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# Local error analysis for the Stokes equations with a punctual source term

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**Abstract** The solution of the Stokes problem with a punctual force in source term is not $H^1 \times L^2$ and therefore the approximation by a finite element method is suboptimal. In the case of Poisson problem with a Dirac mass in the right-hand side, an optimal convergence for the Lagrange finite elements has been shown on a subdomain which excludes the singularity in $L^2$-norm by Kp̈pl and Wohlmuth.

Here we show a quasi-optimal local convergence in $H^1 \times L^2$-norm for a $\mathbb{P}_k/\mathbb{P}_{k-1}$ finite element method, $k \geq 2$, and for the $\mathbb{P}_1 b/\mathbb{P}_1$. The error is still analysed on a sub- domain which does not contain the singularity. The proof is based on local Arnold and Liu error estimates, a weak version of Aubin–Nitsche duality lemma applied to the Stokes problem and discrete inf-sup conditions. These theoretical results are generalized to a wide class of finite element methods, before being illustrated by numerical simulations.

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## 1 Introduction

This paper is about the accuracy of the finite element method to solve the Stokes problem with a punctual force in source term. Let us consider this following problem

$$\begin{cases} -\triangle \mathbf{u} + \nabla p = \delta_{\mathbf{x}_0}\mathbf{F} & \text{in} \quad \Omega, \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open $\mathcal{C}^\infty$ domain or a square, and $\delta_{\mathbf{x}_0}\mathbf{F}$ denotes the punctual force $\mathbf{F}$ located at $\mathbf{x}_0 \in \Omega$ such that $\operatorname{dist}(\mathbf{x}_0, \partial\Omega) > 0$.

Our interest in Problem (1) is motivated by the modeling of the movement of thin structures in a viscous fluid, such as flagella connected to bacteria or cilia involved in the muco-ciliary transport in the lung [11]. Indeed, for instance in the lung, the cilium is very thin and a direct simulation with a graded mesh would be too expensive. In the asymptotics of a zero diameter cilium and an infinite velocity, the cilium is thus replaced by a line Dirac of forces in source term. In order to ease the computations, the line Dirac of forces is approached by a sum of punctual forces distributed along the cilium [15]. Finally, by linearity of the Stokes problem, the analysis of the subsequent problem reduces to Problem (1).

In dimension 2, Problem (1) has no $H^1(\Omega)^2 \times \mathbb{L}^2(\Omega)$-solution. Although the numerical solution can be defined in this case, the $H^1(\Omega)$-error (respectively $\mathbb{L}^2(\Omega)$-error) for the velocity (respectively the pressure) has no sense, and the $\mathbb{L}^2$-estimates of the velocity cannot be derived like in the regular case without suitable modifications.

Let us notice that the scalar version of this problem, the Poisson Problem with a Dirac mass in right-hand side, has already been widely studied. It occurs in many applications from different areas like in optimal control of elliptic problems with state contraints [7] or in the mathematical modeling of electromagnetic fields [13]. Problems of this type are found in controllability for elliptic parabolic equations [8,9,16] and in parameter identification problems with pointwise measurements [19]. In this case, Babu$\tilde{s}$ka proved in [3] for a two-dimensional smooth domain an $\mathbb{L}^2(\Omega)$-convergence of order $h^{1-\varepsilon}$, $\varepsilon > 0$, where $h$ is the mesh size, and Scott has shown in [20] an a priori error estimates of order $2 - \frac{d}{2}$, where the dimension $d$ is 2 or 3. Casas has got the same result in [6] for general Borel measures on the right-hand side.

To the best of our knowledge, there is no finite element method convergence result for the Stokes problem with a punctual force in source term. Moreover, in applications, the punctual force (or the Dirac measure) at the point $\mathbf{x}_0$ is often a model reduction approach and the finite element solution does not need to approximate precisely the exact solution at the point $\mathbf{x}_0$. Thus, it is interesting to estimate the error on a fixed subdomain which does not contain the singularity. In the case of the Poisson problem, Köppl and Wohlmuth recently showed in [14] a quasi-optimal local convergence for low order in $\mathbb{L}^2$-norm for Lagrange finite elements and optimal local convergence for higher orders. In this paper, we establish in dimension 2 local error estimates for the Stokes problem with a punctual force in source term, Problem (1), and show a quasi-optimal convergence in $H^1 \times \mathbb{L}^2$-norm. The proof is based on the Arnold and Liu Theorem [2] that establishes local error estimates for finite element discretizations of the Stokes equations with regular source term. It is written in the case of the $\mathbb{P}_k/\mathbb{P}_{k-1}$ elements for $k \geqslant 2$, and the MINI finite element method $\mathbb{P}_1\mathrm{b}/\mathbb{P}_1$. No graded meshes are required for these results and they imply that there is no pollution effects far from the singularity.

The paper is organized as follows. Our main result is formulated in Sect. 2 followed by the Arnold and Liu Theorem [2], an important tool for the proof presented in Sect. 3. Our theoretical results are generalized in Sect. 4, before being illustrated in Sect. 5 by some numerical simulations.

## 2 Main results

In this section, we first set all the notations used in this paper. Then, we formulate our main result and give an important tool for the proof: the Arnold and Liu Theorem. For the sake of clarity, this result is first set and proved in the particular case of the $\mathbb{P}_k/\mathbb{P}_{k-1}$ finite elements ($k \geqslant 2$) and the $\mathbb{P}_1\mathrm{b}/\mathbb{P}_1$ elements. It will be generalized in Sect. 4.

### 2.1 Notations

For a domain $D$, we will denote by $\|\cdot\|_{s,q,D}$ (respectively $|\cdot|_{s,q,D}$) the norm (respectively semi-norm) of the Sobolev space $W^{s,q}(D)$, and by $\|\cdot\|_{s,D}$ (respectively $|\cdot|_{s,D}$) the norm (respectively semi-norm) of the Sobolev space $H^s(\Omega)$. Letters in bold refer to a vector of $\mathbb{R}^2$ or a function with values in $\mathbb{R}^2$. We also introduce the functional space $L_0^2(\Omega)$, that denotes all functions in $L^2(\Omega)$ with 0 average on $\Omega$.

