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Wavelet Stabilization of the Lagrange Multiplier Method

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Summary We propose here a stabilization strategy for the Lagrange multiplier formulation of Dirichlet problems. The stabilization is based on the use of equivalent scalar products for the spaces $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$, which are realized by means of wavelet functions. The resulting stabilized bilinear form is coercive with respect to the natural norm associated to the problem. A uniformly coercive approximation of the stabilized bilinear form is constructed for a wide class of approximation spaces, for which an optimal error estimate is provided. Finally, a formulation is presented which is obtained by eliminating the multiplier by static condensation. This formulation is closely related to the *Nitsche's method* for solving Dirichlet boundary value problems.

1 Introduction

The interest in the Lagrange multiplier formulation of elliptic Dirichlet boundary value problems, has increased in the last few years, mainly due to its application in the framework of non conforming domain decomposition methods ([21,22,5,11]). Consequently there is a greater and greater need for efficient numerical schemes for the resulting saddle point problems.

Recently, a class of optimal diagonal preconditioners for the wavelet discretization of such saddle point problems has been introduced ([19, 23]). The existence of such preconditioners, along with other desirable properties of wavelet bases (space-frequency localization allows adaptivity with respect to both operator and solution), makes the use of such bases in this context particularily promising.

On the other hand, it is well known that the bilinear form corresponding to such a formulation is not coercive. Existence and uniqueness of the solution are ensured by an *inf-sup* condition ([10]). Consequently, in discretizing such problems, the approximation spaces cannot be chosen arbitrarily, but need themselves to satisfy a discrete *inf-sup* condition. The design of wavelet spaces satisfying such condition can be quite technical and pose limitations which, especially in the case of problems arising from domain decomposition, might be too restrictive, and lead to discard otherwise desirable choices.

An alternative is to use a stabilization technique ([4,3,27], that is, to introduce a second "stabilized" problem (whose form depends in general on the discretization to be used), equivalent to the saddle point problem considered (in the sense that the unique solutions of the two problems coincide), and such that the corresponding "stabilized" bilinear form a_{STAB} is continuous and coercive with respect to two suitable (possibly mesh dependent) norms. The stabilized problem can then be discretized safely also by means of approximation spaces which do not satisfy the *inf-sup* condition.

In doing this for wavelet discretizations, one has to keep in mind the goal of applying the class of efficient multilevel preconditioners of the type described in [19,23]. We recall that the optimal behaviour of such preconditioners relies on an equivalence of the form

$$a_{STAB}(\cdot,\cdot) \sim \|\cdot\|^2 \tag{1}$$

(for a suitable norm $\|\cdot\|$). In other terms, a_{STAB} needs to be continuous *and* coercive with respect to *the same* norm.

This leads us to discard the stabilization techniques proposed up to now, which are based on adding residual terms measured in a mesh dependent norm. In fact the two norms with respect to which the resulting stabilized bilinear form is continuous and coercive are not equivalent, and therefore (1) does not hold.

Instead, we propose here to substitute the mesh dependent scalar products used for instance in [4], with scalar products in suitable Sobolev spaces of fractional exponent ($H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$), which we propose to realize also by means of wavelet bases (not necessarily the ones to be used for discretizing). By doing this we will end up with a stabilized bilinear form satisfying an equivalence of the type (1) (see (36), (37)).

We would like to remark that this approach falls in the line proposed in [2,7]. It also has some relation with [8].

2 Notations and Preliminary Results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\Gamma = \partial\Omega$ of class C^R, $R > n+1$. We indicate by ν the outer unit normal.

We will denote by $\|\cdot\|_{s,\Omega}$ (resp. $\|\cdot\|_{s,\Gamma}$) the norm of the Sobolev space $H^s(\Omega)$ (resp. $H^s(\Gamma)$) for both positive and negative s, while for positive r, $\|\cdot\|_{-r,*}$ will denote the norm of the space $(H^r(\Omega))'$ (dual of $H^r(\Omega)$). We will denote by $< \cdot, \cdot >$ the $L^2(\Gamma)$ scalar product, and, by abuse of notation, also the duality relation between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$. For functions $u \in H^1(\Omega)$ we will write $< u, \phi >$ ($\phi \in H^{-1/2}(\Gamma)$) for $< u|_\Gamma, \phi >$, $u|_\Gamma \in H^{1/2}(\Gamma)$ denoting the trace of u. An analogous notation will be used for the normal derivative.

The letter C will stand here for several positive constants, which, unless otherwise indicated, will be independent of any relevant parameter. In particular C is (unless otherwise stated), independent of the mesh size h and on the level j.

2.1 Wavelets, equivalent norms and equivalent scalar products

Assume we are given a couple $\{\psi_i, i \in I = \cup_{j\geq 0} I_j\}$, $\{\widetilde{\psi}_i, i \in I = \cup_{j\geq 0} I_j\}$, ($I_j$ finite dimensional), of biorthogonal bases for $L^2(\Gamma)$, satisfying the following properties:

(W0) Any function $f \in L^2(\Gamma)$ can be expanded in terms of either one of the two bases as follows:

$$f = \sum_{i\in I} < f, \widetilde{\psi}_i > \psi_i = \sum_{i \in I} < f, \psi_i > \widetilde{\psi}_i. \tag{2}$$

(W1) For all $i \in I$ we have $\psi_i, \widetilde{\psi}_i \in H^R(\Gamma)$, and the following norm equivalences holds:

- For any $f \in H^s(\Gamma)$, $-R \leq s \leq R$:

$$A_s \|f\|^2_{s,\Gamma} \leq \sum_j 2^{2sj} \sum_{i \in I_j} | < f, \psi_i > |^2 \leq B_s \|f\|^2_{s,\Gamma} \tag{3}$$

- For any $f \in H^s(\Gamma)$, $-R \leq s \leq R$:

$$\tilde{A}_s \|f\|^2_{s,\Gamma} \leq \sum_j 2^{2sj} \sum_{i \in I_j} | < f, \widetilde{\psi}_i > |^2 \leq \tilde{B}_s \|f\|^2_{s,\Gamma} \tag{4}$$

with A_s, B_s, $\tilde{A}_s$ and $\tilde{B}_s$ positive constants.

(W2) There exist constants D and N independent of j such that for $i \in I_j$ we have, for some $x_i \in \mathbb{R}^n$, $supp\, \psi_i \subset \Gamma \cup B(x_i, D2^{-j})$ (ball of center x_i and radius $D2^{-j}$), and for each ball $B(k/2^j, 1/2^j) \subset \mathbb{R}^n$ of center $k/2^j$ and radius $1/2^j$ there exists at most N indexes $i \in I_j$ such that $supp\, \psi_i \cap B_{j,k} \neq \emptyset$.

Remark 1 The splitting $I = \cup_{j \geq 0} I_j$ indicates the fact that the basis functions ψ_i and $\widetilde{\psi}_i$ are "living" at different scales. For $i \in I_j$ the function ψ_i (and $\widetilde{\psi}_i$) "lives" at the scale 2^{-j}. This is also reflected from the factor 2^{2js} in norm equivalences (3) and (4).

Remark 2 We remark that (W0) implies that the two bases are "bi-orthogonal" in the following sense:

$$\int_\Gamma \widetilde{\psi}_i \psi_{i'} = \delta_{i,i'}, \qquad i, i' \in I = \cup_{j \geq 0} I_j. \tag{5}$$

Remark 3 Describing how bases satisfying assumptions (W0–2) can be constructed is beyond the goals of this paper. We want to stress out that biorthogonal wavelets fall in the class here described, and are therefore a natural candidate for the use in the stabilization that we will describe. We refer to [17,16,12,13,20] for the description of such bases. A more general study of the class of bases satifying a norm equivalence of the form (3,4), can be found in [18].

