STABLE DISCRETIZATIONS OF CONVECTION-DIFFUSION PROBLEMS VIA COMPUTABLE NEGATIVE-ORDER INNER PRODUCTS *

SILVIA BERTOLUZZA†, CLAUDIO CANUTO , AND ANITA TABACCO‡

Abstract. A new functional framework for consistently stabilizing discrete approximations to convection-diffusion problems was recently proposed by the authors. The key ideas are the evaluation of the residual in an inner product of the type $H^{-1/2}$ (unlike classical SUPG methods, which use elemental weighted L^2-inner products), and the realization of this inner product via explicitely computable multilevel decompositions of function spaces (such as those given by wavelets or hierarchical finite elements). In the present paper, we first provide further motivations for our approach. Next, we carry on a detailed analysis of the method, which covers all regimes (convection-dominated and diffusion-dominated). A consistent part of the analysis justifies the use of easily computable truncated forms of the stabilizing inner product. Numerical results, in close agreement with the theory, are given at the end of the paper.

Key words. Singularly perturbed problems, convection-diffusion problems, stabilized Galerkin methods, multiscale decompositions, wavelets.

AMS subject classifications. 65N30, 35B25, 42C15.

1. Introduction. The standard Galerkin discretization of convection-diffusion boundary value problems, in the convection-dominated regime, may generate spurious numerical oscillations which affect the discrete solution. This instability is allowed by the weakly-coercive nature of the problem in the energy norm (of type H^1). The bilinear form associated with the variational formulation of the problem has a coercivity constant (proportional to the diffusion coefficient) which is much smaller than the continuity constant (basically depending on the size of the velocity field). The loose control of the energy norm allows for discrete solutions having indeed exceedingly large energy, realized precisely through a large high-frequency component. The reason why the exact solution does not suffer from such oscillations – while having itself a large energy norm – is that the exact differential equation yields control on other norms than the energy one, such as the maximum norm or some bounded variation norm. Such norms do not blow up with the inverse of the diffusion parameter and their boundedness does not allow spurious oscillations. Therefore, in order to produce stable discrete solutions, the standard Galerkin scheme needs to be modified by some mechanism which provides an extra control in one of these norms.

A popular strategy for performing this modification is the *stabilization* of the discrete variational formulation, through the addition of consistent terms depending on the strong residual of the equation. The prototype of this class of methods is the SUPG (Streamline Upwind Petrov Galerkin) method ([11]; for an extensive review see, e.g., [22]). Galerkin/least-square methods [19] are a variant of this approach. Under suitable circumstances, it has been recognized ([6]) that the stabilized equations can be obtained by a standard Galerkin projection upon an augmented subspace,

* This work was partially supported by the following funds: EEC TMR Project *Wavelets and multiscale methods in numerical analysis and simulation*, CNR Grant 97.04849.ST74 *Modelli e metodi per la matematica e l'ingegneria*

† Istituto di Analisi Numerica del C.N.R., via Abbiategrasso 209, 27100 Pavia, Italy, e–mail: aivlis@dragon.ian.pv.cnr.it

‡Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy, e–mail: ccanuto,tabacco@polito.it

once the static condensation of the extra unknown (the 'unresolved scales') has been performed: in this way, the arbitrariness inherent in the tuning of the stabilization parameters is eliminated ([10]). These and other variants of the stabilization strategy share the property that a diffusion-independent control on the streamline derivative (the directional derivative along the characteristic field) is obtained for the discrete solution. Most (in some cases, all) of the spurious oscillations are removed, and the effect of the remaining ones is localized around boundary and internal singular manifolds; satisfactory results are obtained in a wide variety of situations of practical interest. The norm in which the streamline derivative is controlled is usually a mesh-dependent, weighted L^2-type discrete norm. While such an expression makes the computation of the stabilization terms particularly efficient (as it guarantees the maximum of localization with the minimum of computational overhead), yet it somehow hiddens the intimate nature of the stabilization. Indeed, the norm involved is obviously not a standard functional norm of some Sobolev type (although inverse inequality arguments and some heuristics relate it to an $H^{-1/2}$-type norm. We also refer to the recent paper [7] for a rigorous estimate in a Besov norm of order $1/2$ for the discrete solution produced by the residual-free bubble method). In addition, the discrete stabilized formulation cannot be directly recognized as the Galerkin approximation of an intrinsic (i.e., discretization-independent) stabilized formulation on the continuous level.

In [5], moving from an abstract setting for the stabilization of the non-coercive or weakly-coercive problems ([4]), we advocated the use of a classical norm of the $H^{-1/2}$-type in the stabilization of convection-diffusion problems. Such a norm is naturally suggested by space interpolation results applied to the bilinear form describing the convective part of the operator. We introduced a stabilized variational formulation of the exact convection-diffusion problem, involving precisely a $H^{-1/2}$-type inner product. Next, we considered a standard Galerkin approximation of the new formulation, and – in a model situation – we derived stability and error estimates for the discrete solutions, in a norm including the $H^{-1/2}$-norm of the streamline derivative. No discretization parameter is involved in such a stabilization procedure. The strong motivation for our approach was the current availability in the literature of a number of results, indicating the possibility of explicitely computing functional norms and inner products of negative and even fractional regularity index, using multilevel decompositions of function spaces ([14, 21, 8]).

The purpose of the present paper is two-fold. On one hand, we shall push forward the analysis of the stabilization method introduced in [5], by considering less restrictive situations and by deriving deeper results. On the other hand, we shall carefully investigate the influence of the multilevel structure of the chosen $H^{-1/2}$-type inner product upon the stabilization mechanism. In this way, the efficiency of the calculation of the stabilization terms is improved.

In Section 2, we review some estimates on the exact solution of the convection-diffusion problem and by interpolatory arguments, we establish a uniform (i.e., diffusion-independent) bound in a Besov norm of order $1/2$. This result obviously implies a uniform bound on the streamline derivative of the solution in the appropriate norm of index $-1/2$, thus providing further support for our choice of the functional framework for the stabilization. Next, we introduce the appropriate functional spaces, as well as the stabilized formulation of the problem. In the next section, we deal with the Galerkin discretization of the stabilized problem. Improving over [5], we discuss the full range of regimes, from 'diffusion-dominated' to 'convection-dominated'. The structure of the stabilized parameter is identified, in order to obtain optimal error estimates in all regimes.

Section 4 is devoted to the numerical realization of the proposed multilevel stabilization mechanism. The $H^{-1/2}$-type inner product is an infinite sum of 'layers', each one depending on a single scale of details. Obviously, it would be impossible to compute all the contributions. Furthermore, contributions coming from finer and finer scales would be exceedingly expensive to compute, whereas contributions from coarse scales, although cheap, would reduce the desired localized structure of the stabilization. We can prove that only a finite number of layers are really needed for the stabilization, and they are associated with scales that concentrate around the finest representable scale within the discretization space. The theoretical prediction is confirmed by the examples described in the last section. At first, we prove that, in a model one-dimensional situation, the use of a particular $H^{-1/2}$-type inner product based on piecewise-linear interpolatory wavelets yields stabilization terms which *coincide* with those produced by the standard SUPG method. This seems to us an interesting result, in that it sheds new light on classical stabilization methods from the '$H^{-1/2}$-perspective'. Next, we use a by now popular family of biorthogonal spline wavelets to construct the multilevel inner product; we investigate numerically the sensitiveness to the stabilization parameter, as well as the finite truncation of the layers.

2. Convection-diffusion problems. Let Ω be a bounded open set in $\mathbb{R}^n$ ($n \geq 1$), whose boundary $\Gamma = \partial\Omega$ is a piecewise $\mathcal{C}^1$ manifold. Let $\nu > 0$ be a given constant, $\boldsymbol{a} \in (W^{1,\infty}(\Omega))^n$ be a given vector field, $b \in L^\infty(\Omega)$ be a given scalar function. For any $f \in H^{-1}(\Omega)$, we are interested in solving the following model problem:

$$\begin{cases} -\nu\Delta u + \boldsymbol{a}\cdot\nabla u + bu = f\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \Gamma. \end{cases} \tag{2.1}$$

Introducing the bilinear form

$$\begin{aligned} &a \;:\; H^1_0(\Omega)\times H^1_0(\Omega) \to \mathbb{R} \\ &a(w,v) = \nu\int_\Omega \nabla w\cdot\nabla v\,dx + \int_\Omega \boldsymbol{a}\cdot\nabla w\, v\,dx + \int_\Omega b\,w\,v\,dx \end{aligned} \tag{2.2}$$

and the duality pairing $\langle\cdot,\cdot\rangle$ between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, we look for the solution $u \in H^1_0(\Omega)$ of the variational problem

$$a(u,v) = \langle f, v\rangle\,, \qquad \forall v \in H^1_0(\Omega). \tag{2.3}$$

Under the assumption of the existence of a constant $\gamma > 0$ such that

$$-\frac{1}{2}\nabla\cdot\boldsymbol{a} + b \geq \gamma \qquad \text{a.e. in } \Omega, \tag{2.4}$$

the form a is continuous and coercive in $H^1_0(\Omega)$. This implies the existence and uniqueness of the solution of Problem (2.3).

2.1. Regularity results. We will keep track of the influence of the constant ν: to this end, we shall use the notation $A \lesssim B$ to indicate $A \leq cB$ for a suitable constant c independent of ν (but possibly depending on $\|\boldsymbol{a}\|_{L^\infty(\Omega)}$, γ and the diameter of Ω); the symbol $A \asymp B$ means $A \lesssim B$ and $B \lesssim A$.