For the numerical solution, let us introduce a family of quasi-uniform simplical triangulations $\mathcal{T}_h$ of $\Omega$, where $h$ is the meshsize. The approximation spaces for velocity and pressure will be denoted by $V_h^k$ and $W_h^k$ respectively, where $k$ is the order of the global method. In practice we will use the $\mathbb{P}_k/\mathbb{P}_{k-1}$ finite elements, for $k \geqslant 2$, defined as

$$\begin{aligned} V_h^k &= \left\{ \mathbf{v}_h \in \mathscr{C}(\bar{\Omega})^2 \,\middle|\, \mathbf{v}_{h|_T} \in P_k[T], \forall T \in \mathcal{T}_h \right\}, \\ W_h^k &= \left\{ p_h \in \mathscr{C}(\bar{\Omega}) \,\middle|\, p_{h|_T} \in P_{k-1}[T], \forall T \in \mathcal{T}_h \right\}, \end{aligned}$$

and for $k = 1$, we will use the MINI finite element method $\mathbb{P}_1\mathrm{b}/\mathbb{P}_1$, where $\mathbb{P}_1\mathrm{b}$ denotes the continous piecewise linear and bubble functions. For a finite element $T$, the bubble function $b$ is defined by

$$b(x) = \begin{cases} \lambda_1^T(x)\lambda_2^T(x)\lambda_3^T(x) & \text{if } x \in T, \\ 0 & \text{else,} \end{cases}$$

where $\lambda_1^T$, $\lambda_2^T$ and $\lambda_3^T$ are the barycentric coordinates of $x$ in relation to the triangle $T$.

We fix two subdomains of $\Omega$, called $\Omega_0$ and $\Omega_1$, such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and $\mathbf{x}_0 \notin \overline{\Omega_1}$ (see Fig. 1). We consider a mesh which satisfies the following condition:

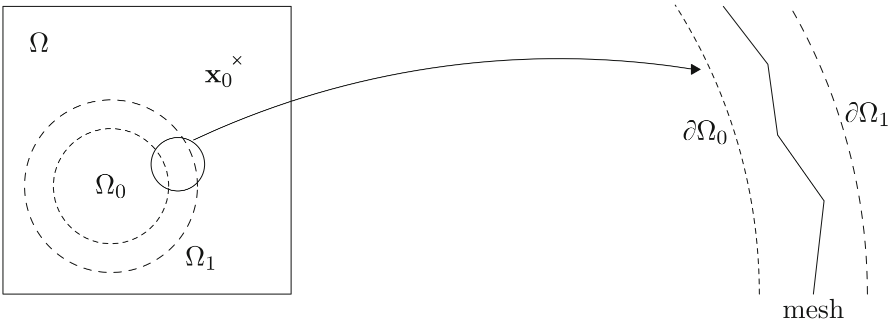


**Fig. 1** Domains $\Omega_0$ and $\Omega_1$

**Assumption 1** For some $h_0$, we have for all $0 < h \leqslant h_0$ (see Fig. 1),

$$\overline{\Omega}_0^m \cap \Omega_1^c = \varnothing, \quad \text{where } \overline{\Omega}_0^m = \bigcup_{\substack{T \in \mathcal{T}_h \\ T \cap \Omega_0 \neq \varnothing}} T.$$

## 2.2 Statement of our main results

Our main result is given by Theorem 1. The rest of the paper is mostly concerned by the proof, the generalization and the illustration of this theorem.

**Theorem 1** *Consider $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ satisfying Assumption 1, $k \geqslant 1$, $1 \leqslant q < 2$, let $(\mathbf{u}, p) \in W_0^{1,q}(\Omega) \times \mathbb{L}_0^q(\Omega)$ be the solution of Problem (1) and $(\mathbf{u}_h, p_h)$ its Galerkin projection onto $V_h^k \times W_h^k$ satisfying $\int_\Omega p_h = 0$ and*

$$\begin{aligned} \int_\Omega \nabla(\mathbf{u} - \mathbf{u}_h) :: \nabla \eta - \int_\Omega (p - p_h) \operatorname{div}(\eta) &= 0 \quad \text{for all } \eta \in V_h^k, \\ \int_\Omega \operatorname{div}(\mathbf{u} - \mathbf{u}_h) \xi &= 0 \quad \text{for all } \xi \in W_h^k. \end{aligned} \tag{2}$$

*Under the assumption that $(\mathbf{u}, p) \in H^{k+1}(\Omega_1)^2 \times H^k(\Omega_1)$, there exists $h_1$ such that if $0 < h \leqslant h_1$, we have,*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega_0} + \|p - p_h\|_{0,\Omega_0} \leqslant C(\Omega_0, \Omega_1, \Omega) h^k \sqrt{|\ln h|}.$$

***Remark 1*** Note that since $V_h^k \subset \mathscr{C}(\Omega)$, the Galerkin projection of the solution of Problem (1) onto $V_h^k \times W_h^k$ is well-defined.

### 2.3 Regularity of the solution $(\mathbf{u}, p)$

In this subsection, we focus on the singularity of the solution, which is the main difficulty in the study of this kind of problems. In dimension 2, Problem (1) has a unique weak solution $(\mathbf{u}, p) \in W^{1,q}(\Omega)^2 \times \mathbb{L}_0^q(\Omega)$ for all $q \in [1, 2[$. Indeed, the 2d Stokeslet denoting $(\mathbf{u}_\delta, p_\delta)$ is defined as (see for instance [18])

$$\begin{aligned}\mathbf{u}_\delta(\mathbf{x}) &= \frac{1}{4\pi}\left(-\ln\|\mathbf{x}\|\mathbf{I}_2 + \frac{\mathbf{x}\otimes\mathbf{x}}{\|\mathbf{x}\|^2}\right)\mathbf{F}, \\ p_\delta(\mathbf{x}) &= \frac{\mathbf{x}\cdot\mathbf{F}}{2\pi\|\mathbf{x}\|^2}.\end{aligned} \tag{3}$$

The Stokeslet $(\mathbf{u}_\delta, p_\delta)$ satisfies in $\mathscr{D}'(\mathbb{R}^2)$

$$\begin{cases} -\triangle\mathbf{u}_\delta + \nabla p_\delta = \delta_0\mathbf{F}, \\ \operatorname{div}(\mathbf{u}_\delta) = 0, \end{cases}$$

so that the Stokeslet $(\mathbf{u}_\delta(\cdot - \mathbf{x}_0), p_\delta(\cdot - \mathbf{x}_0))$ contains the singular part of $(\mathbf{u}, p)$, the solution of Problem (1). As it is done in [1] in the case of the Poisson problem, the solution $(\mathbf{u}, p)$ can be built by using a suitable lift procedure which consists in adding to $(\mathbf{u}_\delta, p_\delta)$ a corrector term $(\mathbf{w}, \pi) \in H^1(\Omega)^2 \times \mathbb{L}_0^2(\Omega)$, which satisfies the following problem:

$$\begin{cases} -\triangle\mathbf{w} + \nabla\pi = 0 & \text{in} \quad \Omega, \\ \operatorname{div}(\mathbf{w}) = 0 & \text{in} \quad \Omega, \\ \mathbf{w} = -\mathbf{u}_\delta(\cdot - \mathbf{x}_0) & \text{on} \quad \partial\Omega. \end{cases}$$