For all integer $M > 0$ we introduce the spaces V_M and $\widetilde{V}_M$ defined respectively as

$$V_M = span < \psi_i, \quad i \in \cup_{j=0}^{M-1} I_j >, \tag{6}$$

$$\widetilde{V}_M = span < \widetilde{\psi}_i, \quad i \in \cup_{j=0}^{M-1} I_j >, \tag{7}$$

and the related projectors $P_M : L^2(\Gamma) \longrightarrow V_M$ and $\widetilde{P}_M : L^2(\Gamma) \longrightarrow \widetilde{V}_M$ defined respectively as

$$P_M f = \sum_{i \in \cup_{j=0}^{M-1} I_j} < f, \widetilde{\psi}_i > \psi_i, \tag{8}$$

$$\widetilde{P}_M f = \sum_{i \in \cup_{j=0}^{M-1} I_j} < f, \psi_i > \widetilde{\psi}_i. \tag{9}$$

For any r, $-R < r < R$ we can introduce the bilinear form $(\cdot,\cdot)_r : H^r(\Gamma) \times H^r(\Gamma) \longrightarrow \mathbb{R}$ defined by

$$(f, g)_r = \sum_{j=0}^{+\infty} 2^{2jr} \sum_{i \in I_j} < f, \psi_i >< g, \psi_i > \tag{10}$$

An immediate consequence of the norm equivalence (3) is the following

Proposition 1 *For all r, $-R < r < R$, the bilinear form $(\cdot,\cdot)_r$ is a scalar product on $H^r(\Gamma)$ inducing an equivalent $H^r(\Gamma)$ norm.*

2.2 New bounds on the trace of the normal derivative

In the following we will need a bound on the trace of the normal derivative in terms of a suitable seminorm which we will construct via the wavelet basis introduced in the previous section.

In order to do that we need to define some functions $\Psi_i \in H^R(\Omega)$, $i \in I$, which are suitable liftings of the wavelet functions ψ_i. More precisely, for each $i \in I_j$ we introduce a function Ψ_i in $H^R(\Omega)$ which we will assume to satisfy the following assumptions:

(B1) $\Psi_i = 2^{j/2}\psi_i$ on Γ;
(B2) for $i \in I_j$, $supp\,\Psi_i \subset B(x_i, D'2^{-j})$ (ball of center x_i and radius $D'2^{-j}$) for some $x_i \in \mathbb{R}^n$, and for each ball $B_{j,k} := B(k/2^j, 1/2^j) \subset \mathbb{R}^n$ of center $k/2^j$ and radius $1/2^j$ there exists at most N' indexes $i \in I_j$ such that $supp\,\Psi_i \cap B_{j,k} \neq \emptyset$ (D' and N' independent of j);
(B3) $\|\Psi_i\|_{s,\Omega} \leq C2^{sj}$, for C independent of j, for all s $0 \leq s \leq R$;
(B4) $\|\Psi_i\|_{-s,*} \leq C2^{-sj}$, for C independent of j, for all s $0 \leq s \leq R$.

Given ψ_i, it is quite easy to construct a lifting Ψ_i satisfying assumptions (B1 – B3). As far as assumption (B4) is concerned, we can give the following sufficient condition.

Lemma 1 *Assume that Ψ_i verifies assumptions (B1–B3) and that there exists a subspace $P \subset L^2(\Omega)$ (which might depend on i) such that the following hold:*

- *For all $u \in H^s(\Omega)$, $0 < s \leq R$*

$$\inf_{p\in P} \|u-p\|_{0,supp\,\Psi_i} \leq C2^{-js}\|u\|_{s,\Omega}, \qquad (i \in I_j), \tag{11}$$

with a constant C independent of i and of j;
- *for all $p \in P$ we have*

$$\int_\Omega \Psi_i p = 0. \tag{12}$$

Then (B4) holds.

Proof We have

$$\|\Psi_i\|_{-r,*} = \sup_{\varphi\in H^r(\Omega)} \frac{|\int_\Omega \Psi_i\varphi|}{\|\varphi\|_{r,\Omega}}. \tag{13}$$

Now, by (12), for any element $p \in P$ we have

$$\int_\Omega \Psi_i \varphi = \int_{supp\,\Psi_i} \Psi_i \varphi = \int_{supp\,\Psi_i} \Psi_i(\varphi - p), \tag{14}$$

so that we can bound, using (B3) and (11),

$$|\int_\Omega \Psi_i \varphi| \leq C\|\Psi_i\|_{0,\Omega} \inf_{p\in P} \|\varphi - p\|_{0,supp\,\Psi_i} \leq C 2^{-jr}\|\varphi\|_{r,\Omega}. \tag{15}$$

By plugging (15) into (13) we get (B4). □

In view of lemma 1 it is now easy to construct liftings Ψ_i of ψ_i satisfying (B1-B4). This can be done for instance by locally mapping a subset of Ω onto a reference square, in such a way that the boundary portion coinciding with the support of ψ_i is mapped onto one edge, building a function with the required properties on the reference square, and then mapping it back on the domain. In this case the space P would consist in the images through the mapping of polynomials of degree $\leq R-1$ on the reference square. For more details see [6].

The following lemma holds.

Lemma 2 *Let $v \in L^2(\Omega)$. Let $\{\Psi_i, \quad i \in I = \cup_{j\geq 0} I_j\}$ be a set of functions satisfying assumptions (B1 – B4). Then it holds*

$$\sum_j 2^{-2j} \sum_{i\in I_j} | < v, \nabla\Psi_i > |^2 \leq C\|v\|^2_{0,\Omega}. \tag{16}$$

The proof of lemma 2 consists in expanding v in terms of a suitable wavelet basis (or frame) for $L^2(\Omega)$, and then applying the Schur Lemma for bounding the sum on the left hand side of (16). For more details see [6].

Let us now introduce the following symmetric positive semi-definite bilinear form $\sigma : L^2(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$.

$$\sigma(f,g) = \sum_j 2^{-2j} \sum_{i\in I_j} < f, \Psi_i >< g, \Psi_i > . \tag{17}$$

Corresponding to σ we can define the seminorm

$$|u|_\sigma = \sqrt{\sigma(u,u)}. \tag{18}$$

The following estimate on the trace of the normal derivative holds:

Lemma 3 *There exists a positive constant C_σ such that for all $u \in H^1(\Omega)$ with $\Delta u \in H^{-1/2+\varepsilon}(\Omega)$, $\varepsilon > 0$ the following bound holds:*

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{-1/2,\Gamma} \leq C_\sigma(\|u\|_{1,\Omega} + |\Delta u|_\sigma). \tag{19}$$

Proof By using norm equivalence (3) we bound the left hand side of (19) with

$$\left\| \frac{\partial u}{\partial \nu} \right\|^2_{-1/2,\Gamma} \leq \sum_j 2^{-j} \sum_{i \in I_j} | < \frac{\partial u}{\partial \nu}, \psi_i > |^2. \tag{20}$$

In order to bound the right hand side of (20), since $\Delta u \in H^{-1/2+\varepsilon}(\Omega)$, $\varepsilon > 0$, we can apply Green's formula to the term $< \frac{\partial u}{\partial \nu}, \psi_i >$, and we get

$$2^{-j/2} | < \frac{\partial u}{\partial \nu}, \psi_i > | \leq 2^{-j} | < \nabla u, \nabla \Psi_i > | + 2^{-j} | < \Delta u, \Psi_i > |. \tag{21}$$

Now, thanks to (16) we have

$$\sum_j 2^{-j2} \sum_{i \in I_j} | < \nabla u, \nabla \Psi_i > |^2 \leq C \|\nabla u\|^2_{0,\Omega} \leq C \|u\|^2_{1,\Omega}. \tag{22}$$

and by plugging (21) in (20), thanks to (22), (17) and (18), we get (19). □

Analogously to lemma 2, the following bound can also be proven (see [6]) by expanding u with respect to a suitable wavelet frame of $L^2(\Omega)$ and then applying the Schur lemma.

Lemma 4 *If the functions Ψ_i, $i \in I = \cup_{j \geq 0} I_j$ satisfy assumptions (B1 – B4) then for $u \in (H^1(\Omega))'$ it holds*

$$|u|_\sigma \leq C \|u\|_{-1,*}. \tag{23}$$

3 A New Coercive Formulation of the Lagrange Multiplier Method

For notational simplicity we will consider the following simple model problem. Given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$ find u verifying

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ u = g \qquad \text{on } \Gamma. \end{cases} \tag{24}$$

For such a problem we consider the Lagrange multiplier variational formulation ([1]) :

Problem 1 Find $u \in H^1(\Omega)$ and $\lambda \in H^{-1/2}(\Gamma)$ such that for all $v \in H^1(\Omega)$ and $\mu \in H^{-1/2}(\Gamma)$ one has

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \lambda v = \int_\Omega f v \\ \displaystyle\int_\Gamma \mu u \qquad\qquad\quad = \int_\Gamma \mu g \end{cases} \tag{25}$$

It is well known that problem 1 has a unique solution (u, λ) satisfying the relation

$$\lambda - \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Gamma. \tag{26}$$

We will denote by $a : (H^1(\Omega) \times H^{-1/2}(\Gamma))^2 \longrightarrow \mathbb{R}$ the bilinear form associated with problem (1), that is

$$a(u, \lambda; v, \mu) = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \lambda v + \int_\Gamma \mu u. \tag{27}$$