Let us first collect some estimates on the solution of Problem (2.1), depending on the regularity of the data f. The standard energy estimate is

$$\nu\|\nabla u\|_{L^2(\Omega)} + \nu^{1/2}\|u\|_{L^2(\Omega)} \lesssim \|f\|_{H^{-1}(\Omega)}; \tag{2.5}$$

if $f \in L^2(\Omega)$, it can be improved as follows

$$\nu^{1/2}\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}; \tag{2.6}$$

then, using the equation, one gets

$$\nu^{3/2}\|\Delta u\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}. \tag{2.7}$$

If $f \in L^p(\Omega)$ for some $p > 2$ and the strengthened assumption

$$-\frac{1}{p}\nabla\cdot\boldsymbol{a} + b \geq \gamma \qquad \text{a.e. } \Omega$$

holds, then multiplication of the equation by $|u|^{p-2}u$ yields

$$((p-1)\nu)^{1/p}\|\,|u|^{\frac{p-2}{2}}\nabla u\,\|^{2/p}_{L^2(\Omega)} + \|u\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}$$

(see [22], Lemma 1.18); if, in addition, $f \in L^\infty(\Omega)$ and $b \geq \gamma$ a.e. in Ω, letting $p \to +\infty$ in the previous estimate we get

$$\|u\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^\infty(\Omega)}. \tag{2.8}$$

On the other hand, if $f \in W^{1,1}(\Omega)$ and γ is large enough, then the following estimate can be obtained ([3])

$$\|u\|_{W^{1,1}(\Omega)} \lesssim \|f\|_{W^{1,1}(\Omega)}. \tag{2.9}$$

By applying the interpolation theory of function spaces, we get the intermediate estimate between (2.8) and (2.9)

$$\|u\|_{[L^\infty(\Omega),W^{1,1}(\Omega)]_{1/2}} \lesssim \|f\|_{[L^\infty(\Omega),W^{1,1}(\Omega)]_{1/2}}, \tag{2.10}$$

where $[X,Y]_\theta$ denotes the intermediate space of order θ between X and Y obtained by the complex interpolation method [23]. Unfortunately, the space $[L^\infty(\Omega),W^{1,1}(\Omega)]_{1/2}$ seems not to coincide with any of the known function spaces of the Besov or Triebel-Lizorkin type, in particular with $H^{1/2}(\Omega)$. (The same situation occurs if we replace the complex interpolation method by the real one.) The best result we can state is contained in next lemma. It involves the Besov spaces $B^s_{p,q}(\Omega)$, whose definition can be found, e.g., in [23], p. 92, eq. (1).

Lemma 2.1. *[24] The following topological inclusions hold*

$$H^{1/2}(\Omega) = B^{1/2}_{2,2}(\Omega) \subset [L^\infty(\Omega),W^{1,1}(\Omega)]_{1/2} \subset B^{1/2}_{2,\infty}(\Omega).$$

Proof. The right-hand side imbedding follows from the topological inclusions $L^\infty(\Omega) \subset B^0_{\infty,\infty}(\Omega)$ (obvious, by the definition of the second space) and $W^{1,1}(\Omega) \subset B^1_{1,\infty}(\Omega)$ ([23], p. 90, eq (10)), from the interpolation formula $[B^0_{\infty,\infty}(\Omega), B^1_{1,\infty}(\Omega)]_{1/2} = B^{1/2}_{2,2}(\Omega)$ ([23], p. 69, eq. (2)) and from the property that the interpolation functor preserves the inclusions.

For the left-hand side, we need other function spaces. Let $F^s_{p,q}(\Omega)$ be the Triebel-Lizorkin spaces (see, e.g., [23], p. 192, eq. (3)); let $\mathfrak{h}_1(\Omega) = F^0_{1,2}(\Omega) \subset L^1(\Omega)$ denote the local Hardy space, and let $\mathfrak{h}^1_1(\Omega) = F^1_{1,2}(\Omega)$ denote its lifted space, defined as

$$\mathfrak{h}^1_1(\Omega) = \{f \in \mathfrak{h}_1(\Omega) : \frac{\partial f}{\partial x_i} \in \mathfrak{h}_1(\Omega),\ i = 1,\ldots,n\}.$$

Firstly, we use the identities

$$[L^\infty(\Omega), L^{4/3}(\Omega)]_{2/3} = L^2(\Omega)\,, \qquad [L^2(\Omega), \mathfrak{h}_1(\Omega)]_{1/2} = L^{4/3}(\Omega)$$

(see [23], p. 204, eq. (8)) and Wolff's Theorem [26] to deduce

$$[L^\infty(\Omega), \mathfrak{h}_1(\Omega)]_{1/2} = L^2(\Omega)\,.$$

Next, by the so-called retraction method of interpolation theory (see [25], Chapter III), we obtain

$$[L^\infty(\Omega), \mathfrak{h}_1^1(\Omega)]_{1/2} = B^{1/2}_{2,2}(\Omega)\,.$$

Observing that $\mathfrak{h}_1^1(\Omega) \subset W^{1,1}(\Omega)$, we get the result. □

PROPOSITION 2.2. *Under the assumptions on the coefficients and the data which yield (2.8) and (2.9), one has*

$$\|u\|_{B^{1/2}_{2,\infty}(\Omega)} \lesssim \|f\|_{H^{1/2}(\Omega)}\,; \tag{2.11}$$

in particular, for any $\eta > 0$,

$$\|u\|_{H^{1/2-\eta}(\Omega)} \lesssim \|f\|_{H^{1/2}(\Omega)}\,,$$

(where the constant implied by the symbol $\lesssim$ depends on η).

Proof. It is enough to combine (2.10) with the previous lemma, and to recall the inclusion $B^s_{p,q}(\Omega) \subset B^r_{p,t}(\Omega)$, $s < r$, and p, q, t arbitrary. □

The simple example $-\nu u'' + u' = 1$ in $\Omega = (0,1)$, $u(0) = u(1) = 0$, shows that $\|u\|_{L^2(\Omega)} \asymp 1$, $\|u\|_{H^{1/2}(\Omega)} \asymp 1$, $\|u'\|_{L^2(\Omega)} \asymp \nu^{-1/2}$ and $\|u''\|_{L^2(\Omega)} \asymp \nu^{-3/2}$, i.e., the estimates (2.6), (2.7) and (2.11) are sharp. In particular, for the convection-diffusion problem (2.1), the bound $\|\boldsymbol{a}\cdot\nabla u\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$ is generally not true. This is in contrast with the behavior of the limit problem (as $\nu \to \infty$)

$$\begin{cases} \boldsymbol{a}\cdot\nabla u^* + bu^* = f\,, & \text{in } \Omega, \\ u^* = 0\,, & \text{on } \Gamma_- \end{cases}$$

(with $\Gamma_- := \{x \in \Gamma : \boldsymbol{a}\cdot\boldsymbol{n} < 0\}$, $\boldsymbol{n}$ denoting the normal vector to Γ pointing outwards), for which it is easy to obtain the estimate (with $\Gamma_+ := \{x \in \Gamma : \boldsymbol{a}\cdot\boldsymbol{n} > 0\}$)

$$\|\boldsymbol{a}\cdot\nabla u^*\|_{L^2(\Omega)} + \|u^*\|_{L^2(\Omega)} + \|(\boldsymbol{a}\cdot\boldsymbol{n})^{1/2}u^*\|_{L^2(\Gamma_+)} \lesssim \|f\|_{L^2(\Omega)}\,.$$

Note, however, that if $\Gamma_- \cup \Gamma_+ = \emptyset$, it is proven in [1] that

$$\|\boldsymbol{a}\cdot\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|\boldsymbol{a}\cdot\nabla f\|_{L^2(\Omega)}.$$

2.2. Stabilized formulations. It is convenient to introduce the following notation:

$$Dv = \boldsymbol{a}\cdot\nabla v \quad \text{and} \quad Lv = -\nu\Delta v + Dv + bv\,.$$

Firstly, we shall provide the reader with some 'semi-heuristic' considerations which lead in a natural way to introducing the proposed stabilized formulations. Next, we shall make the functional setting and the definitions precise.

It is well known that the origin of the troubles in the discretization of Problem (2.1) at small values of ν can be traced back to the large value of the ratio between the continuity and the coerciveness constants of the bilinear form a, which indeed scales as ν^{-1}. Hence, a route to overcoming those troubles can be found in trasforming the original problem into an equivalent one, associated with a new bilinear form (say A), for which the ratio between the characteristic constants is close to 1.

As far as the coerciveness is concerned, Proposition 2.2 suggests to add a term controlling some '$B^{1/2}_{p,q}$-type' norm of the solution to the usual energy norm (the norm which appears on the left-hand side of (2.6)). Indeed, thinking, e.g., of a finite element or finite difference discretization based on a mesh-size $h \gtrsim \nu$, the energy norm does allow the discrete solution to develop spurious grid oscillations with gradients of order h^{-1}, whereas the norm appearing on the left-hand side of (2.11) (and any stronger norm) doesn't.

Let us now deal with continuity. To this end, let us introduce the spaces $\widetilde{X}^0(\Omega) := H^1_0(\Omega)$, equipped with the norm $\|v\|_{\widetilde{X}^0(\Omega)} = \|v\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v\|_{L^2(\Omega)}$, and

$$\widetilde{X}^1(\Omega) := \{v \in H^1_0(\Omega) : Dv \in L^2(\Omega)\}$$

equipped with the norm $\|v\|_{\widetilde{X}^1(\Omega)} = \|v\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v\|_{L^2(\Omega)} + \|Dv\|_{L^2(\Omega)}$. Then, it is easily seen that the two following inequalities hold:

$$\begin{aligned} |a(w,v)| &\lesssim \|w\|_{\widetilde{X}^1(\Omega)}\|v\|_{\widetilde{X}^0(\Omega)}\,, &\quad \forall w \in \widetilde{X}^1(\Omega),\ \forall v \in \widetilde{X}^0(\Omega)\,,\\ |a(w,v)| &\lesssim \|w\|_{\widetilde{X}^0(\Omega)}\|v\|_{\widetilde{X}^1(\Omega)}\,, &\quad \forall w \in \widetilde{X}^0(\Omega),\ \forall v \in \widetilde{X}^1(\Omega)\,. \end{aligned}$$

Thus, we get a symmetric (in w and v) bound using the interpolation theory of function spaces, namely

$$|a(w,v)| \lesssim \|w\|_{[\widetilde{X}^1(\Omega),\widetilde{X}^0(\Omega)]_{1/2}}\|v\|_{[\widetilde{X}^1(\Omega),\widetilde{X}^0(\Omega)]_{1/2}}\,, \qquad \forall w,v \in [\widetilde{X}^1(\Omega),\widetilde{X}^0(\Omega)]_{1/2}\,.$$

By applying Baiocchi's theorem (see [2] or [20], Chapter I, Theorem 14.3), one can check that $[\widetilde{X}^1(\Omega),\widetilde{X}^0(\Omega)]_{1/2}$ coincides (algebraically and topologically) with the intermediate space

$$\widetilde{X}^{1/2}(\Omega) := \{v \in H^1_0(\Omega) : Dv \in [H^2(\Omega), H^{-1}(\Omega)]_{1/2} = (H^{1/2}_{00}(\Omega))'\}$$

equipped with the norm $\|v\|_{\widetilde{X}^{1/2}(\Omega)} = \|v\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v\|_{L^2(\Omega)} + \|Dv\|_{(H^{1/2}_{00}(\Omega))'}$. We note, however, that in space dimension higher than one, the equivalence of norms is not independent of ν; precisely, looking carefully at the proof of Baiocchi's theorem, one gets

$$\begin{aligned} \|v\|_{\widetilde{X}^{1/2}(\Omega)} &\lesssim \|v\|_{[\widetilde{X}^1(\Omega),\widetilde{X}^0(\Omega)]_{1/2}}\\ &\lesssim \|v\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v\|_{L^2(\Omega)} + \rho\|Dv\|_{(H^{1/2}_{00}(\Omega))'} \end{aligned} \tag{2.12}$$

with $\rho \asymp 1$ if $n = 1$ (and $\boldsymbol{a}$ smooth and non-zero), and $\rho \asymp \nu^{-1/2}$ in all other cases.