Then, the solution is given by:

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= \mathbf{u}_\delta(\mathbf{x}) + \mathbf{w}(\mathbf{x}) = \frac{1}{4\pi}\left(-\ln\|\mathbf{x}\|\mathbf{I}_2 + \frac{\mathbf{x}\otimes\mathbf{x}}{\|\mathbf{x}\|^2}\right)\mathbf{F} + \mathbf{w}(\mathbf{x}), \\ p(\mathbf{x}) &= p_\delta(\mathbf{x}) + \pi(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{F}}{2\pi\|\mathbf{x}\|^2} + \pi(\mathbf{x}).\end{aligned}$$

Moreover, it is easy to show that $\mathbf{u}_\delta \notin H_0^1(\Omega)^2$ and $p_\delta \notin \mathbb{L}^2(\Omega)$. Actually, we can specify how the quantity $|\mathbf{u}_\delta|_{1,q,\Omega}$ goes to infinity when $q$ goes to 2, with $q < 2$ (which will be noted $q \underset{<}{\to} 2$). According to (3), estimating $|\mathbf{u}_\delta|_{1,q,\Omega}$ when $q \underset{<}{\to} 2$ is reduced to estimate $|\mathbf{u}_\delta|_{1,q,B}$, where $B$ denotes the ball $B(\mathbf{x}_0, 1)$: we can easily show that there exists $C > 0$ depending only on $\mathbf{F}$ such that

$$\forall 1 \leqslant q < +\infty,\ \mathbf{u}_\delta \in \mathbb{L}^q(\Omega) \quad \text{and} \quad |\nabla\mathbf{u}_\delta| \leqslant \frac{C}{\|\mathbf{x}\|},$$

and so, using polar coordinates, we get for $q < 2$,

$$|\mathbf{u}_\delta|^q_{1,q,\Omega} = \int_B |\nabla \mathbf{u}_\delta(\mathbf{x})|^q \mathrm{d}x \leqslant \int_B \frac{C^q}{\|\mathbf{x}\|^q}\mathrm{d}x$$
$$= C^q \int_0^1 \int_0^{2\pi} \frac{1}{r^{q-1}} \mathrm{d}\theta \mathrm{d}r = 2\pi C^q \frac{1}{2-q}.$$

Finally, there exists $C > 0$ independent of $q$ such that, for $1 \leqslant q < 2$,

$$|\mathbf{u}_\delta|_{1,q,\Omega} \leqslant \frac{C}{\sqrt{2-q}}. \tag{4}$$

In the same way, we can easily show that there exists $C > 0$ independent of $q$ such that, for $1 \leqslant q < 2$,

$$|p_\delta|_{0,q,\Omega} \leqslant \frac{C}{\sqrt{2-q}}. \tag{5}$$

### 2.4 Arnold and Liu Theorem

Before stating Arnold and Liu Theorem, let us enumerate the assumptions that the finite element spaces $\mathcal{V}_h^k$ and $\mathcal{W}_h^k$ have to satisfy so that the theorem is valid.

**Assumption 2** Given two fixed concentric spheres $B_0$ and $B$ with $B_0 \subset\subset B \subset\subset \Omega$, there exists an $h_0$ such that for all $0 < h \leqslant h_0$, we have for some integers $k_1$ and $k_2$:

**B1** For any $1 \leqslant \ell$, for each $\mathbf{v} \in H^\ell(B)^2$, there exists $\eta \in \mathcal{V}_h^k$ such that

$$\|\mathbf{v} - \eta\|_{1,B} \leqslant Ch^{r_1-1}\|\mathbf{v}\|_{\ell,B}, \quad r_1 = \min(k_1+1, \ell).$$

For any $0 \leqslant s$, for each $\pi \in H^s(B)$, there exists $\xi \in \mathcal{W}_h^k$ such that

$$\|\pi - \xi\|_{0,B} \leqslant Ch^{r_2}\|\pi\|_{s,B}, \quad r_2 = \min(k_2+1, s).$$

Moreover, if $\mathbf{v} \in H_0^1(B_0)^2$ (respectively $\pi$ vanishes on $B\backslash\overline{B}_0$) then $\eta$ (respectively $\xi$) can be chosen to satisfy $\eta \in H_0^1(B)^2$ (respectively $\xi$ vanishes on $\Omega\backslash\overline{B}$).

**B2** Let $\varphi \in \mathscr{C}_0^\infty(B_0)$, $\mathbf{v}_h \in \mathcal{V}_h^k$ and $\pi_h \in \mathcal{W}_h^k$, then there exist $\eta \in \mathcal{V}_h^k \cap H_0^1(B)$ and $\xi \in \mathcal{W}_h^k$ with supp $\xi \subset \overline{B}$ such that

$$\|\varphi\mathbf{v}_h - \eta\|_{1,B} \leqslant C(\varphi, B, B_0)h\|\mathbf{v}_h\|_{1,B},$$
$$\|\varphi\pi_h - \xi\|_{0,B} \leqslant C(\varphi, B, B_0)h\|\pi_h\|_{0,B}.$$

**B3** For each $0 < h \leqslant h_0$ there exists a domain $B_h$ with $B_0 \subset\subset B_h \subset\subset B$ such that for any $0 \leqslant \ell$, for all $\mathbf{v}_h \in \mathcal{V}_h^k$ and $\pi_h \in \mathcal{W}_h^k$, we have

$$\|\mathbf{v}_h\|_{1,B_h} \leqslant Ch^{-1-\ell}\|\mathbf{v}_h\|_{-\ell,B_h},$$
$$\|\pi_h\|_{0,B_h} \leqslant Ch^{-\ell}\|\pi_h\|_{-\ell,B_h}.$$

**B4** There exists $\beta > 0$ such that for all $0 < h \leqslant h_0$, there is a domain $B_h$, with $B_0 \subset\subset B_h \subset\subset B$ for which

$$\inf_{\substack{\pi_h \in \mathcal{W}_h^k \\ \operatorname{supp} \pi_h \subset B_h}} \sup_{\substack{\mathbf{v}_h \in \mathcal{V}_h^k \\ \operatorname{supp} \mathbf{v}_h \subset B_h}} \frac{\int_{B_h} \operatorname{div}(\mathbf{v}_h)\pi_h}{|\pi_h|_{0,B_h} |\mathbf{v}_h|_{1,B_h}} \geqslant \beta > 0.$$

We now state the following theorem by Arnold and Liu [2], a key tool in the forthcoming proof of Theorem 1.