The bilinear form a is not coercive. Existence and uniqueness of the solution of problem 1 rely on the inf-sup condition

$$\inf_{\lambda \in H^{-1/2}(\Gamma)} \sup_{u \in H^1(\Omega)} \frac{\int_\Gamma \lambda u}{\|\lambda\|_{-1/2,\Gamma} \|u\|_{1,\Omega}} \geq \beta > 0. \tag{28}$$

The aim of this section is to introduce an equivalent coercive formulation of problem 1. In order to do that we start by considering the following linear space

$$V = \{u \in H^1(\Omega), \quad \text{s.t. } \exists\, \varepsilon = \varepsilon(u) > 0 : \quad \Delta u \in H^{-1/2+\varepsilon}(\Omega)\}, \tag{29}$$

which we equip with the norm

$$\|u\|_V^2 = \|u\|_{1,\Omega}^2 + |\Delta u|_\sigma^2. \tag{30}$$

We remark that V is only a *prehilbertian* space, since it is clearly not complete. We will write $\mathcal{V} = V \times H^{-1/2}(\Gamma)$ and we will denote by

$$|||u, \lambda|||^2 = \|u\|_{1,\Omega}^2 + |\Delta u|_\sigma^2 + \|\lambda\|_{-1/2,\Gamma}^2. \tag{31}$$

the corresponding norm.

With such a notation we can introduce the following regularized problem:

Problem 2 Find $(u,\lambda) \in \mathcal{V} = V \times H^{-1/2}(\Gamma)$ such that for all (v,μ) in $\mathcal{V}$ the following equation holds:

$$a_{STAB}(u,\lambda;v,\mu) = F_{STAB}(v,\mu) \tag{32}$$

with the bilinear form $a_{STAB} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ defined by

$$a_{STAB}(u,\lambda;v,\mu) = a(u,\lambda;v,\mu) + (u,v)_{\frac{1}{2}} + \sigma(\Delta u, \Delta v) \tag{33}$$

$$+\gamma(\lambda - \frac{\partial u}{\partial \nu}, \mu - \frac{\partial v}{\partial \nu})_{-\frac{1}{2}}. \tag{34}$$

and $F_{STAB} : \mathcal{V} \longrightarrow \mathbb{R}$ defined by

$$F_{STAB}(v,\mu) = \int_\Omega fv + \int_\Gamma g\mu + (g,v)_{\frac{1}{2}} - \sigma(f, \Delta v). \tag{35}$$

The following lemma holds.

Lemma 5 *There exists a $\bar\gamma > 0$ such that for all γ, $0 < \gamma < \bar\gamma$ the bilinear form $a_{STAB} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ is continuous and coercive with respect to the norm $|||\cdot|||$, that is for all (u,λ), $(v,\mu) \in \mathcal{V}$ it holds*

$$|a_{STAB}(u,\lambda;v,\mu)| \le L|||u,\lambda||| \cdot |||v,\mu|||, \tag{36}$$

$$|a_{STAB}(v,\mu;v,\mu)| \ge \alpha|||v,\mu|||^2, \tag{37}$$

for some positive constants α and L(depending on γ). Problem 2 has a unique solution which coincides with the solution of problem 1.

Proof It is quite easy to check that the solution (u,λ) of problem 1 belongs to $\mathcal{V}$ and that it is also a solution of problem 2. The uniqueness of the solution immediately descends once (36) and (37) are proven. Continuity descends by applying classical trace theorems ([24]) and (19). The only non immediate claim is the coercivity (37) of the biliner form a_{STAB}.

We start by remarking that for all $u \in H^1(\Omega)$ we have

$$\int_\Omega |\nabla u|^2 + (u,u)_{\frac{1}{2}} \ge C|u|^2_{1,\Omega} + A_{1/2}\|u\|^2_{1/2,\Gamma} \ge C_1\|u\|^2_{1,\Omega}. \tag{38}$$

Recalling that for any norm $\|\cdot\|$ we have $\|a-b\|^2 \ge 1/2\|a\|^2 - \|b\|^2$, we can then give the following lower bounds.

$$a_{STAB}(v,\mu;v,\mu) \ge C_1\|v\|^2_{1,\Omega} + |\Delta v|^2_\sigma + \tag{39}$$

$$+\gamma A_{-1/2}\left\|\mu - \frac{\partial v}{\partial \nu}\right\|^2_{-1/2,\Gamma} \ge \tag{40}$$

$$\ge C_1\|v\|^2_{1,\Omega} + \frac{A_{-1/2}\gamma}{2}\|\mu\|^2_{-1/2,\Gamma} + \tag{41}$$

$$|\Delta v|^2_\sigma - A_{-1/2}\gamma\left\|\frac{\partial v}{\partial \nu}\right\|^2_{-1/2,\Gamma}. \tag{42}$$

We can now apply lemma 3 which implies

$$a_{STAB}(v,\mu;v,\mu) \geq (C_1 - \gamma A_{-1/2}C_\sigma)\|v\|^2_{1,\Omega} \quad (43)$$
$$+(1-\gamma A_{-1/2}C_\sigma)|\Delta v|^2_\sigma + \frac{\gamma}{2}A_{-1/2}\|\mu\|^2_{-1/2,\Gamma} \quad (44)$$

By choosing $\bar{\gamma} > 0$ in such a way that $(C_1 - A_{-1/2}\gamma C_\sigma) > 0$ and $(1 - A_{-1/2}\gamma C_\sigma) > 0$ we get (37). □

4 Stable Discretization

Due to the coercivity of the bilinear form (34), any discretization space can in principle be used in a Galerkin discretization of problem 2. *We remark that the discretization spaces do not need to be in any way related to the multiscale decomposition used to derive the regularized form 2.* However, the bilinear form $(\cdot,\cdot)_{\frac{1}{2}}$ and $(\cdot,\cdot)_{-\frac{1}{2}}$ are infinite sums and therefore they are not computable in practice. In order to write a computable discrete problem we need to approximate them with suitable finite sums. Such approximation needs to be *uniformly coercive* (that is, the coercivity constant of the discrete bilinear form need to be independent of the mesh-size) so that they still lead to well posed problems and so that optimal error estimates can be obtained.

4.1 The general case

We will consider in the following two families V_h and Λ_h of finite dimensional subspaces of V and $H^{-1/2}(\Gamma)$ respectively, depending on the discretization parameter $h = (h_u, h_\lambda)$. In order to construct *uniformly coercive* approximations of the bilinear forms $(\cdot,\cdot)_{\frac{1}{2}}$ and $(\cdot,\cdot)_{-\frac{1}{2}}$ we will assume that the approximation spaces U_h and Λ_h satisfy the following inverse inequalities.

(A1) For some $S > 1/2$ we have $V_h|_\Gamma \subset H^S(\Gamma)$, and the following inverse inequality holds: for all $0 \leq r < s \leq S$ and for all $v \in V_h$ we have

$$\|v\|_{s,\Gamma} \leq C h_u^{r-s}\|v\|_{r,\Gamma}. \quad (45)$$

(A2) For some $T > -1/2$, $\Lambda_h \subset H^T(\Gamma)$ and the following inverse inequality holds:
for all $-1/2 \leq r < s \leq T$ and for all $\eta \in \Lambda_h$ we have

$$\|\eta\|_{s,\Gamma} \leq C h_\lambda^{r-s}\|\eta\|_{r,\Gamma}. \quad (46)$$

Assumption (A2) is somewhat out of the usual framework in which discretization of PDEs is stated. However it can be proved that (A2) is implied by the following more classical assumption

(A2bis) For some $T' > 1/2$, $\Lambda_h \in H^{T'}(\Gamma)$, and for all $0 \le r < s \le T'$ and for all $\eta \in \Lambda_h$ we have

$$\|\eta\|_{s,\Gamma} \le C h_\lambda^{r-s} \|\eta\|_{r,\Gamma}. \tag{47}$$

Proposition 2 *(A2bis) implies (A2) with $T = T'$.*

Proof Let $\eta \in \Lambda_h$. It holds

$$\|\eta\|_{0,\Gamma} = \frac{\int \eta^2}{\|\eta\|_{0,\Gamma}} \le \frac{\|\eta\|_{-1/2,\Gamma}\|\eta\|_{1/2,\Gamma}}{\|\eta\|_{0,\Gamma}} \le C h_\lambda^{-1/2} \|\eta\|_{-1/2,\Gamma}. \tag{48}$$

The result is then proved for $r = 0$ and $s = -1/2$. Combining (48) with (47), (46) is proven for $r = T'$ and $s = -1/2$. By interpolation one gets (46) for the other values of r and s □

Remark 4 Unlike the usual notation, the h_u, h_λ which we use to parametrize the discretization are the "smallest" mesh-sizes, that is the parameters appearing in the inverse inequalities (45) and (46). We make *no assumption* on the "largest" mesh-size (which would appear in a direct approximation estimate). In particular we make no assumption on the ratio between "largest" and "smallest" mesh-size. In our formulation we allow extremely non uniform discretizations, for which we still get optimal error estimates (see corollary 2).