In conclusion, the bilinear form a is defined and continuous on $\widetilde{X}^{1/2}(\Omega)\times\widetilde{X}^{1/2}(\Omega)$, although in general the continuity constant is not independent of ν. However, such a lack of independence is not so relevant for our purposes, since we will show that the form a is ideed uniformly continuous with respect to ν, provided one of the arguments is in $H^{1/2}(\Omega)$ and the other one is in a finite dimensional approximation space (see

Proposition 3.1). This will be enough for our analysis. The important conclusion of the above discussion is that an intermediate space of the type $\widetilde{X}^{1/2}(\Omega)$ is a natural candidate for the functional setting we are looking for. Technically speaking, since we aim at designing a consistent perturbation of the bilinear form a, we shall actually replace Dv by Lv in the definition of the intermediate space and its norm. We will show that the two definitions are equivalent on the discrete level.

REMARK 2.3. The intimate motivation for the constant ρ in (2.12) being not independent of ν is that the operator D is anisotropic (i.e., it decreases regularity only along the streamlines defined by $\frac{dx}{ds} = \boldsymbol{a}(x)$), whereas we measure it in the (dual) norm of an isotropic Sobolev space. It is likely that uniform (with respect to ν) equivalence of norms could be obtained involving anisotropic function spaces (see,e.g., [23], chap.10, as well as the recent work [15]; see also [17]).□

We now come to the precise definition of our stabilized formulation; to this end, we describe the functional setting in which we will work. Let us introduce the family $\mathcal{H}^r(\Omega)$ for $r \geq 0$, setting $\mathcal{H}^0(\Omega) = L^2(\Omega)$, $\mathcal{H}^r(\Omega) = H^r(\Omega) \cap H_0^1(\Omega)$ for an integer $r \geq 1$, and extending the definition to any real $r > 0$ by interpolation; in particular, for $0 \leq r < 3/2$, one has

$$\mathcal{H}^r(\Omega) = \begin{cases} H^r(\Omega), & 0 \leq r < 1/2, \\ H_{00}^{1/2}(\Omega), & r = 1/2, \\ H_0^r(\Omega), & 1/2 < r \leq 1. \end{cases}$$

For $r < 0$, we set $\mathcal{H}^r(\Omega) = (\mathcal{H}^{-r}(\Omega))'$. We choose an inner product on $\mathcal{H}^r(\Omega)$; it will be denoted by $[\xi, \eta]_{\mathcal{H}^r(\Omega)}$, $\forall \xi, \eta \in \mathcal{H}^r(\Omega)$. Section 4 will be devoted to a thorough discussion on the choice of such products and their approximations. Next, for $0 \leq r \leq 1$, we define the space

$$X^r(\Omega) = \{v \in H_0^1(\Omega) : Lv \in \mathcal{H}^{r-1}(\Omega)\}\,;$$

recalling the estimate (2.5), it is natural to equip it with the norm

$$\|v\|_{X^r(\Omega)} = \|v\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v\|_{L^2(\Omega)} + \tilde{\delta}^{1/2}\|Lv\|_{\mathcal{H}^{r-1}(\Omega)}\,,$$

where $\tilde{\delta} > 0$ is a suitable parameter, whose definition will be made precise later on.

Let us assume that f belongs to $\mathcal{H}^{r-1}(\Omega)$ for some $0 \leq r \leq 1$. Then, we introduce the stabilized bilinear form

$$A : X^r(\Omega) \times X^r(\Omega) \to I\!R\,, \quad A(w,v) := a(w,v) + \tilde{\delta}[Lw, Lv]_{\mathcal{H}^{r-1}(\Omega)} \tag{2.13}$$

and the stabilized linear form

$$F : X^r(\Omega) \to I\!R\,, \quad F(v) := \langle f, v\rangle + \tilde{\delta}[f, Lv]_{\mathcal{H}^{r-1}(\Omega)}. \tag{2.14}$$

The trivial fact that the solution u of (2.1) is also the unique solution of the variational problem

$$\begin{cases} u \in X^r(\Omega) \\ A(u,v) = F(v), \quad \forall v \in X^r(\Omega) \end{cases} \tag{2.15}$$

will be important in the subsequent analysis. The form A is coercive in $X^r(\Omega)$; indeed, we have, for a suitable $\bar{\alpha} > 0$ independent of ν,

$$\begin{aligned} A(w,w) = \;& a(w,w) + \delta\|Lw\|^2_{\mathcal{H}^{1-r}(\Omega)} \\ & \geq \nu\|\nabla w\|^2_{L^2(\Omega)} + \gamma\|w\|^2_{L^2(\Omega)} + \tilde{\delta}\|Lw\|^2_{\mathcal{H}^{r-1}(\Omega)} \\ & \geq \bar{\alpha}\|w\|^2_{X^r(\Omega)}, \quad \forall w \in X^r(\Omega). \end{aligned} \tag{2.16}$$

In particular, this implies the uniqueness of the solution of Problem (2.15).

3. Discrete stabilized methods. Let $h \in (0, \operatorname{diam}\Omega]$ be a discretization parameter, and let V_h be a finite dimensional subspace of $H^1_0(\Omega)$. From now on, the symbol $\lesssim$ will indicate a bound independent of both ν *and* h. We distinguish two cases:
i) the *regular case* is characterized by the condition

$$V_h \subset \mathcal{H}^{3/2}(\Omega). \tag{3.1}$$

This occurs, e.g., for a space of $\mathcal{C}^1$–finite elements or splines of order ≥ 2 or sufficiently smooth wavelets. In this case, one has $Lv_h \in \mathcal{H}^{-1/2}(\Omega)$, $\forall v_h \in V_h$, hence, $V_h \subset X^{1/2}(\Omega)$. Thus, we are led to choose $r = 1/2$ in (2.13) – (2.16).
ii) the *non-regular case* is defined by the condition

$$V_h \subset \mathcal{H}^{3/2-\varepsilon}(\Omega) \ \forall\varepsilon > 0, \ \text{but } V_h \not\subset \mathcal{H}^{3/2}(\Omega). \tag{3.2}$$

This is the case of a locally smooth, globally $\mathcal{C}^0$–approximation, such as $\mathcal{C}^0$-finite elements, or spectral elements, or wavelet elements ([12]). Condition (3.2) implies $Lv_h \in \mathcal{H}^{-1/2-\varepsilon}(\Omega)$, $\forall v_h \in V_h$, so that $V_h \subset X^{1/2}(\Omega)$, for all $\varepsilon > 0$. Thus, we fix a sufficiently small $\varepsilon > 0$ and we set $r = 1/2 - \varepsilon$ in (2.13) – (2.16). Furthermore, we choose $\tilde{\delta}$ in the form $\tilde{\delta} = \delta h^{-2\varepsilon}$, where $\delta > 0$ is a suitable parameter whose selection will be discussed in the sequel.

The two cases can be formally unified by setting $\varepsilon = 0$ and $\tilde{\delta} = \delta$ in the regular case. We will proceed with such a notational convention.

After making the functional framework precise, we are led to consider the following Galerkin approximation of problem (2.15):

$$\begin{cases} u_h \in V_h \\ A(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h. \end{cases} \tag{3.3}$$

Existence and uniqueness of the solution of this problem, as well as a stability estimate in the $X^r(\Omega)$–norm, follow from the coerciveness (2.16) of A.

In order to study such an approximation, we will use the following continuity estimate for the bilinear form A.

PROPOSITION 3.1. *For any $w, v \in X^r(\Omega)$,*

$$|A(w,v)| \lesssim (\|w\|_{X^r(\Omega)} + \tilde{\delta}^{-1/2}\|w\|_{\mathcal{H}^{1-r}(\Omega)})\|v\|_{X^{1/2}(\Omega)}.$$

Proof. It is easily seen that

$$\begin{aligned} a(w,v) &= \nu\int_\Omega \nabla w\cdot\nabla v\,dx - \int_\Omega w\,\boldsymbol{a}\cdot\nabla v\,dx - \int_\Omega \nabla\cdot\boldsymbol{a}\,w\,v\,dx + \int_\Omega b\,w\,v\,dx \\ &= -a(v,w) + 2\nu\int_\Omega \nabla w\cdot\nabla v\,dx + 2\int_\Omega (b - \frac{1}{2}\nabla\cdot\boldsymbol{a})\,w\,v\,dx. \end{aligned}$$

Next, let us observe that, since $Lv \in \mathcal{H}^{r-1}(\Omega)$, one has

$$a(v,w) = {}_{H^{-1}(\Omega)}\langle Lv, w\rangle_{H^1_0(\Omega)} = {}_{\mathcal{H}^{r-1}(\Omega)}\langle Lv, w\rangle_{\mathcal{H}^{1-r}(\Omega)}.$$

Thus,

$$\begin{aligned} |a(w,v)| &\lesssim \|Lv\|_{\mathcal{H}^{r-1}(\Omega)}\|w\|_{\mathcal{H}^{1-r}(\Omega)} + \nu\|\nabla v\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \\ &\lesssim \big(\tilde{\delta}\|Lv\|^2_{\mathcal{H}^{r-1}(\Omega)} + \nu\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}} \times \\ &\qquad \times\big(\tilde{\delta}^{-1}\|w\|^2_{\mathcal{H}^{1-r}(\Omega)} + \nu\|\nabla w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}. \end{aligned}$$

Recalling the definition of the form $A(w,v)$, we easily conclude. □

The previous result is the basis for the following a-priori error estimate.