**Theorem** (Arnold and Liu [2]) *Consider* $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, $V_h^k$ *and* $W_h^k$ *satisfy Assumption* 2. *Suppose that* $(\mathbf{v}, \pi) \in H^1(\Omega)^2 \times \mathbb{L}^2(\Omega)$ *satisfies* $(\mathbf{v}, \pi)_{|\Omega_1} \in H^\ell(\Omega_1)^2 \times H^{\ell-1}(\Omega_1)$ *for some* $\ell > 0$. *Suppose that* $(\mathbf{v}_h, \pi_h) \in V_h^k \times W_h^k$ *satisfies* $\int_\Omega (\pi - \pi_h) = 0$ *and*

$$\int_\Omega \nabla(\mathbf{v} - \mathbf{v}_h) :: \nabla\eta - \int_\Omega (\pi - \pi_h)\operatorname{div}(\eta) = 0 \quad \textit{for all } \eta \in V_h^k,$$
$$\int_\Omega \operatorname{div}(\mathbf{v} - \mathbf{v}_h)\xi = 0 \quad \textit{for all } \xi \in W_h^k.$$

*Let* $t$ *be a nonnegative integer. Then there exist a constant* $C > 0$ *and a real* $h_1 > 0$ *depending only on* $\Omega_1$, $\Omega_0$, *and* $t$, *such that if* $0 < h \leqslant h_1$ *we have*

$$\begin{aligned} \|\mathbf{v} - \mathbf{v}_h\|_{1,\Omega_0} + \|\pi - \pi_h\|_{0,\Omega_0} \leqslant C(&h^{r_1-1}\|\mathbf{v}\|_{\ell,\Omega_1} + h^{r_2-1}\|\pi\|_{\ell-1,\Omega_1} \\ &+ \|\mathbf{v} - \mathbf{v}_h\|_{-t,\Omega_1} + \|\pi - \pi_h\|_{-t-1,\Omega_1}), \end{aligned}$$

*where* $r_1 = \min(k_1 + 1, \ell)$, $r_2 = \min(k_2 + 2, \ell)$, *and* $k_1$, $k_2$ *as in Assumption* **B1**.

Assumption **B1** is quite standard and satisfied by a wide class of finite element spaces, including all finite element spaces defined on quasi-uniform meshes [10]. These elements also satisfy the inverse property defined by assumption **B3** (see [17], §2). The parameters $k_1$ and $k_2$ play respectively the role of the order of approximation of the spaces $\mathcal{V}_h^k$ and $\mathcal{W}_h^k$. In our paper, for $k \geqslant 2$, we will have $k_1 = k$ and $k_2 = k - 1$, and for $k = 1$, we will have $k_1 = k_2 = k = 1$. Assumption **B2** is less common but also satisfied by a wide variety of approximation spaces, including the $\mathbb{P}_1$b-finite elements [2]. Actually, for Lagrange finite elements, a more general property than assumption **B2** is shown in [4]: let $0 \leqslant s \leqslant \ell \leqslant k$, $\varphi \in \mathscr{C}_0^\infty(B)$ and $\mathbf{v}_h \in V_h^k$, then there exists $\eta \in V_h^k$ such that

$$\|\varphi\mathbf{v}_h - \eta\|_{s,B} \leqslant C(\varphi)h^{\ell-s+1}\|\mathbf{v}_h\|_{\ell,B}. \tag{6}$$

Applied for $s = \ell = 1$, inequality (6) gives assumption **B2**. When $B_h = \Omega$, Assumption **B4** is the standard stability condition or discrete inf-sup condition of the Stokes elements. It usually holds as long as $B_h$ is a union of elements.

## 3 Proof of Theorem 1

This section is devoted to the proof of our main result: Theorem 1. First, we show a weak version of Aubin–Nitsche duality lemma (Lemma 1), then we establish two discrete inf-sup conditions (Lemmas 2, 3), and finally we use these results to prove Theorem 1.

### 3.1 Aubin–Nitsche duality lemma with a Dirac source term

The proof of Theorem 1 is based on Arnold and Liu Theorem. In order to estimate the quantities $\|\mathbf{u}-\mathbf{u}_h\|_{-t,\Omega_1}$ and $\|p-p_h\|_{-t-1,\Omega_1}$, we will first show a weak version of Aubin–Nitsche Lemma in the case of the Stokes Problem with a Dirac source term.

**Lemma 1** *Consider* $\mathbf{f} \in W^{-1,q}(\Omega)^2 = (W_0^{1,q'}(\Omega)^2)'$, $1 < q < 2$, *and let* $(\mathbf{w},\pi) \in W_0^{1,q}(\Omega) \times \mathbb{L}^q(\Omega)$ *be the unique solution of*

$$\begin{cases} -\triangle \mathbf{w} + \nabla \pi = \mathbf{f} & in \quad \Omega, \\ \operatorname{div}(\mathbf{w}) = 0 & in \quad \Omega, \\ \mathbf{w} = 0 & on \quad \partial\Omega. \end{cases}$$

*Let* $(\mathbf{w}_h, \pi_h)$ *be the Galerkin projection of* $(\mathbf{w},\pi)$ *in* $V_h^k \times W_h^k$. *For any integer* $0 \leqslant t \leqslant k-1$,

$$\begin{aligned} &\|\mathbf{w}-\mathbf{w}_h\|_{-t,\Omega} + \|\pi-\pi_h\|_{-t-1,\Omega} \\ &\quad \leqslant Ch^{2(1/q'-1/2)} h^{t+1} \left( |\mathbf{w}-\mathbf{w}_h|_{1,q,\Omega} + |\pi-\pi_h|_{0,q,\Omega} \right), \end{aligned} \tag{7}$$

*where* $C = C(t, q', \Omega)$ *is uniformly bounded in* $q'$ *when* $q' \to 2$.

For the proof of this Lemma we will need the following result.