Given discretization spaces V_h and Λ_h satisfying assuptions (A1) and (A2) it is possible to express an $H^{1/2}(\Gamma)$ equivalent norm of the traces of elements of V_h and an $H^{-1/2}(\Gamma)$ equivalent norm of the elements of Λ_h by means of computable bilinear forms (that is, of finite sums).

We first stress out that if one truncates the infinite sum appearing in (3) the following bounds do trivially hold. For any J and M and for any $g \in H^{1/2}(\Gamma)$ and $\lambda \in H^{-1/2}(\Gamma)$ we have

$$\sum_{j \le J} 2^j \sum_{i \in I_j} | < g, \psi_i > |^2 \le B_{1/2} \|g\|^2_{1/2,\Gamma} \tag{49}$$

and

$$\sum_{j \le M} 2^{-j} \sum_{i \in I_j} | < \lambda, \psi_i > |^2 \le B_{-1/2} \|\lambda\|^2_{-1/2,\Gamma}. \tag{50}$$

A reverse inequality is the object of the following lemma.

Lemma 6 *Let U_h be a family of finite dimensional subspaces of $H^s(\Gamma)$, $-R < s < R$, depending on a parameter h. Assume that $U_h \subset H^{s+t}(\Gamma)$ for some $t > 0$, with $s + t < R$, and that the following inverse inequality holds for a constant C independent of h : $f \in U_h$ implies*

$$\|f\|_{s+t,\Gamma} \leq Ch^{-t}\|f\|_{s,\Gamma}. \tag{51}$$

Then for all h there exists $J(h)$ such that the following bound holds for all $f \in U_h$:

$$\sum_{j\leq J(h)} 2^{2js} \sum_{i\in I_j} | < f, \psi_i > |^2 \geq C\|f\|^2_{s,\Gamma}, \tag{52}$$

with C positive constant independent of h.

Proof Let $f \in U_h$. By (3), $f \in H^{s+t}(\Gamma)$ implies that for any j we have

$$2^{2js} \sum_{i\in I_j} | < f, \psi_i > |^2 \leq 2^{-2jt} B_{s+t}\|f\|^2_{s+t,\Gamma}. \tag{53}$$

Let J be fixed. By applying (51) and (53) we get

$$\sum_{j>J} 2^{2js} \sum_{i\in I_j} | < f, \psi_i > |^2 \leq B_{s+t}\|f\|^2_{s+t,\Gamma} \sum_{j>J} 2^{-2jt} \leq \tag{54}$$

$$\leq Ch^{-2t}2^{-2Jt}\|f\|^2_{s,\Gamma}. \tag{55}$$

Plugging (55) in (3) we obtain

$$A_s\|f\|^2_{s,\Gamma} \leq \sum_{j\leq J} 2^{2js} \sum_{i\in I_j} | < f, \psi_i > |^2 + Ch^{-2t}2^{-2Jt}\|f\|^2_{s,\Gamma}, \tag{56}$$

that is

$$\sum_{j\leq J} 2^{2js} \sum_{i\in I_j} | < f, \psi_i > |^2 \geq (A_s - Ch^{-2t}2^{-2Jt})\|f\|^2_{s,\Gamma}. \tag{57}$$

By choosing $J = J(h)$ in such a way that $(A_s - Ch^{-t}2^{-2Jt}) > 0$, we get (52). □

By applying lemma 6 with U_h equals respectively to $V_h|_\Gamma$ and Λ_h the following bounds are proven.

Corollary 1 *Let V_h and Λ_h satisfy respectively assumptions (A1) and (A2). Then $\exists$ $J(h_u)$ and $M(h_\lambda)$ such that $\forall u_h \in V_h$*

$$\sum_{j\leq J(h_u)} 2^j \sum_{i\in I_j} | < u, \psi_i > |^2 \geq C\|u_h\|^2_{1/2,\Gamma} \tag{58}$$

and $\forall\lambda \in \Lambda_h$

$$\sum_{j\leq M(h_\lambda)} 2^{-j} \sum_{i\in I_j} | < \lambda, \psi_i > |^2 \geq C\|\lambda_h\|^2_{-1/2,\Gamma} \tag{59}$$

We can then introduce the following bilinear forms

$$(u,v)_{\frac{1}{2},J} = \sum_{j\leq J} 2^j \sum_{i\in I_j} < u, \psi_i >< v, \psi_i > \tag{60}$$

$$(\lambda,\mu)_{-\frac{1}{2},M} = \sum_{j\leq M} 2^{-j} \sum_{i\in I_j} < \lambda, \psi_i >< \mu, \psi_i > \tag{61}$$

Remark that thanks to corollary 1, for $J = J(h_u)$ and $M = M(h_\lambda)$ the bilinear forms $(\cdot,\cdot)_{\frac{1}{2},J}$ and $(\cdot,\cdot)_{-\frac{1}{2},M}$ act on the elements of respectively V_h and Λ_h as an $H^{1/2}$ and an $H^{-1/2}$ scalar product respectively.

We also introduce a truncated form of the operator $\sigma(\cdot,\cdot)$ and of the seminorm $|\cdot|_\sigma$, namely we set

$$\sigma_M(u,v) = \sum_{j\leq M} 2^{-2j} \sum_{i\in I_j} < f, \Psi_i >< g, \Psi_i >, \tag{62}$$

$$|u|_{\sigma,M} = \sqrt{\sigma_M(u,u)}. \tag{63}$$

Analogously to lemma (19) it is possible to prove (the proof being essentially the same)

Lemma 7 *Let* $u \in H^1(\Omega)$ *with* $\Delta u \in L^2(\Omega)$, *then the following bound holds*

$$(\frac{\partial u}{\partial\nu}, \frac{\partial u}{\partial\nu})_{-\frac{1}{2},M} \leq C(\|u\|^2_{1,\Omega} + |\Delta u|^2_{\sigma,M}). \tag{64}$$

We can now introduce the following "mesh dependent" norm on $\mathcal{V}$:

$$|||u,\lambda|||^2_M = \|u\|^2_{1,\Omega} + |\Delta u|^2_{\sigma,M} + \|\lambda\|^2_{-1/2,\Gamma}. \tag{65}$$

We can then introduce the following stable discrete formulation of problem 1.

Problem 3 Find (u_h,λ_h) in $\mathcal{V}_h = V_h \times \Lambda_h$ such that for all (v_h,μ_h) in $\mathcal{V}_h$ one has

$$a^h_{STAB}(u_h,\lambda_h;v_h,\mu_h) = F^h_{STAB}(v_h,\mu_h) \tag{66}$$

with $a^h_{STAB} : \mathcal{V}\times\mathcal{V} \longrightarrow \mathbb{R}$ defined by

$$a^h_{STAB}(u,\lambda;v,\mu) = a(u,\lambda;v,\mu) + (u,v)_{\frac{1}{2},J} + \sigma_M(\Delta u,\Delta v) \tag{67}$$

$$+\gamma(\lambda - \frac{\partial u}{\partial\nu}, \mu - \frac{\partial v}{\partial\nu})_{-\frac{1}{2},M}, \tag{68}$$

and $F^h_{STAB} : \mathcal{V} \longrightarrow \mathbb{R}$, by

$$F^h_{STAB}(v,\mu) = \int_\Omega fv + \int_\Gamma \mu g + (g, v_h)_{\frac{1}{2},J} - \sigma_M(f, \Delta v), \qquad (69)$$

with $J = J(h_u)$ and $M = M(h_\lambda)$ given by corollary 1.