THEOREM 3.2. *Let u be the solution of Problem* (2.1) *and let u_h be the solution of the stabilized Galerkin Problem* (3.3). *Then*

$$\|u-u_h\|_{X^r(\Omega)} \lesssim \inf_{w_h\in V_h}\left(\|u-w_h\|_{X^r(\Omega)}+\tilde{\delta}^{-1/2}\|u-w_h\|_{\mathcal{H}^{1-r}(\Omega)}\right). \tag{3.4}$$

Proof. Let w_h be any element in V_h. Using (2.16), the Galerkin orthogonality property $A(u-u_h,v_h)=0$, $\forall v_h\in V_h$, and Proposition 3.1, we have

$$\begin{aligned}\|w_h-u_h\|^2_{X^r(\Omega)} &\lesssim A(w_h-u_h,w_h-u_h)=A(w_h-u,w_h-u_h)\\ &\lesssim \left(\|u-w_h\|_{X^r(\Omega)}+\tilde{\delta}^{-1/2}\|u-w_h\|_{\mathcal{H}^{1-r}(\Omega)}\right)\|w_h-u_h\|_{X^{1/2}(\Omega)}.\end{aligned}$$

Then, the bound easily follows. □

In order to estimate the rate of convergence of our stabilized approximation, let us furtherly assume that for some $\bar{s}\geq 3/2$, there exists a linear operator $\Pi_h:\mathcal{H}^{\bar{s}}(\Omega)\to V_h$ such that

$$\|v-\Pi_h v\|_{H^s(\Omega)} \lesssim h^{\bar{s}-s}|v|_{\mathcal{H}^{\bar{s}}(\Omega)} \qquad \forall v\in\mathcal{H}^{\bar{s}}(\Omega),\ 0\leq s\leq 1+r. \tag{3.5}$$

COROLLARY 3.3. *Assume that $u\in H^{\bar{s}}(\Omega)$. Then, the following convergence estimate holds:*

$$\|u-u_h\|_{X^r(\Omega)} \lesssim \left[h^{\bar{s}}+\nu^{1/2}h^{\bar{s}-1}+\delta^{1/2}\left(\nu h^{\bar{s}-3/2}+h^{\bar{s}-1/2}+h^{\bar{s}}\right)+\delta^{-1/2}h^{\bar{s}-1/2}\right]|u|_{\mathcal{H}^{\bar{s}}(\Omega)}.$$

Proof. Take $w_h=\Pi_h u$ in (3.4) and repeatedly use (3.5) with several values of s to bound all the norms which appear on the right-hand side. □

From now on, we make the assumption

$$\frac{\nu\delta}{h} \lesssim 1. \tag{3.6}$$

Introducing a 'mesh' Péclet number

$$I\!\!Pe \asymp \frac{\|\boldsymbol{a}\|_{L^\infty(\Omega)}h}{\nu}, \tag{3.7}$$

we can give an interpretation of (3.6) depending on the particular regime we are dealing with. Indeed, in the so-called 'convection-dominated' regime, i.e., if $I\!\!Pe\geq 1$, our assumptions is satisfied by choosing δ of the order of $\|\boldsymbol{a}\|^{-1}_{L^\infty(\Omega)}$, hence, independent of h. Conversely, in the 'diffusion-dominated' regime (i.e., $I\!\!Pe<1$), (3.6) forces δ to vanish with h; in particular, δ can be chosen in the form $\delta\asymp h/\nu$. This yields a smooth transition between the artificially stabilized formulation (2.15) and the original (non-stabilized) variational problem (2.3).

REMARK 3.4. It would also be possible to consider a stabilization with δ independent of h in the 'diffusion-dominated' regime. However, this would result in a overly diffusive method: indeed, in the limit case $\boldsymbol{a}\equiv 0$, the resulting method would be a Galerkin/Least-Squares approximation for a Laplacian-type operator, with the least-square term in $\mathcal{H}^{-1/2}(\Omega)$.□

REMARK 3.5. The definition (3.7) of the Péclet number is clearly a global one. This is coherent with the intrinsically global form of our stabilization terms in (2.13) and (2.14). It is well known that optimal results are obtained in finite elements by a local definition of the stabilization constants, depending on a local mesh Péclet number. In Subsection 4.3, we shall indicate how to modify our stabilized formulation in order to deal with the (common) situation of coexistence of locally convection-dominated and diffusion-dominated regimes.□

Using (3.6), the estimate of Corollary 3.3 simplifies as follows:

$$\|u-u_h\|_{X^r(\Omega)} \lesssim \left[h^{\bar{s}} + \delta^{1/2} h^{\bar{s}-1/2} + \delta^{-1/2} h^{\bar{s}-1/2}\right] |u|_{\mathcal{H}^{\bar{s}}(\Omega)}. \tag{3.8}$$

In the convection-dominated regime, i.e., $\delta \asymp \|\boldsymbol{a}\|^{-1}_{L^\infty(\Omega)} \asymp 1$, we get

$$\|u-u_h\|_{X^r(\Omega)} \lesssim h^{\bar{s}-1/2} |u|_{\mathcal{H}^{\bar{s}}(\Omega)};$$

on the other hand, in the diffusion-dominated regime with $\delta \asymp h/\nu \lesssim 1$, we get

$$\|u-u_h\|_{X^r(\Omega)} \lesssim \nu^{1/2} h^{\bar{s}-1} |u|_{\mathcal{H}^{\bar{s}}(\Omega)}.$$

All these estimates, in addition to the standard error control in the energy norm, yield an extra control on the error term

$$\tilde{\delta}^{1/2} \|Lu - Lu_h\|_{\mathcal{H}^{r-1}(\Omega)}.$$

We shall now derive a more significant error control involving the streamline derivative D; precisely, we are going to estimate the term

$$\tilde{\delta}^{1/2} \|Du - Du_h\|_{\mathcal{H}^{r-1}(\Omega)}.$$

To this end, in addition to (3.6), we assume that the following inverse inequality holds:

$$\|v_h\|_{\mathcal{H}^{r+1}(\Omega)} \lesssim h^{-r} \|\nabla v_h\|_{L^2(\Omega)}, \qquad \forall v_h \in V_h. \tag{3.9}$$

Next lemma is crucial in the sequel.

LEMMA 3.6. *Assume that* (3.6) *and* (3.9) *hold true. Then, for all* $v_h \in V_h$,

$$\tilde{\delta}^{1/2} \|Dv_h\|_{\mathcal{H}^{r-1}(\Omega)} \lesssim \tilde{\delta}^{1/2} \|Lv_h\|_{\mathcal{H}^{r-1}(\Omega)} + \nu^{1/2} \|\nabla v_h\|_{L^2(\Omega)} + \tilde{\delta}^{1/2} \|v_h\|_{\mathcal{H}^{r-1}(\Omega)}.$$

A similar estimate with D and L exchanged holds as well. Thus, in the 'regular case' with $\delta \lesssim 1$, if $\mathcal{L}$ denotes either the operator L or the operator D, the norms

$$\|v_h\|_{L^2(\Omega)} + \nu^{1/2} \|\nabla v_h\|_{L^2(\Omega)} + \delta^{1/2} \|\mathcal{L} v_h\|_{\mathcal{H}^{-1/2}(\Omega)} \tag{3.10}$$

are all uniformly equivalent on V_h.

Proof. From the identity $Dv_h = Lv_h + \nu \Delta v_h - b v_h$, we get

$$\tilde{\delta}^{1/2} \|Dv_h\|_{\mathcal{H}^{r-1}(\Omega)} \lesssim \tilde{\delta}^{1/2} \|Lv_h\|_{\mathcal{H}^{r-1}(\Omega)} + \tilde{\delta}^{1/2} \nu \|v_h\|_{\mathcal{H}^{r+1}(\Omega)} + \tilde{\delta}^{1/2} \|v_h\|_{\mathcal{H}^{r-1}(\Omega)}.$$

Using (3.9) and (3.6), we have $\tilde{\delta}^{1/2} \nu \|v_h\|_{\mathcal{H}^{r+1}(\Omega)} \lesssim \nu^{1/2} \|\nabla v_h\|_{L^2(\Omega)}$, whence the announced inequality easily follows. The norm equivalence follows from this and the companion estimate, using the trivial bound $\|v_h\|_{\mathcal{H}^{r-1}(\Omega)} \lesssim \|v_h\|_{L^2(\Omega)}$. □

We now apply the triangle inequality to $Du - Du_h = (Du - D\Pi_h u) + (D\Pi_h u - u_h)$ and we use the previous lemma, together with (3.5) and (3.8). After some algebra, we obtain the error estimate for the streamline derivative:

$$\tilde{\delta}^{1/2} \|Du - Du_h\|_{\mathcal{H}^{r-1}(\Omega)} \lesssim \tilde{\delta}^{1/2} \left[h^{\bar{s}-1/2} + \delta^{1/2} h^{\bar{s}-1/2} + \delta^{-1/2} h^{\bar{s}-1/2}\right] |u|_{\mathcal{H}^{\bar{s}}(\Omega)}.$$

We conclude by stating the final error estimate in the convection-dominated case.

COROLLARY 3.7. *Assume* (3.6) *and* (3.9)*; furthermore, choose* δ *satisfying* $\delta \asymp \|\boldsymbol{a}\|^{-1}_{L^\infty(\Omega)}$. *The following error estimate holds:*

$$\|u-u_h\|_{L^2(\Omega)}+\nu^{1/2}\|\nabla u-\nabla u_h\|_{L^2(\Omega)}+\delta^{1/2}\|Du-Du_h\|_{\mathcal{H}^{-1/2-\varepsilon}(\Omega)} \lesssim h^{\bar{s}-1/2}\,|u|_{\mathcal{H}^{\bar{s}}(\Omega)}. \square$$

4. Inner products and their approximation. For the sake of simplicity, hereafter we place ourselves in the 'regular case' described at the beginning of the previous section. We also assume that (3.9) holds and δ satisfies the bounds $\delta \lesssim 1$ and (3.6). This occurs if δ fulfils

$$\delta \asymp \begin{cases} \|\boldsymbol{a}\|^{-1}_{L^\infty(\Omega)} & \text{if } I\!\!P e \geq 1, \\ h/\nu & \text{if } I\!\!P e \leq 1, \end{cases}$$

where $I\!\!Pe$ satisfies (3.7). Obviously, all the subsequent results can be easily extended to the 'non–regular case', at the expense of a slightly heavier notation.