**Proposition 1** (Girault and Raviart [12], Corollary A.2, page 97) *Let* $\mathcal{T}_h$ *be a family of quasi-uniform simplicial triangulations of* $\Omega \subset \mathbb{R}^2$, *where* $h$ *is the meshsize. For any* $0 \leqslant m \leqslant t+1 \leqslant k$, *for any mesh element* $T$ *in the family, for any* $v \in W^{k+1,q'}(\Omega)$, $q' \geqslant 2$ *real,*

$$|v - \Pi_h v|_{m,q',T} \leqslant Ch^{2(1/q'-1/2)} h^{t+2-m} |v|_{t+2,2,T}, \tag{8}$$

*where* $\Pi_h v$ *is the* $P_k$*-interpolant of the function* $v$ *and* $C$ *is a constant that only depends on* $q'$, $t$ *and* $m$, *uniformly bounded in* $q'$ *when* $q' \to 2$.

Let us now prove Lemma 1.

***Proof*** We aim at estimating, for $t \geqslant 0$, the $H^{-t}(\Omega)$-norm and the $H^{-t-1}(\Omega)$-norm respectively of the errors $\mathbf{w}-\mathbf{w}_h$ and $\pi - \pi_h$:

$$\|\mathbf{w}-\mathbf{w}_h\|_{-t,\Omega} = \sup_{\varphi \in \mathscr{C}_0^\infty(\Omega)^2} \frac{1}{\|\varphi\|_{t,\Omega}} \left| \int_\Omega (\mathbf{w}-\mathbf{w}_h)\cdot\varphi \right| \tag{9}$$

$$\|\pi - \pi_h\|_{-t-1,\Omega} = \sup_{\psi \in \mathscr{C}_0^\infty(\Omega)} \frac{1}{\|\psi\|_{t+1,\Omega}} \left| \int_\Omega (\pi - \pi_h)\psi \right| \tag{10}$$

The Galerkin projection $(\mathbf{w}_h, \pi_h)$ satisfies $\int_\Omega \pi - \pi_h = 0$ and

$$\begin{aligned} \int_\Omega \nabla(\mathbf{w} - \mathbf{w}_h)::\nabla\eta - \int_\Omega (\pi - \pi_h)\mathrm{div}(\eta) = 0 \quad & \text{for all } \eta \in V_h^k, \\ \int_\Omega \mathrm{div}(\mathbf{w} - \mathbf{w}_h)\xi = 0 \quad & \text{for all } \xi \in W_h^k. \end{aligned} \tag{11}$$

Consider $\varphi \in \mathscr{C}_0^\infty(\Omega)^2$ and let $(\mathbf{w}^\varphi, \pi^\varphi)$ be the solution of

$$\begin{cases} -\triangle \mathbf{w}^\varphi + \nabla \pi^\varphi = \varphi & \text{in} \quad \Omega, \\ \mathrm{div}(\mathbf{w}^\varphi) = 0 & \text{in} \quad \Omega, \\ \mathbf{w}^\varphi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

On a smooth domain, the unique solution to this problem belongs to $H^{t+2}(\Omega) \times H^{t+1}(\Omega)$, for any $t \geq 0$ (see for instance [21], Chapter I, §2), and we have the estimate

$$\|\mathbf{w}^\varphi\|_{t+2,\Omega} + \|\pi^\varphi\|_{t+1,\Omega} \leqslant C\|\varphi\|_{t,\Omega}, \tag{12}$$

with $C = C(t, \Omega)$. This is not true on general polygonal domains, but it holds on rectangular domains, as can be proven following [17] (see Example 3 in §7).

In dimension 2, by the Sobolev injections established for instance in [5], we have

$$H^{t+2}(\Omega) \subset W^{1,q'}(\Omega), \qquad H^{t+1}(\Omega) \subset \mathbb{L}^{q'}(\Omega), \tag{13}$$

for all $q'$ in $[2, +\infty[$. Thus

$$\begin{aligned} \int_\Omega (\mathbf{w} - \mathbf{w}_h) \cdot \varphi &= -\int_\Omega (\mathbf{w} - \mathbf{w}_h) \cdot \triangle \mathbf{w}^\varphi + \int_\Omega (\mathbf{w} - \mathbf{w}_h) \cdot \nabla \pi^\varphi \\ &= \int_\Omega \nabla(\mathbf{w} - \mathbf{w}_h)::\nabla \mathbf{w}^\varphi - \int_\Omega \mathrm{div}(\mathbf{w} - \mathbf{w}_h)\pi^\varphi. \end{aligned}$$

By adding (11) in the last equality, we get for any $\eta \in V_h^k$ and any $\xi \in W_h^k$,

$$\begin{aligned} \int_\Omega (\mathbf{w} - \mathbf{w}_h) \cdot \varphi = \int_\Omega \nabla(\mathbf{w} - \mathbf{w}_h)::\nabla(\mathbf{w}^\varphi - \eta) - \int_\Omega \mathrm{div}(\mathbf{w} \\ - \mathbf{w}_h)(\pi^\varphi - \xi) + \int_\Omega \mathrm{div}(\eta)(\pi - \pi_h). \end{aligned}$$

By definition of $\mathbf{w}^\varphi$, $\operatorname{div}(\mathbf{w}^\varphi) = 0$ on $\Omega$, so

$$\begin{aligned}\int_\Omega (\mathbf{w} - \mathbf{w}_h) \cdot \varphi &= \int_\Omega \nabla(\mathbf{w} - \mathbf{w}_h)::\nabla(\mathbf{w}^\varphi - \eta) - \int_\Omega \operatorname{div}(\mathbf{w} - \mathbf{w}_h)(\pi^\varphi - \xi) \\ &\quad + \int_\Omega \operatorname{div}(\eta - \mathbf{w}^\varphi)(\pi - \pi_h) \\ &\leqslant |\mathbf{w} - \mathbf{w}_h|_{1,q,\Omega} \left(|\mathbf{w}^\varphi - \eta|_{1,q',\Omega} + |\pi^\varphi - \xi|_{0,q',\Omega}\right) \\ &\quad + |\pi - \pi_h|_{0,q,\Omega} |\mathbf{w}^\varphi - \eta|_{1,q',\Omega}. \end{aligned} \tag{14}$$

Now let us deal with the pressure estimate. For any $\psi \in \mathscr{C}_0^\infty(\Omega)$, we denote by $\widetilde{\psi}$ the function

$$\widetilde{\psi} = \psi - \frac{1}{|\Omega|} \int_\Omega \psi.$$

By definition, it is easy to see that $\widetilde{\psi}$ satisfies

$$\int_\Omega \widetilde{\psi} = 0 \ \text{ and } \ \forall t \geqslant 0,\ \|\widetilde{\psi}\|_{t+1,\Omega} \leqslant C(\Omega) \|\psi\|_{t+1,\Omega}.$$

