bar{\gamma}_3$:

$$\begin{aligned} R_j^1\eta(x,y) &= \sum_{j'=j_0}^{j-1}\sum_{k=1}^{2^{j'}}\left(\int_{\gamma_1}\eta\tilde{\psi}_{j'k}\right)\varphi^*_{j',0}(x)\psi_{j'k}(y) \\ &\quad + \sum_{j'=j_0}^{j-1}\sum_{k=1}^{2^{j'}}\left(\int_{\gamma_3}\eta\tilde{\psi}_{j'k}\right)\varphi^*_{j',2^j+1}(x)\psi_{j'k}(y) \\ &\quad + \varphi^*_{j_0 0}(x)P_{j_0}(\eta|_{\gamma_1})(y) + \varphi^*_{j_0 2^{j_0}+1}(x)P_{j_0}(\eta|_{\gamma_3})(y). \end{aligned}$$

while R_j^2 is a lifting from $\gamma_2 \cup \gamma_4$:

$$\begin{aligned} R_j^2\eta(x,y) &= \sum_{j'=j_0}^{j-1}\sum_{k=1}^{2^{j'}}\left(\int_{\gamma_4}\eta\tilde{\psi}^*_{j'k}\right)\psi^0_{j'k}(x)\varphi^*_{j',0}(y) \\ &\quad + \sum_{j'=j_0}^{j-1}\sum_{k=1}^{2^{j'}}\left(\int_{\gamma_2}\tilde{\psi}^*_{j'k}\right)\psi^0_{j'k}(x)\varphi^*_{j',2^j+1}(y) \\ &\quad + \pi_{j_0}(\eta|_{\gamma_4})(x)\varphi^*_{j_0,0}(y) + \pi_{j_0}(\eta|_{\gamma_2})(x)\varphi^*_{j_0 2^{j_0}+1}(y) \end{aligned}$$

Using again the norm equivalences of Theorems 3.5 and 3.7, it is not difficult to check that $\check{R}_j$ verifies

$$\|\check{R}_j\eta\|_{1,\hat{\Omega}} \lesssim \|\eta\|_{1/2,\hat{\Gamma}}.$$

Since $\check{R}_j$ does not in general verify the seminorm inequality $|\check{R}_j\eta|_{1,\hat{\Omega}} \lesssim |\eta|_{1/2,\hat{\Gamma}}$, we need to modify it. More precisely, denoting by $<\eta>_{\hat{\Gamma}} = \frac{1}{4}\int_{\hat{\Gamma}}\eta$ the average value of η on $\hat{\Gamma}$, we set

$$\mathcal{R}_j(\eta) = <\eta>_{\hat{\Gamma}} + \check{R}_j(\eta - <\eta>_{\hat{\Gamma}}). \tag{136}$$

Now, thanks to Poincaré inequality we have

$$|\mathcal{R}_j(\eta)|_{1,\hat{\Omega}} \lesssim |\check{R}_j(\eta - <\eta>_{\hat{\Gamma}})|_{1,\hat{\Omega}} \lesssim \|\eta - <\eta>_{\hat{\Gamma}}\|_{1/2,\hat{\Gamma}} \lesssim |\eta|_{1/2,\hat{\Gamma}}.$$

$\mathcal{R}_j$ satisfies then the assumptions of section 2.6.

Thanks to the properties of wavelet bases, the construction of an *easily invertible operator* $\tilde{a}^0 : \mathcal{V}_j^0 \times \mathcal{V}_j^0 \to \mathbb{R}$, satisfying assumption (59) and (60) is easy. More precisely, setting

$$\tilde{a}^0(u,v) = \sum_{j'=j_0}^{j-1} \sum_{\kappa \in \mathbb{I}_j^*} 2^{2j'} u_{j',\kappa}\ v_{j',\kappa} + \int_{\hat{\Omega}} \Pi_{j_0} u\ \Pi_{j_0} v \tag{137}$$

where

$$u_{j',\kappa} =< u | \tilde{\Psi}^*_{j',\kappa} > \qquad \text{and} \qquad v_{j',\kappa} =< v | \tilde{\Psi}^*_{j',\kappa} >$$

are the coefficients of u and v in the wavelet basis $\{\Psi^0_{j',\kappa}\}$, due to the norm equivalence (126) it is not difficult to check that $\tilde{a}^0$ verifies indeed the bounds (59) and (60).