In order to provide an efficient computable stabilization term, we start by considering a representation of the $\mathcal{H}^{-1/2}(\Omega)$–inner product in terms of a suitable multilevel decomposition of $L^2(\Omega)$. Precisely, we are given a nested sequence $\mathcal{V}_{j_0} \subset \ldots \subset \mathcal{V}_j \subset \mathcal{V}_{j+1} \subset \ldots \subset H^1_0(\Omega)$ of subspaces, which allows the characterization of the family of spaces $\mathcal{H}^r(\Omega)$ in the range $0 \leq r \leq 1$ in the following sense: there exists linear approximation operators $P_j : L^2(\Omega) \to \mathcal{V}_j$ (with $P_{j_0-1} \equiv 0$) and a parameter $\theta > 1$ such that, setting $Q_j = P_j - P_{j-1}$, one has, for $0 \leq r \leq 1$,

$$\|v\|^2_{\mathcal{H}^r(\Omega)} \asymp \sum_{j=j_0}^{+\infty} \theta^{2rj}\|Q_j v\|^2_{L^2(\Omega)}\,, \qquad \forall v \in \mathcal{H}^r(\Omega)\,. \tag{4.1}$$

Furthermore, we also assume that the characterization extends to the dual spaces, according to the relations

$$\|f\|^2_{\mathcal{H}^{-r}(\Omega)} \asymp \sum_{j=j_0}^{+\infty} \theta^{-2rj}\|\widetilde{Q}_j v\|^2_{L^2(\Omega)}\,, \qquad \forall f \in \mathcal{H}^{-r}(\Omega)\,, \tag{4.2}$$

where $\widetilde{Q}_j = \widetilde{P}_j - \widetilde{P}_{j-1} : H^{-1}(\Omega) \to L^2(\Omega)$ is the adjoint of Q_j. Examples of such decompositions are provided by wavelet bases (see, e.g., [14]), and by various hierarchical finite elements ([21], [9], [8]). Taking into account (4.2), it is natural to define the $\mathcal{H}^{-1/2}(\Omega)$–inner product which enters in (2.13) and (2.14) as follows

$$[f,g]_{\mathcal{H}^{-1/2}(\Omega)} = \sum_{j=j_0}^{+\infty} \theta^{-j}(\widetilde{Q}_j f, \widetilde{Q}_j g)_{L^2(\Omega)}\,. \tag{4.3}$$

Obviously, this inner product is not computable, since in general an infinite number of non–zero terms appear on the right–hand side. Therefore, we are led to modify this expression, by retaining the minimal number of terms while still preserving the control on the $X^{1/2}$–norm of the error.

At first, we will be concerned with the problem of truncating the series from above. Subsequently, in a less general context, we will show that the computational cost of the procedure can be furtherly reduced by discarding also some lower levels without loss of accuracy.

The following notation will be useful in the sequel: for $j_0 \le j_1 \le j_2 \le +\infty$, we set

$$[f,g]_{j_1,j_2} := \sum_{j=j_1}^{j_2} \theta^{-j}(\widetilde{Q}_j f, \widetilde{Q}_j g)_{L^2(\Omega)}$$

and $\|f\|_{j_1,j_2} := [f,g]_{j_1,j_2}^{1/2}$; note that $[f,g]_{\mathcal{H}^{-1/2}(\Omega)} = [f,g]_{j_0,+\infty}$.

4.1. Truncation from above. We shall need the following inverse inequality on the convection derivative: there exists $t > 0$ such that

$$\|Dv_h\|_{\mathcal{H}^{(t-1)/2}(\Omega)} \lesssim h^{-t/2}\|Dv_h\|_{\mathcal{H}^{-1/2}(\Omega)}\,, \qquad \forall v_h \in V_h\,. \tag{4.4}$$

This assumption implies the following coercivity estimate.

PROPOSITION 4.1. *There exists a minimal level $\bar{J}(h)$ such that*

$$\|Dv_h\|_{\mathcal{H}^{-1/2}(\Omega)} \lesssim \|Dv_h\|_{j_0,\bar{J}(h)}\,, \qquad \forall v_h \in V_h\,.$$

Proof. It is not restrictive to assume $t \le 1$, so that the norm of $\mathcal{H}^{(t-1)/2}(\Omega)$ can be characterized by (4.2). Then, for an index $J \ge j_0$ to be determined later on, we have

$$\begin{aligned}\sum_{j>J} \theta^{-j}\|\widetilde{Q}_j Dv_h\|^2_{L^2(\Omega)} &\le \theta^{-t(J+1)} \sum_{j>J} \theta^{(t-1)j}\|\widetilde{Q}_j Dv_h\|^2_{L^2(\Omega)} \\ &\lesssim \theta^{-t(J+1)}\|Dv_h\|^2_{\mathcal{H}^{(t-1)/2}(\Omega)} \lesssim \theta^{-t(J+1)}h^{-t}\|Dv_h\|^2_{\mathcal{H}^{-1/2}(\Omega)}\,.\end{aligned}$$

It follows that, for a suitable constant $c > 0$,

$$\sum_{j\le J} \theta^{-j}\|\widetilde{Q}_j Dv_h\|^2_{L^2(\Omega)} \ge [1 - c(\theta^{J+1}h)^{-t}]\|Dv_h\|^2_{\mathcal{H}^{-1/2}(\Omega)}\,,$$

and it is enough to choose $J = \bar{J}(h)$ sufficiently large so that $(\theta^{J+1}h)^t \ge 2c$. $\square$

REMARK 4.2. The proof shows that the larger t is, the smaller is J. Thus, the validity of (4.4) for a t larger than 1 allows us to truncate at a lower level, provided the characterization (4.2) extends to negative values of r, precisely to $r = (1-t)/2$. $\square$

The previous proposition allows us to replace the exact representation (4.3) of the $\mathcal{H}^{-1/2}(\Omega)$–inner product by the truncated one

$$[f,g]_{j_0,J(h)} = \sum_{j=j_0}^{J(h)} \theta^{-j}(\widetilde{Q}_j f, \widetilde{Q}_j g)_{L^2(\Omega)}, \qquad \text{for any } J(h) \ge \bar{J}(h),$$

when the stabilized method (2.15) and its discrete counterpart (3.3) are considered. All the estimates given in the previous section extend to the resulting perturbed problem.

4.2. Truncation from below. Hereafter, we aim at replacing the inner product (4.3) in (2.15) and (3.3) by the furtherly truncated expression $[f,g]_{j(h),J(h)}$ for some $j_0 \le j(h) \le J(h)$. More precisely, we will show that this is possible for any $j(h)$ such that

$$\mathcal{V}_{j(h)} \subset V_h\,. \tag{4.5}$$

In the sequel, we shall make slightly more restrictive assumptions on the multilevel decomposition $\{\mathcal{V}_j\}_{j\geq j_0}$ than before. Precisely, we suppose that there exists a (sufficiently small) $\sigma > 0$ such that

$$\mathcal{V}_j \subset \mathcal{H}^{3/2+\sigma}(\Omega)\,, \qquad \forall j \geq j_0\,, \tag{4.6}$$

and the norm representation (4.1) extends to the range $0 \leq r \leq \frac{3}{2}+\sigma$. Note that the inclusion (4.5) implies the bound

$$h\theta^{j(h)} \lesssim 1\,. \tag{4.7}$$

Indeed, applying the inverse inequality (3.9) to any non–zero $Q_{j(h)}v$, $v \in L^2(\Omega)$, and using the norm representation (4.1) with $r = 3/2$, we get

$$\begin{aligned}\theta^{\frac{3}{2}j(h)}\|Q_{j(h)}v\|_{L^2(\Omega)} \asymp \|Q_{j(h)}v\|_{\mathcal{H}^{3/2}(\Omega)} &\lesssim h^{-1/2}\|\nabla Q_{j(h)}v\|_{L^2(\Omega)}\\ &\asymp h^{-1/2}\theta^{j(h)}\|Q_{j(h)}v\|_{L^2(\Omega)}\,,\end{aligned}$$

whence (4.7) follows.

From now on, let $j(h) \geq j_0$ be any level index for which (4.5) holds; let $J(h)$ be any level index satisfying $J(h) \geq \max(j(h), \bar{J}(h))$. Let us define the perturbed stabilized bilinear form

$$A_h : X^{1/2}(\Omega) \times X^{1/2}(\Omega) \to I\!R\,, \qquad A_h(w,v) := a(w,v) + \delta[Lw, Lv]_{j(h),J(h)} \tag{4.8}$$

as well as the perturbed stabilized linear form

$$F_h : X^{1/2}(\Omega) \to I\!R\,, \qquad F_h(v) := \langle f, v\rangle + \delta[f, Lv]_{j(h),J(h)}. \tag{4.9}$$

(Compare with (2.13) and (2.14), and recall that in the *regular case* $r = 1/2$ and $\tilde{\delta} = \delta$.) Let us observe that the exact solution of Problem (2.1) also satisfies the variational problem

$$A_h(u,v) = F_h(v), \qquad \forall v \in X^{1/2}(\Omega). \tag{4.10}$$

Thus, we are led to consider the solution $u_h^* \in V_h$ of the approximate problem

$$A_h(u_h^*, v_h) = F_h(v_h), \qquad \forall v_h \in V_h. \tag{4.11}$$

Our goal is to estimate the error $u - u_h^*$ in the $X^{1/2}(\Omega)$–norm. To this end, we first establish two technical results.

LEMMA 4.3. *Under the assumptions of this subsection, one has*

$$\|L(u-u_h^*)\|_{j_0,j(h)-1} \lesssim \|L(u-u_h^*)\|_{j(h),J(h)}\,.$$

Proof. We have

$$\|L(u-u_h^*)\|^2_{j_0,j(h)-1} = \sum_{j=j_0}^{j(h)-1} \theta^{-j}\|\widetilde{Q}_j L(u-u_h^*)\|^2_{L^2(\Omega)}\,. \tag{4.12}$$

By substracting (4.11) from (4.10), we get the relations

$$a(u-u_h^*, v_h) + \delta[L(u-u_h^*), Lv_h]_{j(h),J(h)} = 0\,, \qquad \forall v_h \in V_h\,. \tag{4.13}$$

By (4.5), for any $v \in L^2(\Omega)$ and $j < j(h)$, we have $Q_j v \in V_h$. In addition, we observe that

$$a(u-u_h^*, Q_j v) = \langle L(u-u_h^*), Q_j v\rangle = (\widetilde{Q}_j L(u-u_h^*), v)_{L^2(\Omega)} .$$

This and (4.13), imply that

$$(\widetilde{Q}_j L(u-u_h^*), v)_{L^2(\Omega)} + \delta[L(u-u_h^*), LQ_j v]_{j(h),J(h)} = 0 , \qquad \forall v \in L^2(\Omega) .$$

Setting $g = L(u-u_h^*)$ and taking $v = \widetilde{Q}_j g$ in the previous equation, we get

$$(\widetilde{Q}_j g, \widetilde{Q}_j g)_{L^2(\Omega)} = -\delta[g, LQ_j \widetilde{Q}_j g]_{j(h),J(h)} ,$$

whence

$$\|\widetilde{Q}_j g\|^2_{L^2(\Omega)} \le \delta \sum_{\ell=j(h)}^{J(h)} \theta^{-\ell} |(\widetilde{Q}_\ell g, \widetilde{Q}_\ell L Q_j \widetilde{Q}_j g)_{L^2(\Omega)}| = \delta \sum_{\ell=j(h)}^{J(h)} \theta^{-\ell} |\langle Q_\ell \widetilde{Q}_\ell g, LQ_j \widetilde{Q}_j g\rangle| . \tag{4.14}$$