We can now establish the result for the pressure: consider $\psi \in \mathscr{C}_0^\infty(\Omega)$ and let $(\mathbf{w}^\psi, \pi^\psi) \in H^{t+2}(\Omega) \times H^{t+1}(\Omega)$, for any $t \geq 0$, be the solution of

$$\begin{cases} -\triangle \mathbf{w}^\psi + \nabla \pi^\psi = 0 & \text{in } \ \Omega, \\ \operatorname{div}(\mathbf{w}^\psi) = \widetilde{\psi} & \text{in } \ \Omega, \\ \mathbf{w}^\psi = 0 & \text{on } \ \partial\Omega, \end{cases}$$

see [21] (Chapter I, §2) for the existence and the uniqueness of the solution on a smooth domain and the following estimate, that holds also on rectangular domains:

$$\|\mathbf{w}^\psi\|_{t+2,\Omega} + \|\pi^\psi\|_{t+1,\Omega} \leqslant C \|\widetilde{\psi}\|_{t+1,\Omega} \leqslant C \|\psi\|_{t+1,\Omega}, \tag{15}$$

with $C = C(t, \Omega)$. Moreover, $\displaystyle\int_\Omega (\pi - \pi_h) = 0$, so that

$$\int_\Omega (\pi - \pi_h)\psi = \int_\Omega (\pi - \pi_h)\widetilde{\psi} + \frac{1}{|\Omega|} \int_\Omega \psi \int_\Omega \pi - \pi_h = \int_\Omega (\pi - \pi_h)\widetilde{\psi}.$$

By the Sobolev injections recalled in (13), and the Galerkin projection property (11), we can write for all $\eta \in V_h^k$,

$$\begin{aligned}\int_\Omega (\pi - \pi_h)\psi &= \int_\Omega (\pi - \pi_h)\widetilde{\psi} \\ &= \int_\Omega (\pi - \pi_h)\operatorname{div}(\mathbf{w}^\psi) \\ &= \int_\Omega (\pi - \pi_h)\operatorname{div}(\mathbf{w}^\psi - \eta) + \int_\Omega \nabla(\mathbf{w} - \mathbf{w}_h)::\nabla\eta. \end{aligned}$$

Then, for all $\mathbf{v} \in W_0^{1,q}(\Omega)$

$$\int_\Omega \nabla \mathbf{w}^\psi :: \nabla \mathbf{v} - \int_\Omega \pi^\psi \mathrm{div}(\mathbf{v}) = 0,$$

so, with $\mathbf{v} = \mathbf{w} - \mathbf{w}_h$, and for any $\xi \in W_h^k$,

$$\begin{aligned}
\int_\Omega (\pi - \pi_h)\psi &= \int_\Omega (\pi - \pi_h)\mathrm{div}(\mathbf{w}^\psi - \eta) + \int_\Omega \nabla(\mathbf{w} - \mathbf{w}_h)::\nabla(\eta - \mathbf{w}^\psi) \\
&\quad + \int_\Omega \pi^\psi \mathrm{div}(\mathbf{w} - \mathbf{w}_h) \\
&= \int_\Omega (\pi - \pi_h)\mathrm{div}(\mathbf{w}^\psi - \eta) + \int_\Omega \nabla(\mathbf{w} - \mathbf{w}_h)::\nabla(\eta - \mathbf{w}^\psi) \\
&\quad + \int_\Omega (\pi^\psi - \xi)\mathrm{div}(\mathbf{w} - \mathbf{w}_h) \\
&\leqslant |\pi - \pi_h|_{0,q,\Omega} |\mathbf{w}^\psi - \eta|_{1,q',\Omega} \\
&\quad + |\mathbf{w} - \mathbf{w}_h|_{1,q,\Omega} \left( |\mathbf{w}^\psi - \eta|_{1,q',\Omega} + |\pi^\psi - \xi|_{0,q',\Omega} \right). \qquad (16)
\end{aligned}$$

Finally, for any $(\eta_1, \xi_1) \in V_h^k \times W_h^k$,

$$\begin{aligned}
\int_\Omega (\mathbf{w} - \mathbf{w}_h) \cdot \varphi &\leqslant |\mathbf{w} - \mathbf{w}_h|_{1,q,\Omega} \left( |\mathbf{w}^\varphi - \eta_1|_{1,q',\Omega} + |\pi^\varphi - \xi_1|_{0,q',\Omega} \right) \\
&\quad + |\pi - \pi_h|_{0,q,\Omega} |\mathbf{w}^\varphi - \eta_1|_{1,q',\Omega},
\end{aligned}$$

and for any $(\eta_2, \xi_2) \in V_h^k \times W_h^k$,

$$\begin{aligned}
\int_\Omega (\pi - \pi_h)\psi &\leqslant |\pi - \pi_h|_{0,q,\Omega} |\mathbf{w}^\psi - \eta_2|_{1,q',\Omega} \\
&\quad + |\mathbf{w} - \mathbf{w}_h|_{1,q,\Omega} \left( |\mathbf{w}^\psi - \eta_2|_{1,q',\Omega} + |\pi^\psi - \xi_2|_{0,q',\Omega} \right).
\end{aligned}$$

Let us now estimate $|\mathbf{w}^\varphi - \eta_1|_{1,q',\Omega}$, $|\mathbf{w}^\psi - \eta_2|_{1,q',\Omega}$, $|\pi^\varphi - \xi_1|_{0,q',\Omega}$ and $|\pi^\psi - \xi_2|_{0,q',\Omega}$. Up to now and until the end of this proof, we will take

$$\begin{aligned}
\eta_1 = \Pi_h \mathbf{w}^\varphi \text{ and } \eta_2 = \Pi_h \mathbf{w}^\psi \in V_h^k, \\
\xi_1 = \widetilde{\Pi}_h \pi^\varphi \text{ and } \xi_2 = \widetilde{\Pi}_h \pi^\psi \in W_h^k,
\end{aligned}$$

where $\Pi_h \mathbf{v}$ is the $P_k$-interpolant of the function $\mathbf{v}$ and $\widetilde{\Pi}_h v$ is the $P_{k-1}$-interpolant of the function $v$. By (8), with $m = 1$, $0 \leqslant t \leqslant k - 1$, for all $T \in \mathcal{T}_h$,