The *bilinear forms* $b_{\ell,i}$ can be defined analogously. In fact, working on the reference segment $]0,1[$, b can be defined by taking advantage of the properties of wavelets. Any function in $\mathcal{V}_j^0$ can in fact be expressed in the wavelet basis ψ^0_{jk}, and hence we can define

$$b(\eta,\xi) = \sum_{j'=j_0}^{j-1} \sum_{k=1}^{2^{j'}} 2^{j'} \eta_{j',k}\ \xi_{j',k} + \int_0^1 \pi_{j_0}(\eta)\ \pi_{j_0}(\xi) \tag{138}$$

where

$$\eta_{j',k} =< \eta | \tilde{\psi}^*_{j'k} >, \qquad \xi_{j,k} =< \xi | \tilde{\psi}^*_{j'k} >$$

denote the coefficient of respectively η and ξ in the wavelet basis. Again, thanks to the norm equivalence (117), the bilinear form b satisfies the equivalence (72).

The *preconditioner* $\hat{S}$, which is defined as the Galerkin discretization of the bilinear form $\hat{a}_X$ defined according to (135), will have a block diagonal structure, where most of the blocks (except the very small blocks corresponding to the cross point unknowns) will themselves have a diagonal form, provided that the wavelet basis, rather than the usual nodal basis, is used in the Galerkin discretization.

Applying $\hat{S}^{-1}$ to a given vector (with $N^V + N^E + N^0$ degrees of freedom, respectively counting vertex, edge and interior unknowns) reduces then to the following steps:

- computing η^V by solving the "coarse" linear system having as unknowns the (multiple, one for each subdomain) values of such function at the cross points (last term of equation (135) , and then extending it linearly on each "trace edge" $(0((N^V)^3)$ operations)
- computing η^E by solving on each "trace side" a linear system corresponding to the bilinear operator $b_{\ell,i}$ (138). When expressed in the wavelet basis $\{\psi^0_{jk}\}$, the corresponding matrix is diagonal. Expressing the right hand side in the wavelet basis reduces to applying a Fast Wavelet Transform ($O(N^E)$ operations).
- Computing u^0 by solving, in each subdomain, the linear system corresponding to the bilinear form $\tilde{a}_\ell$, with homogeneous boundary conditions (137). Again, since the corresponding linear system is diagonal if the wavelet basis is used, this reduces to expressing the right hand side in the wavelet basis by applying a two dimensional Fast Wavelet Transform ($O(N^0)$ operations).
- η_δ is then reconstructed on "trace sides" as $\eta^E + \eta^V$, and on "slave sides" by applying the projector π, ($O(N^S)$ operations, N^S denoting the dimension of M_δ).
- The lifting $\mathcal{R}_j(\eta)$ is then computed (136). Thanks to its simple expression, this operation can be performed in $O(N^E + N^V + N^S)$ operations. Remark that the value of η at "multiplier sides" affects the result in the interior of the subdomains, therefore it has to be (directly or indirectly) computed.
- The result u_δ is then assembled as $u_h = u^0 + \mathcal{R}_j(\eta_\delta)$. By construction u_h verifies the jump constraint, and therefore can be expressed by using the constrained degrees of freedom, namely the (multiple) values at cross points, the coefficients of its trace at "trace edges" and the coefficients of the basis functions $\Phi_{j,\mathbf{k}}$ of $\mathcal{V}_j$.

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