Each inner product can be bounded as follows:

$$|\langle Q_\ell \widetilde{Q}_\ell g, LQ_j \widetilde{Q}_j g\rangle| \le \|Q_\ell \widetilde{Q}_\ell g\|_{\mathcal{H}^{1/2-\sigma}(\Omega)} \ \|LQ_j \widetilde{Q}_j g\|_{\mathcal{H}^{-1/2+\sigma}(\Omega)} .$$

Now, using the norm representation (4.1), we obtain

$$\|Q_\ell \widetilde{Q}_\ell g\|_{\mathcal{H}^{1/2-\sigma}(\Omega)} \asymp \theta^{(\frac{1}{2}-\sigma)\ell} \|Q_\ell \widetilde{Q}_\ell g\|_{L^2(\Omega)} \ \lesssim \ \theta^{(\frac{1}{2}-\sigma)\ell} \|\widetilde{Q}_\ell g\|_{L^2(\Omega)} .$$

On the other hand,

$$\begin{aligned}
\|LQ_j \widetilde{Q}_j g\|_{\mathcal{H}^{-1/2+\sigma}(\Omega)} &\le \nu \|\Delta Q_j \widetilde{Q}_j g\|_{\mathcal{H}^{-1/2+\sigma}(\Omega)} + \|DQ_j \widetilde{Q}_j g\|_{\mathcal{H}^{-1/2+\sigma}(\Omega)} \\
&\lesssim \nu \|Q_j \widetilde{Q}_j g\|_{\mathcal{H}^{3/2+\sigma}(\Omega)} + \|Q_j \widetilde{Q}_j g\|_{\mathcal{H}^{1/2+\sigma}(\Omega)} \\
&\lesssim [\nu \theta^{(\frac{3}{2}+\sigma)j} + \theta^{(\frac{1}{2}+\sigma)j}] \ \|Q_j \widetilde{Q}_j g\|_{L^2(\Omega)} \\
&\lesssim \theta^{(\frac{1}{2}+\sigma)j} \|\widetilde{Q}_j g\|_{L^2(\Omega)} ,
\end{aligned}$$

where, in the last inequality, we have used (3.6) and (4.7). Hence, (4.14) yields

$$\|\widetilde{Q}_j g\|_{L^2(\Omega)} \ \lesssim \ \delta \sum_{\ell=j(h)}^{J(h)} \theta^{(\frac{1}{2}+\sigma)(j-\ell)} \|\widetilde{Q}_\ell g\|_{L^2(\Omega)} ,$$

or, equivalently,

$$\theta^{-\frac{1}{2}j} \|\widetilde{Q}_j g\|_{L^2(\Omega)} \ \lesssim \ \delta \sum_{\ell=j(h)}^{J(h)} \theta^{\sigma(j-\ell)} \theta^{-\frac{1}{2}\ell} \|\widetilde{Q}_\ell g\|_{L^2(\Omega)} .$$

Recalling (4.12), we obtain the result. □

The following notation will be useful:

$$\|v\|_h = \|v\|_{L^2(\Omega)} + \nu^{1/2} \|\nabla v\|_{L^2(\Omega)} + \delta^{1/2} \|Lv\|_{j(h),J(h)} .$$

LEMMA 4.4. *For any $w_h \in V_h$, one has*

$$\|w_h - u_h^*\|_h^2 \ \lesssim \ (\|u - w_h\|_{X^{1/2}(\Omega)} + \delta^{-1/2} \|u - w_h\|_{\mathcal{H}^{1/2}(\Omega)}) \ \|w_h - u_h^*\|_{X^{1/2}(\Omega)} .$$

Proof. The estimate can be derived by standard arguments using coerciveness and continuity inequalities of the form A_h:

$$\begin{aligned} A_h(w,w) &\gtrsim \|w\|_h^2 \\ A_h(w,v_h) &\lesssim (\|w\|_{X^{1/2}(\Omega)} + \delta^{-1/2}\|w\|_{\mathcal{H}^{1/2}(\Omega)})\|v_h\|_{X^{1/2}(\Omega)}\,, \end{aligned}$$

which hold for any $w \in X^{1/2}(\Omega)$ and any $v_h \in V_h$. □

THEOREM 4.5. *Let the assumptions of this subsection be satisfied. Then, the following error estimate holds:*

$$\|u - u_h^*\|_{X^{1/2}(\Omega)} \lesssim \inf_{w_h \in V_h} \Big(\|u - w_h\|_{X^{1/2}(\Omega)} + \delta^{-1/2}\|u - w_h\|_{\mathcal{H}^{1/2}(\Omega)}\Big)\,.$$

Proof. Let w_h be any element in V_h. The triangle inequality and Lemma 3.6 yield

$$\begin{aligned} \|u - u_h^*\|^2_{X^{1/2}(\Omega)} &\lesssim \|u - w_h\|^2_{X^{1/2}(\Omega)} + \|w_h - u_h^*\|^2_{X^{1/2}(\Omega)} \\ &\lesssim \|u - w_h\|^2_{X^{1/2}(\Omega)} + \|w_h - u_h^*\|^2_{L^2(\Omega)} + \\ &\qquad + \nu\|\nabla(w_h - u_h^*)\|^2_{L^2(\Omega)} + \delta\|Dw_h - Du_h^*\|^2_{\mathcal{H}^{-1/2}(\Omega)}\,. \end{aligned}$$

Applying Proposition 4.1, one gets

$$\begin{aligned} \|u - u_h^*\|^2_{X^{1/2}(\Omega)} &\lesssim \|u - w_h\|^2_{X^{1/2}(\Omega)} + \|w_h - u_h^*\|^2_{L^2(\Omega)} + \\ &\qquad + \nu\|\nabla(w_h - u_h^*)\|^2_{L^2(\Omega)} + \delta\|Dw_h - Du_h^*\|^2_{j_0,J(h)}\,. \end{aligned}$$

At this point, we observe that Lemma 3.6 still holds if we replace $\|\mathcal{L}v_h\|_{\mathcal{H}^{-1/2}(\Omega)}$ by $\|\mathcal{L}v_h\|_{j_0,J(h)}$ in (3.10). This remark and the triangle inequality imply

$$\begin{aligned} \|u - u_h^*\|^2_{X^{1/2}(\Omega)} &\lesssim \|u - w_h\|^2_{X^{1/2}(\Omega)} + \|w_h - u_h^*\|^2_{L^2(\Omega)} + \\ &\quad + \nu\|\nabla(w_h - u_h^*)\|^2_{L^2(\Omega)} + \delta\|Lw_h - Lu_h^*\|^2_{j_0,J(h)} \\ &\lesssim \|u - w_h\|^2_{X^{1/2}(\Omega)} + \|w_h - u_h^*\|^2_{L^2(\Omega)} + \nu\|\nabla(w_h - u_h^*)\|^2_{L^2(\Omega)} + \\ &\quad + \delta\|Lw_h - Lu_h^*\|^2_{j_0,j(h)-1} + \delta\|Lw_h - Lu_h^*\|^2_{j(h),J(h)}\,. \end{aligned}$$

Next, we use the triangle inequality and Lemma 4.3 to get

$$\|Lw_h - Lu_h^*\|^2_{j_0,j(h)-1} \lesssim \|u - u_h^*\|^2_{X^{1/2}(\Omega)} + \|Lw_h - Lu_h^*\|^2_{j(h),J(h)}\,.$$

Thus, using again the modified version of Lemma 3.6, we obtain

$$\begin{aligned} \|u - u_h^*\|^2_{X^{1/2}(\Omega)} &\lesssim \|u - w_h\|^2_{X^{1/2}(\Omega)} + \|w_h - u_h^*\|^2_{L^2(\Omega)} + \\ &\qquad + \nu\|\nabla(w_h - u_h^*)\|^2_{L^2(\Omega)} + \delta\|Lw_h - Lu_h^*\|^2_{j(h),J(h)} \\ &\lesssim \|u - w_h\|^2_{X^{1/2}(\Omega)} + \|w_h - u_h^*\|^2_h\,. \end{aligned}$$

By Lemma 4.4 and the inequality $ab \le \frac{a^2}{4\eta} + \eta b^2$ ($\eta > 0$), we have

$$\begin{aligned} \|u - u_h^*\|^2_{X^{1/2}(\Omega)} &\lesssim \|u - w_h\|^2_{X^{1/2}(\Omega)} + \delta^{-1}\|u - w_h\|^2_{\mathcal{H}^{1/2}(\Omega)} + \eta\|w_h - u_h^*\|^2_{X^{1/2}(\Omega)} \\ &\le \|u - w_h\|^2_{X^{1/2}(\Omega)} + \delta^{-1}\|u - w_h\|^2_{\mathcal{H}^{1/2}(\Omega)} + \eta\|u - u_h^*\|^2_{X^{1/2}(\Omega)}\,. \end{aligned}$$

Hence, $(1 - C\eta)\|u - u_h^*\|^2_{X^{1/2}(\Omega)} \lesssim \|u - w_h\|^2_{X^{1/2}(\Omega)} + \delta^{-1}\|u - w_h\|^2_{\mathcal{H}^{1/2}(\Omega)}$, where C is an appropriate constant. Choosing η small enough, we obtain the desired result. □

4.3. Localization. In the sequel, we indicate how to enhance the performance of the method just described. To this end, we propose to resort to multiscale decompositions with good localization properties. In this way, we will be able to locally tune the amount of stabilization added, in dependence of the discretization and the local behavior of the coefficients of the operator.

The abstract multilevel decomposition setting introduced at the beginning of this section can be modified to deal with other representations of the inner product, which are well-known.

For instance, assume that we have a hierarchical-type basis $\{\psi_{jk} : j \geq j_0, k \in \mathcal{K}_j\}$ of some $\mathcal{H}^{r_0}(\Omega)$ ($r_0 \geq 0$) such that for all r in some interval $[r_0, r_1]$, the following norm equivalences hold:

$$\|v\|^2_{\mathcal{H}^r(\Omega)} \asymp \sum_{j \geq j_0} \theta^{2rj} \sum_{k \in \mathcal{K}_j} |v_{jk}|^2, \qquad \forall v = \sum_{j \geq j_0} \sum_{k \in \mathcal{K}_j} v_{jk}\psi_{jk} \in \mathcal{H}^r(\Omega)\,.$$

Under suitable assumptions, it is possible to prove that for all $r \in [r_0, r_1]$, the following negative norm equivalences hold:

$$\|f\|^2_{\mathcal{H}^{-r}(\Omega)} \asymp \sum_{j \geq j_0} \theta^{-2rj} \sum_{k \in \mathcal{K}_j} |\ {}_{\mathcal{H}^{-r_1}(\Omega)}\langle f, \psi_{jk}\rangle_{\mathcal{H}^{r_1}(\Omega)}|^2, \qquad \forall f \in \mathcal{H}^{-r}(\Omega).$$

Therefore, if we are in the 'regular case' and if $[1/2, 1] \subseteq [r_0, r_1]$, one can replace (4.3) by

$$[f, g]_{\mathcal{H}^{-1/2}(\Omega)} = \sum_{j \geq j_0} \theta^{-j} \sum_{k \in \mathcal{K}_j} \langle f, \psi_{jk}\rangle\langle g, \psi_{jk}\rangle\,, \tag{4.15}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Obvious modifications apply in the 'non-regular' case.