$$\begin{aligned}
|\mathbf{w}^\varphi - \eta_1|_{1,q',T} &\leqslant C h^{2(1/q'-1/2)} h^{t+1} |\mathbf{w}^\varphi|_{t+2,2,T}, \\
|\mathbf{w}^\psi - \eta_2|_{1,q',T} &\leqslant C h^{2(1/q'-1/2)} h^{t+1} |\mathbf{w}^\psi|_{t+2,2,T}, \qquad (17)
\end{aligned}$$

and with $m = 0$,

$$
\begin{aligned}
|\pi^{\varphi} - \xi_1|_{0,q',T} &\leqslant Ch^{2(1/q'-1/2)} h^{t+1} |\pi^{\varphi}|_{t+1,2,T},\\
|\pi^{\psi} - \xi_2|_{0,q',T} &\leqslant Ch^{2(1/q'-1/2)} h^{t+1} |\pi^{\psi}|_{t+1,2,T},
\end{aligned}
$$

with $C = C(q')$ a constant which is uniformly bounded in $q'$ when $q' \to 2$. We denote the triangles of the mesh by $\{T_i\}_{i=1,\cdots,N}$, and we set

$$
a = (a_i)_i \text{ and } b = (b_i)_i, \text{ where } a_i = |\mathbf{w}^{\varphi} - \eta_1|_{1,q',T_i} \text{ and } b_i = |\mathbf{w}^{\varphi}|_{t+2,2,T_i}.
$$

By (17), we have, for all $i$ in $[\![1, N]\!]$,

$$
a_i \leqslant Ch^{2(1/q'-1/2)} h^{t+1} b_i.
$$

We recall the norm equivalence in $\mathbb{R}^N$ for $0 < r < s$,

$$
\|x\|_{\ell^s} \leqslant \|x\|_{\ell^r} \leqslant N^{1/r-1/s} \|x\|_{\ell^s},
$$

with here $N \sim Ch^{-2}$. As $2 < q'$, we have $\|b\|_{\ell^{q'}} \leqslant \|b\|_{\ell^2}$. Then, we can write

$$
\begin{aligned}
|\mathbf{w}^{\varphi} - \eta_1|_{1,q',\Omega} = \|a\|_{\ell^{q'}} &\leqslant Ch^{t+1} h^{2(1/q'-1/2)} \|b\|_{\ell^{q'}}\\
&\leqslant Ch^{t+1} h^{2(1/q'-1/2)} \|b\|_{\ell^2}\\
&\leqslant Ch^{t+1} h^{2(1/q'-1/2)} |\mathbf{w}^{\varphi}|_{t+2,2,\Omega}.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
|\mathbf{w}^{\psi} - \eta_2|_{1,q',\Omega} &\leqslant Ch^{t+1} h^{2(1/q'-1/2)} |\mathbf{w}^{\psi}|_{t+2,2,\Omega},\\
|\pi^{\varphi} - \xi_1|_{0,q',\Omega} &\leqslant Ch^{t+1} h^{2(1/q'-1/2)} |\pi^{\varphi}|_{t+1,2,\Omega},\\
|\pi^{\psi} - \xi_2|_{0,q',\Omega} &\leqslant Ch^{t+1} h^{2(1/q'-1/2)} |\pi^{\psi}|_{t+1,2,\Omega},
\end{aligned}
$$

and by (12) and (15), we get

$$
\begin{aligned}
|\mathbf{w}^{\varphi} - \eta_1|_{1,q',\Omega} &\leqslant Ch^{t+1} h^{2(1/q'-1/2)} \|\varphi\|_{t,2,\Omega},\\
|\mathbf{w}^{\psi} - \eta_2|_{1,q',\Omega} &\leqslant Ch^{t+1} h^{2(1/q'-1/2)} \|\psi\|_{t+1,2,\Omega},\\
|\pi^{\varphi} - \xi_1|_{0,q',\Omega} &\leqslant Ch^{t+1} h^{2(1/q'-1/2)} \|\varphi\|_{t,2,\Omega},\\
|\pi^{\psi} - \xi_2|_{0,q',\Omega} &\leqslant Ch^{t+1} h^{2(1/q'-1/2)} \|\psi\|_{t+1,2,\Omega},
\end{aligned}
$$

Finally, the proof is ended by combining (9), (10), (14), (16), and the last inequalities. □

Let us now introduce a small parameter $0 < \varepsilon < 1$, that is designed to tend to 0. More precisely, it will be taken as $\varepsilon = |\ln h|^{-1}$ at the end of the proof of Theorem 1.

In the following, we will use estimates of the solutions and finite element errors of problem (1) in $W^{1,q}$ and $L^q$ norms, for $q = q_\varepsilon = 2/(1+\varepsilon)$; therefore, we will track the dependence on $q_\varepsilon$ of the constants in the different estimates in order to ensure that they are bounded uniformly in $\varepsilon$ when $\varepsilon \to 0$.

**Corollary 1** *Let* $(\mathbf{u}_h, p_h) \in V_h^k \times W_h^k$ *be the Galerkin projection of the solution* $(\mathbf{u}, p)$ *of Problem* (1)*, for any* $0 < \varepsilon < 1$*,*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{-k+1,\Omega} &+ \|p - p_h\|_{-k,\Omega} \\ &\leqslant Ch^{-\varepsilon}h^k \left(|\mathbf{u}-\mathbf{u}_h|_{1,q_\varepsilon,\Omega} + |p-p_h|_{0,q_\varepsilon,\Omega}\right),\end{aligned}$$

*where* $q_\varepsilon = 2/(1+\varepsilon) \in [1, 2[$*, and* $C$ *is a constant uniformly bounded with respect to* $\varepsilon$ *when* $\varepsilon \to 0$*.*

***Proof*** We will apply Lemma 1 with $\mathbf{f} = \delta_{\mathbf{x}_0}\mathbf{F}$, $\mathbf{w} = \mathbf{u}$, $\pi = p$ and $t = k-1$. We can explicit inequality (7):

$$2\left(\frac{1}{q'_\varepsilon} - \frac{1}{2}\right) = 2\left(\frac{1-\varepsilon}{2} - \frac{1}{2}\right) = -\varepsilon, \tag{18}$$

where

$$q'_\varepsilon = \frac{2}{1-\varepsilon}.$$

It follows

$$\|\mathbf{u}-\mathbf{u}_h\|_{-k+1,\Omega} + \|p - p_h\|_{-k,\Omega} \leqslant Ch^{-\varepsilon}h^k \left(|\mathbf{u}-\mathbf{u}_h|_{1,q_\varepsilon,\Omega} + |p-p_h|_{0,q_\varepsilon,\Omega}\right),$$

where $C = C(k, q'_\varepsilon, \Omega)$ is the constant appearing in inequality (7), and thus it is uniformly bounded with respect to $\varepsilon$ as $\varepsilon \to 0$ (thus when $q'_\varepsilon \to 2$). □

### 3.2 Discrete inf-sup conditions in $\mathbb{L}^{q_\varepsilon}$-norm

Section 3.3 is devoted to estimate of $|\mathbf{u}-\mathbf{u}_h|_{1,q_\varepsilon,\Omega}$ and $|p-p_h|_{0,q_\varepsilon,\Omega}$. In that prospect, we need to establish two discrete inf-sup conditions.