Also in the discrete case one can prove coercivity and continuity of the bilinear form. Namely, the following lemma holds:

Lemma 8 *There exists a $\bar{\gamma}$ such that for all γ, $0 < \gamma < \bar{\gamma}$ the bilinear form $a^h_{STAB} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ is continuous on $\mathcal{V}$ and coercive uniformly on h_u, h_λ on $\mathcal{V}_h$ with respect to the norm $|||\cdot|||_M$, ($M = M(h_\lambda)$), that is for all (u,λ) and (v,μ) in $\mathcal{V}$ we have*

$$|a^h_{STAB}(u,\lambda; v,\mu)| \leq L_1 |||u,\lambda|||_M \cdot |||v,\mu|||_M, \qquad (70)$$

and for all (u_h, λ_h) in $\mathcal{V}_h$

$$a^h_{STAB}(u_h,\lambda_h; u_h,\lambda_h) \geq \alpha_1 |||u_h,\lambda_h|||^2_M, \qquad (71)$$

(the constant α_1 depending on γ). Moreover problem 3 is consistent with problem 1, that is (u,λ) solution of problem 1 implies

$$a^h_{STAB}(u,\lambda; v,\mu) = F^h_{STAB}(v,\mu), \qquad \forall (v,\mu) \in \mathcal{V}. \qquad (72)$$

In view of (58), (59), and (64) the proof of lemma 8 is essentially the same as the proof of lemma 5. Since the space V_h is finite dimensional, it is also complete and therefore continuity and coercivity imply the existence and uniqueness of the discrete solution.

An immediate corollary of lemma 8 is the following:

Corollary 2 *Let (u,λ) be the solution of problem 1 and let (u_h,λ_h) be the unique solution of problem 3. Then the following error estimate holds*

$$|||u-u_h, \lambda-\lambda_h|||_M \leq C \inf_{(v_h,\eta_h)\in\mathcal{V}_h} |||u-v_h, \lambda-\eta_h|||_M \qquad (73)$$

Proof Let $(v_h,\eta_h) \in \mathcal{V}_h$. (71) implies that

$$\begin{aligned} |||u_h-v_h, \lambda_h-\eta_h|||^2_M &\leq \frac{1}{\alpha_1} a^h_{STAB}(u_h-v_h, \lambda_h-\eta_h; u_h-v_h, \lambda_h-\eta_h) \\ &= \frac{1}{\alpha_1} a^h_{STAB}(u-v_h, \lambda-\eta_h; u_h-v_h, \lambda_h-\eta_h) \\ &\leq \frac{L_1}{\alpha_1} |||u-v_h, \lambda-\eta_h|||_M \cdot |||u_h-v_h, \lambda_h-\eta_h|||_M. \end{aligned}$$

Dividing both sides by $|||u_h - v_h, \lambda_h - \eta_h|||_M$, since

$$|||u - u_h, \lambda - \lambda_h|||_M \leq |||u - v_h, \lambda - \eta_h|||_M + |||u_h - v_h, \lambda_h - \eta_h|||_M,$$

we get that

$$|||u - u_h, \lambda - \lambda_h|||_M \leq C|||u - v_h, \lambda - \eta_h|||_M,$$

for any $(v_h, \eta_h) \in \mathcal{V}_h$, which implies (73) □

More interesting is the following error estimate:

Corollary 3 *Let (u, λ) be the solution of problem 1 and let (u_h, λ_h) be the unique solution of problem 3. Then the following error estimate holds*

$$\|u - u_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \leq C \left(\inf_{\eta_h \in \Lambda_h} (\|\lambda - \eta_h\|_{-1/2,\Gamma}) \right. \tag{74}$$
$$\left. + \inf_{v_h \in V_h} \left(\|u - v_h\|_{1,\Omega} + \|\Delta(u - v_h)\|_{-1,*} \right) \right).$$

Proof By observing that $\|v\|_{1,\Omega} + \|\mu\|_{-1/2,\Gamma} \leq |||v, \mu|||_M$ for all M and that $|||v, \mu|||_M \leq |||v, \mu|||$, in view of (22), the bound (75) descends from (73).

□

We would like to remark that the bound (73) obtained for the stabilized method does not depend directly on the parameters h_u and h_λ in the inverse estimates (A1) and (A2). In particular, when non uniform discretizations are concerned, one still gets an optimal error estimate. Instead, these parameters affect the computational complexity of the method. In this respect we observe that a more efficient way of defining a^h_{STAB} in the case of non uniform discretizations can be introduced by adjusting the truncation level in the sums (60) and (61) to the local, rather than the global, mesh size, as we will see in section 4.3

4.2 Convergence of the Method

The aim of this section is to give some sufficient condition on the discretization spaces V_h and Λ_h in order to guarantee convergence with a given rate. We will denote by H_λ and H_u respectively the "largest mesh size" of the spaces Λ_h and V_h respectively.

As far as the space Λ_h for the approximation of the multiplier is concerned we can proceed as it is usually done. More precisely we can make the following assumption:

(C1) for some $S > 1/2$, for all s, $0 < s < S$ it holds for all $\lambda \in H^s(\Gamma)$

$$\inf_{\eta_h \in \Lambda_h} \|\lambda - \eta_h\|_{0,\Gamma} \le C H_\lambda^s \|\lambda\|_{s,\Gamma}. \tag{75}$$

Then, by a standard duality argument ([25]), (C1) implies that for all s, $0 \le s \le S$, $\lambda \in H^s(\Gamma)$ implies

$$\inf_{\eta_h \in \Lambda_h} \|\lambda - \eta_h\|_{-1/2,\Gamma} \le C H_\lambda^{s+1/2} \|\lambda\|_{s,\Gamma}. \tag{76}$$

As far as V_h is concerned, we are dealing with an approximation in the norm $\|\cdot\|_V$, which is not standard. Thanks to corollary 3, it is enough to consider the norm $\|\cdot\|_{1,\Omega} + \|\Delta \cdot\|_{-1,*}$ (which is also not standard). A sufficient condition for approximation with order $(s-1)$ is such a norm is given by the following theorem. Among the discretization spaces satisfying the assumptions of such a theorem we recall for instance the *Clough-Tocher Element* ([14,15]).

Theorem 1 *Assume $V_h \subset H^2(\Omega)$. Define $V_h^0 = V \cap H_0^2(\Omega)$, $T_h = V_h|_\Gamma$, $N_h = \frac{\partial}{\partial\nu}(V_h)|_\Gamma$, and let $\pi_h^V : L^2(\Omega) \to V_h^0$, $\pi_h^T : L^2(\Gamma) \to T_h$ and $\pi_h^N : L^2(\Gamma) \to N_h$ denote the three L^2 projections onto V_h^0, T_h and N_h respectively. Assume that for some $R \ge 3$ it holds*

(i) for all $t \in [0,2]$, for all s, $t < s \le R$

$$u \in H^t(\Omega) \cap H_0^2 \quad \Rightarrow \quad \|u - \pi_h^V(u)\|_{t,\Omega} \le C H_u^{s-t} \|u\|_{s,\Omega} \tag{77}$$

(ii) for all $t \in [0,3/2]$, for all s, $t < s \le R - 1/2$

$$\eta \in H^t(\Gamma) \qquad \Rightarrow \qquad \|\eta - \pi_h^T(\eta)\|_{t,\Gamma} \le C H_u^{s-t} \|\eta\|_{s,\Gamma} \tag{78}$$

(iii) for all $t \in [0,1/2]$, for all s, $t < s \le R - 3/2$

$$\eta \in H^t(\Gamma) \qquad \Rightarrow \qquad \|\eta - \pi_h^N(\eta)\|_{t,\Gamma} \le C H_u^{s-t} \|\eta\|_{s,\Gamma}. \tag{79}$$

Then, for all s, $2 < s \le R$ it holds

$$\inf_{v_h \in V_h} \left(\|u - v_h\|_{1,\Omega} + \|\Delta(u - v_h)\|_{-1,*} \right) \le C H_u^{s-1} \|u\|_{s,\Omega}. \tag{80}$$

Remark 5 Properties (ii) and (iii) in the previous theorem could be proven to descend from (i), but for simplicity we prefer to take them as an assumption.