As an example (indeed related to the 'non-regular' case), let us mention the multilevel decomposition based on the piecewise linear *interpolatory wavelets* on the unit interval $(0, 1)$, which are defined as $\psi_{jk}(x) = 2^{(j+1)/2}(1 - |2^{j+1}x - 2k - 1|)_+$, $j \geq 0$, $k = 0, \ldots, 2^j - 1$. In this case, the previous representations hold with $\theta = 2$, for any choice of $r_0 \leq r \leq r_1$ in the interval $(1/2, 3/2)$ (see [14]).

The representation of the inner product given in (4.15) allows one to replace the global choice of the stabilization parameter δ considered so far, by a local one. In this way, the stabilization terms will be better suited to treat the possible coexistence of convection-dominated and diffusion-dominated regimes within the domain, as well as the variation of the coefficients.

To be precise, let us assume that the hierarchical basis functions are compactly supported, with $S_{jk} := \operatorname{supp}\psi_{jk}$ and $\operatorname{diam} S_{jk} \asymp \theta^{-j}$. Let us introduce a 'local' Pèclet number $I\!Pe_{jk}$ associated with the basis function ψ_{jk} as $I\!Pe_{jk} \asymp (\|\boldsymbol{a}\|_{L^\infty(S_{jk})}\theta^{-j})/\nu$, and let us define a local stabilization parameter δ_{jk} satisfying

$$\delta_{jk} \asymp \begin{cases} \|\boldsymbol{a}\|^{-1}_{L^\infty(S_{jk})}, & \text{if } I\!Pe_{jk} \geq 1, \\ \theta^{-j}/\nu & \text{if } I\!Pe_{jk} < 1. \end{cases} \tag{4.16}$$

Then, we replace the stabilization term $\delta[Lw, Lv]_{\mathcal{H}^{-1/2}(\Omega)}$ by the expression

$$[\![Lw, Lv]\!] := \sum_{j \geq j_0} \theta^{-j} \sum_{k \in \mathcal{K}_j} \delta_{jk}\langle Lw, \psi_{jk}\rangle\langle Lv, \psi_{jk}\rangle\,.$$

The analysis provided in the previous sections can be extended to deal with the local form of the stabilization as well. In particular, one can show that the new inner product can be truncated to a finite sum

$$[\![Lw, Lv]\!]_{j(h),J(h)} := \sum_{j=j(h)}^{J(h)} \theta^{-j} \sum_{k\in\mathcal{K}_j} \delta_{jk}\langle Lw, \psi_{jk}\rangle\langle Lv, \psi_{jk}\rangle$$

without affecting the stability and convergence results. The details will be presented elsewhere.

5. Examples. We shall now present some (preliminary) results on the behavior of the proposed method for simple model problems. The main goal of this section is to illustrate some features of the stabilization, such as the effect of the choice of the multilevel decomposition as well as the sensitivity to the parameters involved. The application of our method to more severe test cases will be presented in a forthcoming paper.

5.1. Piecewise linear interpolatory wavelets. Set $n = 1$ and let Ω be the interval $(0,1)$. For a fixed integer $m \geq 1$, set $h = 2^{-m}$ and consider the subspace $V_h \subset H_0^1(0,1)$ of the continuous piecewise linear functions on the uniform partition of $(0,1)$ into subintervals of size h.

With this choice, we are placing ourselves in the 'non-regular case'. A natural multilevel decomposition for stabilizing the Galerkin discretization associated with the subspace V_h, is the one based on piecewise linear interpolatory wavelets, introduced in Section 4.3. For any $j \geq 0$, set $\mathcal{W}_j := \mathrm{span}\,\{\psi_{jk} \,:\, 0 \leq k \leq 2^j - 1\}$ and

$$\mathcal{V}_j := \bigoplus_{j'<j} \mathcal{W}_{j'} \subset H_0^1(0,1). \tag{5.1}$$

Note that $\mathcal{V}_j$ is the space of the continuous piecewise linear functions on the uniform partition of $(0,1)$ into subintervals of size 2^{-j}. In particular, $\mathcal{V}_m = V_h$, so that condition (4.5) is fulfilled. Therefore, we are led to add the following stabilization term to the bilinear form $a(w,v)$:

$$[\![Lw, Lv]\!]_{m,J} := h^{-2\varepsilon} \sum_{j=m}^{J} 2^{-(1+2\varepsilon)j} \sum_{k=0}^{2^j-1} \delta_{jk}\langle Lw, \psi_{jk}\rangle\langle Lv, \psi_{jk}\rangle\,, \tag{5.2}$$

where J is some integer $\geq m$ and we allow a local choice of δ, as discussed at the end of the previous section. An analogous term is added to the linear form $\langle f, v\rangle$.

The interest of the present example is that, under suitable circumstances, these terms *coincide* with those defined by the well-known SUPG method ([11, 16]). Such a result provides a new perspective from which classical stabilization methods can be viewed. To establish the result, let us choose $J = m$, so that (5.2) simplifies as

$$[\![Lw, Lv]\!]_{m,m} = \sum_{k=0}^{2^m-1} h\delta_{mk}\langle Lw, \psi_{mk}\rangle\langle Lv, \psi_{mk}\rangle.$$

Moreover, let us assume that, in the differential operator L, the zeroth order coefficient b is zero and the first order coefficient $\boldsymbol{a}$ is constant in the interior of each element

$I_\ell := [\ell h, (\ell+1)h]$ $(\ell = 0, \dots, 2^m - 1)$. Observing that ψ_{mk} is supported precisely in the element I_k, we have, for all $w \in V_h$,

$$\langle Lw, \psi_{mk} \rangle = Dw_{|I_k} \int_{I_k} \psi_{mk}(x)\, dx = h^{1/2} Dw_{|I_k}.$$

We conclude that

$$[\![Lw, Lv]\!]_{m,m} = \sum_{k=0}^{2^m-1} \delta_{mk} \frac{h}{2} \int_{I_k} Dw(x)\, Dv(x)\, dx.$$

Choosing δ_{mk} as in (4.16) with a suitable definition of the constant implied by the symbol $\asymp$, the right-hand side is precisely the expression that one adds to the bilinear form $a(w, v)$ in order to accomplish the SUPG stabilization.

It is interesting to observe that the same result can be achieved even if J is chosen to be larger than m, or even equal to $+\infty$, in (5.2). Indeed, one has

$$[\![Lw, Lv]\!]_{m,+\infty} = \sum_{k=0}^{2^m-1} \Big(\sum_{j \geq m} \delta_{j,[k]} 2^{-(j-m)}\Big) \frac{h}{2} \int_{I_k} Dw(x)\, Dv(x)\, dx\,,$$

where $\delta_{j,[k]}$ denotes the constant value of δ_{jh} on I_k. Recalling again the definition (4.16), it is easily seen that each term in brackets behaves like the corresponding δ_{mk}.

5.2. Spline biorthogonal wavelets. Let us consider the same one-dimensional problem and the same approximation subspace V_h $(h = 2^{-m})$ as for the previous example. Instead of using piecewise linear interpolatory wavelets for constructing the multilevel $H^{-1/2}$-type inner product, we explore the behavior of a family of piecewise linear biorthogonal wavelets, chosen among the spline biorthogonal wavelet systems introduced by Cohen, Daubechies and Feauveau [13]. Precisely, the chosen wavelets belong to the system identified by the parameters $L = 2$, $\tilde{L} = 2$, i.e., they are splines of order 2 having exactly 2 vanishing moments. Figure 5.1 shows the mother wavelet $\psi = \psi_{00}$. All other wavelets on the real line are defined by the dilation

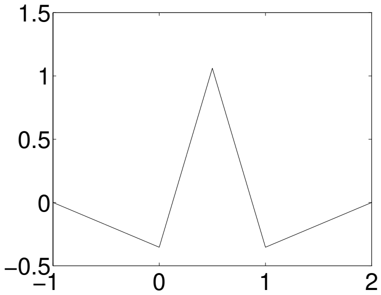

FIG. 5.1. *The spline biorthogonal mother wavelet* ψ_{00}

and translation relations $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$. Near the endpoints, wavelets are modified to satisfy the boundary conditions, while retaining vanishing moments and optimal localization of the support ([18]). We note that, with such a construction and after setting $\mathcal{W}_j := \mathrm{span}\,\{\psi_{jk} : 0 \leq k \leq 2^j - 1\}$, the spaces $\mathcal{V}_j$ defined by formula (5.1) coincide with those of the previous example; thus, again, $\mathcal{V}_m = V_h$.

In order to numerically test our stabilization, we consider the problem

$$-\nu u_{xx} + a u_x = f \quad \text{in } (0,1), \qquad u(0) = u(1) = 0,$$

with $\nu = 10^{-5}$ and $a(x) = 2f(x) = 2$ for $x < 1/2$, $a(x) = 2f(x) = -2$ for $x > 1/2$. The solution exhibits an internal layer around the point $x = 1/2$. For the numerical discretization, we choose $h = 2^{-6}$; since $\|a\|_{L^\infty(0,1)} = 2$, we are in the convection-dominated regime. Thus, we consider the following truncated inner product:

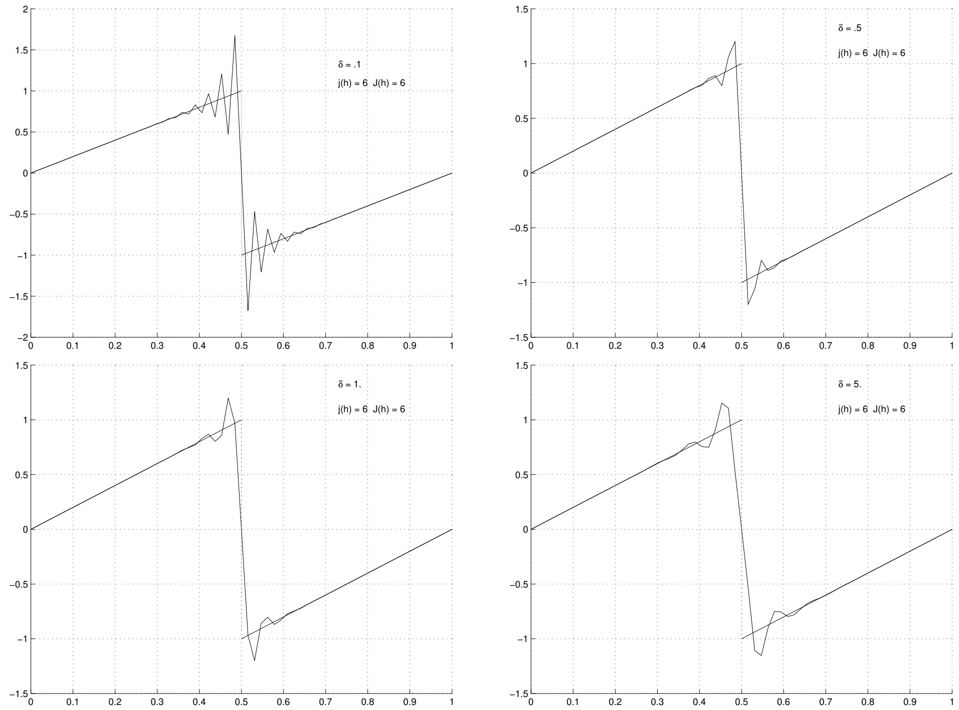

FIG. 5.2. *Sensitiveness to the parameter δ. Top left: $\delta = .1$; top right: $\delta = .5$; bottom left: $\delta = 1.$; bottom right: $\delta = 5$. All cases are obtained with $j(h) = J(h) = 6$.*

$$[\![Lw, Lv]\!]_{j(h),J(h)} := \delta \sum_{j=j(h)}^{J(h)} 2^{-j-2\varepsilon(j-m)} \sum_{k=0}^{2^j-1} \langle Lw, \psi_{jk}\rangle \langle Lv, \psi_{jk}\rangle \,,$$

where $j(h) \le J(h)$ are integers around $m = 6$ and $\varepsilon > 0$ is a (very small) fixed parameter. According to the theory, we can choose $j(h) = m$; so, we start by considering the case $j(h) = J(h) = m$, i.e., we stabilize through the only scale which is next to those fully resolved in the discretization space V_h. Figure 5.2 shows the discrete solutions obtained with several values of δ, thus illustrating the sensitiveness to this parameter. As predicted by the theory, δ has to be in the order of unity to guarantee enough stability. The sharper result is obtained precisely with $\delta = 0.5 = \|a\|^{-1}_{L^\infty(0,1)}$. Values of δ significantly larger than 1 do not overdiffuse the solution, but they rather expand the effect of the stabilization around the singular point.

Next, after fixing $\delta = 0.5$, we explore the effect of varying the level bounds $j(h)$ and $J(h)$. The results are reported in Figure 5.3. The two top plots demonstrate that

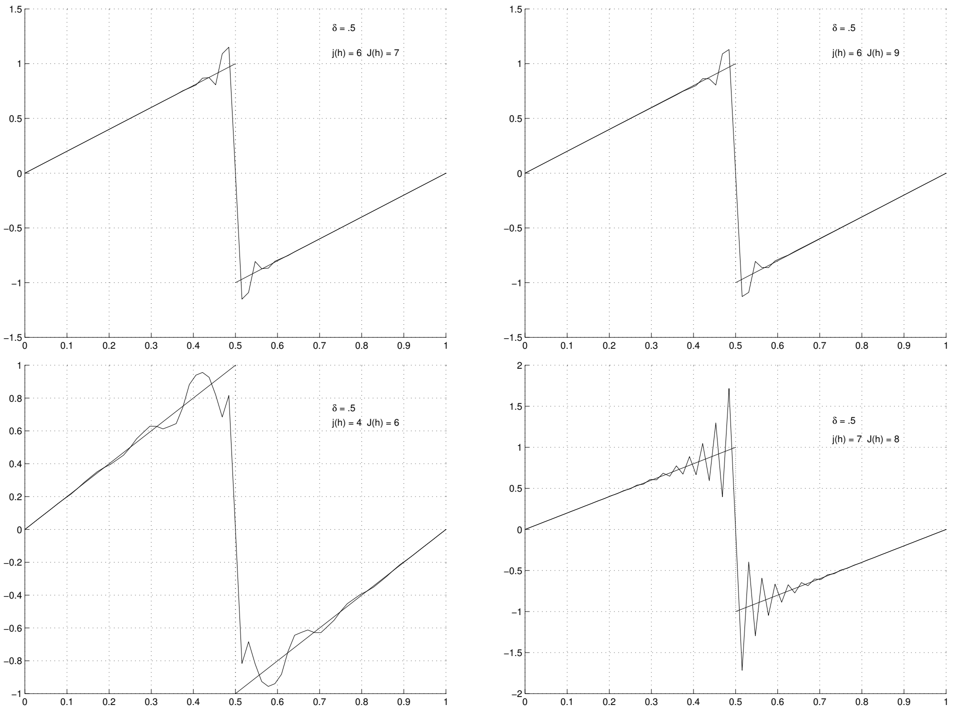

Fig. 5.3. *Sensitiveness to the parameters $j(h)$ and $J(h)$. Top left: $j(h) = 6, J(h) = 7$; top right: $j(h) = 6, J(h) = 9$; bottom left: $j(h) = 4, J(h) = 6$; bottom right: $j(h) = 7, J(h) = 8$. All cases are obtained with $\delta = .5$.*

incrementing the inner product by adding higher levels than $j = m$ does not substantially modify the quality of the stabilization. The situation dramatically changes if one adds lower levels than $j = m$ (bottom left plot): coarse scale effects pollute the discrete solution far away from the singularity point. On the other hand, higher levels than $j = m$ alone are not sufficient to provide the correct amount of stabilization (bottom right plot).

These experiments illustrate the qualitative structure of the stabilization mechanism. From the quantitative point of view, it is apparent that the results are not as good as those produced by the standard SUPG technique with the optimal choice of the stabilization parameter. The presence of (relatively moderate) over- and undershoots indicates that this multilevel stabilization is of non-monotone nature. Furthermore, the effects of the singularity are felt several elements away from it. This is due to the fact that each wavelet is supported over three elements of the corresponding level (see again Figure 5.1).

REFERENCES

[1] O. Axelsson and W. Layton, Defect correction methods for convection-dominated convection-diffusion equations, RAIRO J. Numer. Anal., 24 (1990), pp. 423-455.

[2] C. Baiocchi, Un teorema di interpolazione; applicazioni ai problemi ai limiti per le equazioni a derivate parziali, Ann. Mat. Pura Appl., 73 (1966), pp. 235-252.

[3] C. Bardos, D. Brezis and H. Brezis, Perturbations singulieres et prolongements maximaux d'operateurs positives, Archive Rat. Mech. Anal., 53 (1973), pp. 69-100.

[4] S. Bertoluzza, Stabilization by multiscale decomposition, Appl. Math. Lett., 11 (1998), pp. 129-134.

[5] S. Bertoluzza, C. Canuto and A. Tabacco, Negative norm stabilization of convection-diffusion problems, to appear in Appl. Math. Lett.

[6] F. Brezzi, M-O. Bristeau, L. Franca, M. Mallet and G. Rogèr, A relationship between stabilized finite element methods and the Galerkin method with bubble functions, Comput. Methods Appl. Mech. Engrg., 96 (1992), pp. 117-129.

[7] F. Brezzi, D. Marini and E. Süli, Residual-free bubbles for advection-diffusion problems: The general error analysis, I.A.N.-C.N.R. Report 1115, Pavia, 1998.

[8] J. H. Bramble, R.D. Lazarov and J. E. Pasciak, A least-squares approach based on a discrete minus one inner product for first order systems, Math. Comp., 66 (1997), pp. 935-955.

[9] J. H. Bramble, J. E. Pasciak and J. Xu, Parallel multilevel preconditioners, Math. Comp., 55 (1990), pp. 1-22.

[10] F. Brezzi, L.P. Franca, T.J.R. Hughes and A. Russo, $b = \int g$, Comput. Methods Appl. Mech. Engrg., 145 (1997), pp. 329-340.

[11] A.N. Brooks and T.J.R. Hughes, Streamline upwind/Petrov Galerkin formulations for convective-dominated flows with particular emphasis on the incompressible Navier-Stokes equations, Comput. Methods Appl. Mech. Engrg., 32 (1982), pp. 199-259.

[12] C. Canuto, A. Tabacco and K. Urban, *The Wavelet Element Method, Part I: Construction and analysis*, Appl. Comput. Harm. Anal., 6 (1999), pp. 1-52.

[13] A. Cohen, I. Daubechies e J. Feauveau, Biorthogonal bases of compactly supported wavelet, Comm. Pure Appl. Math., **45** (1992), pp. 485-560.

[14] W. Dahmen, Stability of multiscale transformations, J. Fourier Anal. and Appl., 4 (1996), pp. 341-362.

[15] W. Dörfler, Uniform a priori estimates for singularly perturbed elliptic equations in multi-dimensions, Preprint University of Freiburg, no.24, 1998.

[16] L.P. Franca, S.L. Frey and J.T.R. Hughes, Stabilized finite element methods: I. Application to the advective-diffusive model, Comput. Methods Appl. Mech. Engrg., 92 (1992), pp. 253-276.

[17] G. Garrigós and A. Tabacco, Wavelet characterizations of anisotropic Besov spaces, Dipartimento di Matematica, Politecnico di Torino, Report n. 27, 1999.

[18] S. Grivet Talocia and A. Tabacco, Wavelets on the interval with optimal localization, to appear in Math. Mod. Meths. Appl. Sci..

[19] T.J.R. Hughes, L.P. Franca and G.M. Hulbert, A new finite element formulation for computational fluid dynamics: VIII. The Galerkin/least-squares method for advective-diffusive equations, Comput. Methods Appl. Mech. Engrg., 73 (1989), pp. 173-189.

[20] P.J. Lions and E. Magenes, *Problèmes aux Limites non Homogènes*, Dunod, Paris 1968.

[21] P. Oswald, *Multilevel Finite Element Approximation. Theory and Applications*, Teubner Stuttgart 1994.

[22] H.-G. Roos, M. Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations. Convection-Diffusion and Flow problems*, Springer, Berlin, 1996.

[23] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1983.

[24] H. Triebel, Personal communication.

[25] H. Triebel, *Fourier Analysis and Function Spaces*, Teubner Texte Math. 7, Teubner, Leipzig, 1977.

[26] T. Wolff, A note on interpolation spaces, pp.199-204 in *Harmonic Analysis*, F. Ricci and G. Weiss eds., Lectures Notes in Mathematics 908, Springer-Verlag, Berlin-Heidelberg-New York, 1982.