Proof Let us define a suitable projector $\Pi_h : H^2(\Omega) \to V_h$ as follows:

$$\int_\Omega \Delta(\Pi_h u) \Delta v_h = \int_\Omega \Delta u \Delta v_h, \qquad \text{for all } v_h \in V_h^0 \tag{81}$$

$$\Pi_h u = \pi_h^T(u|_\Gamma) \quad \text{and,} \quad \frac{\partial \Pi_h u}{\partial \nu} = \pi_h^N\left(\left.\frac{\partial u}{\partial \nu}\right|_\Gamma\right), \qquad \text{on } \Gamma. \tag{82}$$

It is easy to see that (81) and (82) define a unique element of V_h. By a standard argument ([14]) we easily obtain the bound

$$\|u - \Pi_h u\|_{2,\Omega} \leq C H_u^{s-2} \|u\|_s. \tag{83}$$

By duality we can then bound the error in $H^1(\Omega)$. In fact, denoting by $e_h = u - \Pi_h(u)$ we have

$$\|e_h\|_{1,\Omega} = \sup_{\varphi \in (H^1(\Omega))'} \frac{\int_\Omega e_h \varphi}{\|\varphi\|_{-1,*}}. \tag{84}$$

Since $L^2(\Omega)$ is dense in $(H^1(\Omega))'$ we can restrict the supremum in (84) to elements of $L^2(\Omega)$. Given $\varphi \in L^2(\Omega)$, let us then consider the solution v_φ of

$$\begin{cases} \Delta^2 v_\varphi = \varphi & \text{in } \Omega, \\ v_\varphi = 0 & \text{on } \Gamma, \\ \dfrac{\partial v_\varphi}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases} \tag{85}$$

We observe that we have $v_\varphi \in H^3(\Omega)$ with

$$\|v_\varphi\|_{3,\Omega} \leq C\|\varphi\|_{-1,\Omega} \leq C\|\varphi\|_{-1,*}. \tag{86}$$

Moreover (86), (19) and Lemma 4 imply

$$\left\|\frac{\partial}{\partial \nu}(\Delta v_\varphi)\right\|_{-1/2,\Gamma} \leq C\|\Delta v_\varphi\|_{1,\Omega} + \|\Delta^2 v_\varphi\|_{-1,*} \leq C\|\varphi\|_{-1,*}. \tag{87}$$

We can write

$$\int_\Omega e_h \varphi = \int e_h \Delta^2 v_\varphi = \int_\Omega \Delta e_h \Delta v_\varphi + \int_\Gamma \frac{\partial}{\partial \nu}(\Delta v_\varphi) e_h - \int_\Gamma (\Delta v_\varphi) \frac{\partial}{\partial \nu} e_h.$$

Let us bound the three integrals at the right hand side. For all $v_h \in V_h^0$ we have

$$\int_\Omega \Delta e_h \Delta v_\varphi = \int_\Omega \Delta e_h \Delta(v_\varphi - v_h) \leq C\|e_h\|_{2,\Omega}\|v_\varphi - v_h\|_{2,\Omega}$$

from where, using (83) and (77) we get

$$\int_\Omega \Delta e_h \Delta v_\varphi \leq C\|e_h\|_{2,\Omega}\|v_\varphi - \pi_h^V v\|_{2,\Omega} \tag{88}$$

$$\leq C H_u^{s-2}\|u\|_{s,\Omega} H_u \|v_\varphi\|_{3,\Omega} \tag{89}$$

$$\leq C H_u^{s-1}\|u\|_{s,\Omega}\|\varphi\|_{-1,*}. \tag{90}$$

As far as the second term is concerned we have (using (87))

$$\int_{\Gamma} \frac{\partial}{\partial \nu}(\Delta v_{\varphi}) e_h \leq C \|\varphi\|_{-1,*} \|e_h\|_{1/2,\Gamma}.$$

Then, since $e_h|_{\Gamma} = u|_{\Gamma} - \pi_h^T(u|_{\Gamma})$ we have by (78)

$$\|e_h\|_{1/2,\Gamma} \leq C H_u^{s-1} \|u\|_{s,\Omega}$$

from which we get

$$\int_{\Gamma} \frac{\partial}{\partial \nu}(\Delta v_{\varphi}) e_h \leq C H_u^{s-1} \|u\|_{s,\Omega} \|\varphi\|_{-1,*}. \tag{91}$$

As far as the third term is concerned we can write

$$\int_{\Gamma} (\Delta v_{\varphi}) \frac{\partial e_h}{\partial \nu} \leq C \|\Delta v_{\varphi}\|_{1/2,\Gamma} \left\| \frac{\partial e_h}{\partial \nu} \right\|_{-1/2,\Gamma} \leq C \|v_{\varphi}\|_{3,\Omega} \left\| \frac{\partial e_h}{\partial \nu} \right\|_{-1/2,\Gamma}$$

Now we observe that $\frac{\partial e_h}{\partial \nu} = \frac{\partial u}{\partial \nu} - \pi_h^N(\frac{\partial u}{\partial \nu})$, and then, since π_h^N is the $L^2(\Gamma)$ orthogonal projection, by using again the standard duality argument of ([25]), we get

$$\left\| \frac{\partial e_h}{\partial \nu} \right\|_{-1/2,\Gamma} \leq C H_u^{s-1} \left\| \frac{\partial u}{\partial \nu} \right\|_{s-1/2,\Gamma} \leq C H_u^{s-1} \|u\|_{s,\Gamma}, \tag{92}$$

from which, using (86) we get

$$\int_{\Gamma} (\Delta v_{\varphi}) \frac{\partial e_h}{\partial \nu} \leq C H_u^{s-1} \|u\|_{s,\Gamma} \|\varphi\|_{-1,*}. \tag{93}$$

Combining (90),(91) and (93) with (84), we get that

$$\|u - \Pi_h u\|_{1,\Omega} \leq C H_u^{s-1} \|u\|_{s,\Omega}. \tag{94}$$

Let us now bound $\|\Delta(e_h)\|_{-1,*}$. It holds

$$\|\Delta(e_h)\|_{-1,*} = \sup_{\varphi \in H^1(\Omega)} \frac{\int_{\Omega} \Delta(e_h)\varphi}{\|\varphi\|_{1,\Omega}}.$$

Now we have

$$\begin{aligned}\int_{\Omega} \Delta(e_h)\varphi &= -\int_{\Omega} \nabla e_h \nabla \varphi + \int_{\Gamma} \frac{\partial e_h}{\partial \nu}\varphi \\ &\leq C \|e_h\|_{1,\Omega} \|\varphi\|_{1,\Omega} + \left\| \frac{\partial e_h}{\partial \nu} \right\|_{-1/2,\Gamma} \|\varphi\|_{1,\Omega},\end{aligned}$$

whence, using (92) and (94)

$$\|\Delta(e_h)\|_{-1,*} \leq C H_u^{s-1} \|u\|_{s,\Omega}.$$

□

The bounds (76) and (80) are optimal in the case of a uniform discretization. We observe that in the case of a non uniform discretization a better estimate can be obtained. For instance, in the case of a *Clough-Tocher* finite element discretization corresponding to a triangulation $\mathcal{T}$ composed by triangles T which are regular in shape (h_T/ρ_T bounded independently of the triangle $T \in \mathcal{T}$, h_T and ρ_T being respectively the diameter T and the radius of the largest ball contained in T), one can prove

$$\inf_{v_h \in V_h} \Big(\|u - v_h\|_{1,\Omega} + \|\Delta(u - v_h)\|_{-1,*} \Big) \le C \sum_{T \in \mathcal{T}} h_T^{s-1} \|u\|_{s,T}$$

4.3 More on the case of Non-Uniform Discretizations

As we already remarked, due to the good localization properties of wavelets, the result of lemma 6 can be sharpened to better take into account the case of non uniform discretizations. In fact, in such a case, a more efficient uniformly coercive approximation of the scalar products $(\cdot,\cdot)_{\frac{1}{2}}$ and $(\cdot,\cdot)_{-\frac{1}{2}}$ can be constructed by locally adjusting the truncation level to the local mesh size, that is by taking into account more levels where the mesh is finer and less levels where it is coarser.

Assume that we are given a discretization space $U_h \subset H^{s+t}(\Gamma)$ ($s + t \ge 0$) corresponding to a non-uniform discretization. More precisely, for $k = 1, \dots, N$ let $h_1 > h_2 > \dots > h_N$ be a decreasing sequence of mesh-size parameters, satisfying the decay property

$$h_{k+1} \le \frac{h_k}{2}. \tag{95}$$

Let U_h satisfy the following assumption: there exist subsets $\Gamma_k \subset \Gamma_{k+1} \subset \Gamma_N = \Gamma$ such that for each function u_h in U_h one has

$$\|u_h\|_{s+t,\Gamma_k} \le C h_k^{-t} \|u_h\|_{s,\Gamma_k} \tag{96}$$

with C independent of the h_k's. Moreover let $\Gamma_0 = \emptyset$.Roughly speaking, Γ_k is the portion of Γ corresponding to a discretization with mesh-size not finer than h_k, while $\Gamma_k \setminus \Gamma_{k-1}$ is the subset of Γ corresponfind to a discretization with mesh size $\sim h_k$.

For each $j = 1, \dots, +\infty$ and for all $k = 0, \dots, N$ let

$$I_j^{(k)} = \{ i \in I_j : supp\,(\psi_i) \subset \bar{\Gamma}_k \}. \tag{97}$$

(Remark that for $k = N$ we have $I_j^{(k)} = I_j$). $I_j^{(k)}$ is the set of indices of those wavelet functions at level j which do not "see" mesh sizes finer than h_k.

We aim at approximating $(\cdot,\cdot)_s$ with an approximation of the form

$$(\cdot,\cdot)_s \sim \sum_{k=0}^{N-1}\sum_{j=j_k}^{j_{k+1}-1} 2^{2js} \sum_{i\in I_j\setminus I_j^{(k)}} |<u_h,\psi_i>|^2 \tag{98}$$

(in such a way the truncation level is adapted to the local mesh size).

We can prove the following lemma.

Lemma 9 *There exist $j_1 < j_2 < \ldots < j_N$ (j_k depending on h_k) such that for all $u_h \in U_h$ we have (setting $j_0 = 0$)*

$$\sum_{k=0}^{N-1}\sum_{j=j_k}^{j_{k+1}-1} 2^{2js} \sum_{i\in I_j\setminus I_j^{(k)}} |<u_h,\psi_i>|^2 \geq C\|u_h\|^2_{s,\Gamma} \tag{99}$$

the constant C being independent of the mesh parameters h_k's.

Lemma 9 is immediate once the following lemma is proven.

Lemma 10 *Given $\delta > 0$ there exist $j_1,\ldots,j_N$ such that, setting $j_{N+1} = +\infty$,*

$$\sum_{k=0}^{N}\sum_{j=j_k}^{j_{k+1}-1} 2^{2js} \sum_{i\in I_j^{(k)}} |<u_h,\psi_i>|^2 \leq \delta\|u_h\|^2_{s,\Gamma}. \tag{100}$$

Proof We start by observing that we have:

$$|<u_h,\psi_i>|^2 \leq \|u_h\|^2_{s+t,supp\,\psi_i}\|\psi_i\|^2_{-s-t,\Gamma} \leq \tag{101}$$

$$\leq C\|u_h\|^2_{s+t,supp\,\psi_i}2^{-2(s+t)j}, \tag{102}$$

from which, since, in view of the definition of $I_j^{(k)}$, assumption (W2) implies that

$$\sum_{i\in I_j^{(k)}} \|\cdot\|^2_{s+t,supp\,\psi_i} \leq C\|\cdot\|_{s+t,\Gamma_k},$$

we get

$$2^{2sj}\sum_{i\in I_j^{(k)}} |<u_h,\psi_i>|^2 \leq 2^{-2tj}\|u\|^2_{s+t,\Gamma_k} \leq \tag{103}$$

$$\leq C2^{-2tj}h_k^{-2t}\|u\|^2_{s,\Gamma_k}. \tag{104}$$

After choosing a splitting of δ as

$$\delta = \sum_{k=1,N}\delta_k, \qquad \text{with} \qquad \delta_k \geq \delta_{k+1} \geq 0, \tag{105}$$

set

$$j_k = \min\{j : j \geq \frac{\log_2 C - \log_2 \delta_k}{2t} - \log_2 h_k\}. \tag{106}$$

We remark that thanks to (95) and (105) we have $j_k < j_{k+1}$. (106) implies that for $j \geq j_k$ we have $C 2^{-2jt} h_k^{-2t} \leq \delta_k$. Hence we have

$$\sum_{j=j_k}^{j_{k+1}} 2^{2js} \sum_{i \in I_j^{(k)}} |< u_h, \psi_i >|^2 \leq \sum_{j=j_k}^{\infty} 2^{2js} \sum_{i \in I_j^{(k)}} |< u_h, \psi_i >|^2 \tag{107}$$

$$\leq C h_k^{-2t} \|u\|_{s,\Gamma_k}^2 \sum_{j=j_k}^{\infty} 2^{-2tj} \tag{108}$$

$$\leq C h_k^{-2t} \|u\|_{s,\Gamma_k}^2 2^{-2tj_k} \tag{109}$$

$$\leq \delta_k \|u\|_{s,\Gamma_k}^2 \tag{110}$$

By adding over $k = 1, \ldots, N$ in view of (105) we get (100). □

Lemma 9 descends from lemma 10 as (52) follows from (57).

Once bound (99) holds, all the results of the previous section can be easily carried out to the case of the non uniform approximation of $(\cdot,\cdot)_{\frac{1}{2}}$ and $(\cdot,\cdot)_{-\frac{1}{2}}$ with an expression of the form (98).

Remark 6 Lemma 10 provides with one way of constructing suitable j_k's, which is not necessarily the optimal one. It would be interesting to consider the problem of selecting an optimal splitting (105) which, depending on the size of the Γ_k, leads to the least expensive approximation (98) (in terms of number of terms in the sum at the right hand side). In this respect we observe that the restriction $\delta_k \leq \delta_{k+1}$ in the splitting can be dropped, which might result in the need of introducing a different splitting $h'_1 > h'_2 > \cdots > h'_{N'}$ of the range of mesh-sizes.

5 Elimination of the Multiplier

It has been observed [27],[9] that several stabilized discretizations of the Lagrange multiplier method leads - for a suitable choice of the multiplier and after its elimination by static condensation - to a problem in the variable u which is closely related to the *Nitsche's Method* for the treatment of boundary condition, introduced in [26]. This is true also in our case, as shown in the following sections.

5.1 The continuous case

Let us consider the continuous regularized formulation (32). For notational simplicity let us rewrite problem (32) in the following form: find $(u,\lambda) \in \mathcal{V}$ such that for all $(v,\mu) \in \mathcal{V}$ we have

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \lambda v - \gamma(\lambda - \frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu})_{-\frac{1}{2}} + b(u,v) = F(v), \\ \int_\Gamma \mu u \qquad + \gamma(\lambda - \frac{\partial u}{\partial \nu}, \mu)_{-\frac{1}{2}} \qquad = \int_\Gamma \mu g, \end{cases} \tag{111}$$

where b and F are defined respectively by

$$b(u,v) = (u,v)_{\frac{1}{2}} + \sigma(\Delta u, \Delta v), \tag{112}$$

$$F(v) = \int_\Omega f v + (g,v)_{\frac{1}{2}} - \sigma(f, \Delta v), \tag{113}$$

We can now eliminate the multiplier by static condensation: remarking that we have $H^{-1/2}(\Gamma) = span < \widetilde{\psi}_i, \quad i \in I >$, we plug $\mu = \widetilde{\psi}_i, i \in I_j$, as test function in the second equation and we get (using the definition of $(\cdot,\cdot)_{-\frac{1}{2}}$, (10) and (5))

$$\gamma 2^{-j} < \lambda, \psi_i > = \gamma 2^{-j} < \frac{\partial u}{\partial \nu}, \psi_i > + < g - u, \widetilde{\psi}_i > . \tag{114}$$

It is easy to check that this implies:

$$\gamma(\lambda - \frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu})_{-\frac{1}{2}} = \int_\Gamma (g-u) \frac{\partial v}{\partial \nu}, \tag{115}$$

and

$$\int_\Gamma \lambda v = \int_\Gamma \frac{\partial u}{\partial \nu} v + \frac{1}{\gamma}[g-u, v]_{\frac{1}{2}}, \tag{116}$$

with $[\cdot,\cdot]_{\frac{1}{2}}$ defined by

$$[g,v]_{\frac{1}{2}} = \sum_{j \geq 0} 2^j \sum_{i \in I_j} < g, \widetilde{\psi}_i >< v, \widetilde{\psi}_i > . \tag{117}$$

Remark 7 We remark that the assumptions made on $\{\psi_i\}$ imply that $[\cdot,\cdot]_{\frac{1}{2}}$ is also an equivalent $H^{1/2}(\Gamma)$ scalar product.

By plugging (115) and (116) into the first equation of (111) we get the following formulation of the BVP: find $u \in V$ such that for all $v \in V$ we have

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \frac{\partial u}{\partial \nu} v + \int_\Gamma u \frac{\partial v}{\partial \nu} + \\ + \frac{1}{\gamma}[u,v]_{\frac{1}{2}} + b(u,v) = F(v) + \int_\Gamma g \frac{\partial v}{\partial \nu} + \frac{1}{\gamma}[g,v]_{\frac{1}{2}}.\end{aligned} \tag{118}$$

(118) presents a strong similarity with a classical method introduced by Nitsche [26] for the treatment of Dirichlet boundary conditions. In such a method, in the framework of finite elements, an equation of the form

$$\begin{aligned}\int_\Omega \nabla u_h \cdot \nabla v_h - \int_\Gamma \frac{\partial u_h}{\partial \nu} v_h + \int_\Gamma u_h \frac{\partial v_h}{\partial \nu} + \mathcal{B}_h(u_h, v_h) = \\ = \int_\Omega f v_h + \int_\Gamma g \frac{\partial v_h}{\partial \nu} + \mathcal{F}_h(v_h)\end{aligned}$$

is connsidered, where $\mathcal{B}_h$ and $\mathcal{F}_h$ are respectively a suitable mesh dependent bilinear form and a suitable mesh dependent linear form. In the continuous formulation obtained here, $\mathcal{B}_h$ and $\mathcal{F}_h$ are substituted respectively by $\frac{1}{\gamma}[u,v]_{\frac{1}{2}} + (u,v)_{\frac{1}{2}} + \sigma(\Delta u, \Delta v)$ and $\frac{1}{\gamma}[g,v]_{\frac{1}{2}} + (g,v)_{\frac{1}{2}} - \sigma(f, \Delta v)$.

5.2 The discrete case

It is also interesting to consider what happens if the above procedure is applied to the discrete formulation 3. In particular one can choose a discretization space V_h, satisfying assumption (A1), and set $\Lambda_h = \widetilde{V}_M$ for some $M > 0$, (which is equivalent to taking $h_\lambda = 2^{-M}$). It is simple to check that for such a particular choice of discretization, one can set $M(h_\lambda) = M$. The resulting discrete problem then reads:

$$\begin{cases} \int_\Omega \nabla u_h \cdot \nabla v_h - \int_\Gamma \lambda_h v_h + \\ \qquad - \gamma(\lambda_h - \frac{\partial u_h}{\partial \nu}, \frac{\partial v_h}{\partial \nu})_{-\frac{1}{2},M} + \\ \qquad\qquad + b_h(u_h, v_h) = F_h(v_h) \\ \int_\Gamma \mu_h u_h \qquad + \gamma(\lambda_h - \frac{\partial u_h}{\partial \nu}, \mu_h)_{-\frac{1}{2},M} = \int_\Gamma \mu_h g, \end{cases} \tag{119}$$

where b_h and F_h are defined respectively by

$$b_h(u,v) = (u,v)_{\frac{1}{2},J} + \sigma_M(\Delta u, \Delta v) \tag{120}$$

$$F_h(v) = \int_\Omega fv + (g,v)_{\frac{1}{2},J} - \sigma_M(f,\Delta v). \tag{121}$$

Now we can plug $\widetilde{\psi}_i$, $i \in \cup_{j=0}^M I_j$ as test function in the second equation: for $i \in I_j$, and thanks to (10) and (5) we get

$$\gamma 2^{-j} < \lambda_h - \frac{\partial u_h}{\partial \nu}, \psi_i > = < g - u_h, \widetilde{\psi}_i > . \tag{122}$$

We now remark that $\lambda_h = \sum_{i\in\cup_{j=0}^M I_j} < \lambda_h, \psi_i > \widetilde{\psi}_i$. This implies

$$\lambda_h = \widetilde{P}_M\left(\frac{\partial u_h}{\partial \nu}\right) + \frac{1}{\gamma}\sum_{j\leq M}\sum_{i\in I_j} 2^j < g - u_h, \widetilde{\psi}_i > \widetilde{\psi}_i. \tag{123}$$

Hence we have

$$\int \lambda_h v_h = \int \widetilde{P}_M\left(\frac{\partial u_h}{\partial \nu}\right) v_h + \frac{1}{\gamma}[g-u_h, v_h]_{1/2,M} \tag{124}$$

with $[g,v]_{1/2,M} = \sum_{j\leq M} 2^j \sum_{i\in I_j} < g, \widetilde{\psi}_i >< v, \widetilde{\psi}_i >$. On the other hand we have

$$\begin{aligned} \gamma \textstyle\sum_{j\leq M} 2^{-j} \sum_{i\in I_j} < \lambda_h - \frac{\partial u_h}{\partial \nu}, \psi_i >< \frac{\partial v_h}{\partial \nu}, \psi_i > = \\ \textstyle\sum_{j\leq M}\sum_{i\in I_j} < g - u_h, \widetilde{\psi}_i >< \frac{\partial v_h}{\partial \nu}, \psi_i > = \\ \textstyle\int P_M(g-u_h)\widetilde{P}_M\left(\frac{\partial v_h}{\partial \nu}\right), \end{aligned}$$

(remark that (5) implies $\int P_M uv = \int P_M u\widetilde{P}_M v = \int u\widetilde{P}_M v$.). The first equation in (119) then becomes:

$$\begin{aligned} \int \nabla u_h \nabla v_h - \int_\Gamma \widetilde{P}_M\left(\frac{\partial u_h}{\partial \nu}\right) P_M(v_h) + \\ + \int_\Gamma \widetilde{P}_M\left(\frac{\partial v_h}{\partial \nu}\right) P_M(u_h) + \frac{1}{\gamma}[u_h, v_h]_{1/2,M} + b_h(u_h,v_h) = \\ = F_h(v_h) + \frac{1}{\gamma}[g, v_h]_{1/2,M} + \int_\Gamma \widetilde{P}_M\left(\frac{\partial v_h}{\partial \nu}\right) P_M(g) \end{aligned} \tag{125}$$

In this discrete formulation, the terms $\int_\Gamma u_h \frac{\partial v_h}{\partial \nu}$ and $\int_\Gamma v_h \frac{\partial u_h}{\partial \nu}$ which appear in the Nitsche's formulation of a Dirichlet Bounday

Value Problem, have been substituted respectively by the approximations $\int_\Gamma P_M(u_h)\widetilde{P}_M\left(\frac{\partial v_h}{\partial\nu}\right)$ and $\int_\Gamma \widetilde{P}_M\left(\frac{\partial u_h}{\partial\nu}\right)P_M(v_h)$ where the parameter M can, a priori, be chosen arbitrarily (in the sense that for any choice of M, equation (125) is well posed). However, the choice of M affects the accuracy of the resulting scheme. In fact, the solution u_h of such an equation (125) verifies the following error estimate.

Corollary 4 *Assume that the wavelet functions ψ_i have L zero moments, that is we have*

$$\int_\Gamma \psi_i p = 0, \qquad \textit{for all polynomial } p,\ \deg(p) \le L-1. \tag{126}$$

Then for all s, $-1/2 < s \le L$, $\frac{\partial u}{\partial\nu} \in H^s(\Gamma)$ implies

$$\|u-u_h\|_1 \le C\left(\inf_{v_h\in V_h}\left(\|u-v_h\|_{1,\Omega} + \|\Delta(u-v_h)\|_{-1,*}\right) + \right.$$
$$\left. +2^{-M(s+1/2)}\left\|\frac{\partial u}{\partial\nu}\right\|_{s,\Gamma}\right). \tag{127}$$

Proof By construction u_h is the first component of the solution (u_h,λ_h) of problem 3 with $\Lambda_h = \tilde{V}_M$. The bound (127) immediately descends from (75) after observing that (126) implies

$$\inf_{\eta_h\in\Lambda_h}\left(\|\lambda-\eta_h\|_{-1/2,\Gamma}\right) \le \|\lambda - \tilde{P}_M(\lambda)\|_{-1/2,\Gamma} \le C2^{-M(s+1/2)}\|\lambda\|_{s,\Gamma}, \tag{128}$$

which in view of (26) gives us (127). □

As a consequence in formulation (125), the truncation level M has to be chosen in such a way that

$$2^{-M(s+1/2)}\left\|\frac{\partial u}{\partial\nu}\right\|_{s,\Gamma} \sim \inf_{v_h\in V_h}\left(\|u-v_h\|_{1,\Omega} + \|\Delta(u-v_h)\|_{-1,*}\right) \tag{129}$$

6 Conclusions

We introduced a stabilized Lagrange multiplier formulation of Dirichlet boundary value problem. In order to do that we took advantage of the features of wavelet bases, which allow to write down equivalent scalar products in Sobolev spaces of fractional index, (both positive and negative), in terms of wavelet coefficients. The stabilized bilinear form is continuous and coercive with respect to the norm $|||\cdot|||$

defined by (31). It is then a good starting point for an application of the multilevel preconditioners introduced in [19].

Discretization of the stabilized formulation is possible, and an approximation of the corresponding bilinear form (consisting in truncating at a suitable level the infinite sums appearing in its definition (34)), is shown to be uniformly coercive for a large class of approximation spaces. An optimal error estimate in the natural norm is then derived.

Finally, the problem in the unknown u resulting after the elimination of the multiplier from a suitable choice of the approximation space Λ_h is computed. As already observed in the case of other stabilizations of the Lagrange multiplier formulation, such a problem is closely related to the *Nitsche's method* for imposing Dirichlet boundary conditions ([26]).

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