**Lemma 2** *For* $q_\varepsilon = 2/(1+\varepsilon)$ *and* $q'_\varepsilon = 2/(1-\varepsilon)$*, the approximation space* $\mathring{V}_h^k$ *defined by*

$$\mathring{V}_h^k = \left\{\mathbf{v}_h \in V_h^k \,\middle|\, \int_\Omega \operatorname{div}(\mathbf{v}_h)p_h = 0, \forall p_h \in W_h^k\right\},$$

*satisfies the following discrete inf-sup condition:*

$$\inf_{\mathbf{u}_h\in\mathring{V}_h^k}\ \sup_{\mathbf{v}_h\in\mathring{V}_h^k} \frac{\int_\Omega \nabla\mathbf{u}_h :: \nabla\mathbf{v}_h}{|\mathbf{u}_h|_{1,q_\varepsilon,\Omega}|\mathbf{v}_h|_{1,q'_\varepsilon,\Omega}} \geqslant Ch^\varepsilon,$$

*where* $C = C(k, q'_\varepsilon, \Omega)$ *is uniformly bounded with respect to* $\varepsilon$ *when* $\varepsilon \to 0$*.*

For the proof of this Lemma we will need the following result.

**Proposition 2** (Ciarlet, Theorem 3.2.6, page 140 [10]) *Let $\mathcal{T}_h$ a family of quasi-uniform simplicial triangulations of $\Omega \subset \mathbb{R}^d$, where $h$ is the meshsize. For $\mathbf{v}_h \in V_h^k$, $1 \leqslant r, s < +\infty$, $0 \leqslant \ell \leqslant m$,*

$$\left( \sum_{T \in \mathcal{T}_h} |\mathbf{v}_h|_{m,r,T}^r \right)^{1/r} \leqslant C h^{-d[\max\{0, 1/s - 1/r\}]} h^{-(m-\ell)} \left( \sum_{T \in \mathcal{T}_h} |\mathbf{v}_h|_{\ell,s,T}^s \right)^{1/s},$$

*where $C$ is a constant that only depends on $m$, $r$, $s$, $\ell$ and $T$, and is uniformly bounded with respect to $r$ when $r \to 2$.*

Let us now prove Lemma 2.

***Proof*** The bilinear form $a : (\mathbf{u}, \mathbf{v}) \mapsto \int_\Omega \nabla \mathbf{u} :: \nabla \mathbf{v}$ is continuous and coercive on $H_0^1(\Omega)$, so for $\mathring{V}_h^k$ vector subspace of $H_0^1(\Omega)$, we have the inf-sup condition:

$$\inf_{\mathbf{u}_h \in \mathring{V}_h^k} \sup_{\mathbf{v}_h \in \mathring{V}_h^k} \frac{\int_\Omega \nabla \mathbf{u}_h :: \nabla \mathbf{v}_h}{|\mathbf{u}_h|_{1,\Omega} |\mathbf{v}_h|_{1,\Omega}} \geqslant \alpha > 0,$$

where $\alpha$ only depends on $\Omega$. We apply Proposition 2 to any $\mathbf{v}_h \in \mathring{V}_h^k \subset \mathscr{C}(\Omega)$, with $d = 2$, $m = \ell = 1$, $s = 2$ and $r = q_\varepsilon'$ to get:

$$|\mathbf{v}_h|_{1,q_\varepsilon',\Omega} \leqslant C(q_\varepsilon') h^{-2(1/2 - 1/q_\varepsilon')} |\mathbf{v}_h|_{1,2,\Omega} = C(q_\varepsilon') h^{-\varepsilon} |\mathbf{v}_h|_{1,2,\Omega},$$

since $q_\varepsilon' = 2/(1-\varepsilon)$, and $C(q_\varepsilon') > 0$ is uniformly bounded with respect to $\varepsilon$ when $\varepsilon \to 0$. Moreover, for any $\mathbf{u}_h \in \mathring{V}_h^k$,

$$\begin{aligned} |\mathbf{u}_h|_{1,q_\varepsilon,\Omega} \leqslant |\Omega|^{\frac{\varepsilon}{2}} |\mathbf{u}_h|_{1,2,\Omega} &\leqslant \frac{|\Omega|^{\frac{\varepsilon}{2}}}{\alpha} \sup_{\mathbf{v}_h \in \mathring{V}_h^k} \frac{\int_\Omega \nabla \mathbf{u}_h :: \nabla \mathbf{v}_h}{|\mathbf{v}_h|_{1,2,\Omega}} \\ &\leqslant \frac{|\Omega|^{\frac{\varepsilon}{2}}}{C(q_\varepsilon')\alpha} h^{-\varepsilon} \sup_{\mathbf{v}_h \in \mathring{V}_h^k} \frac{\int_\Omega \nabla \mathbf{u}_h :: \nabla \mathbf{v}_h}{|\mathbf{v}_h|_{1,q_\varepsilon',\Omega}} \end{aligned}$$

Finally,

$$\inf_{\mathbf{u}_h \in \mathring{V}_h^k} \sup_{\mathbf{v}_h \in \mathring{V}_h^k} \frac{\int_\Omega \nabla \mathbf{u}_h :: \nabla \mathbf{v}_h}{|\mathbf{u}_h|_{1,q_\varepsilon,\Omega} |\mathbf{v}_h|_{1,q_\varepsilon',\Omega}} \geqslant C h^\varepsilon,$$

where $C = \alpha C(q_\varepsilon') |\Omega|^{-\frac{\varepsilon}{2}}$ is uniformly bounded with respect to $\varepsilon$ when $\varepsilon \to 0$. □

The second discrete inf-sup condition we need is given by the following lemma:

**Lemma 3** *For $q_\varepsilon = 2/(1+\varepsilon)$ and $q_\varepsilon' = 2/(1-\varepsilon)$, the approximations spaces $V_h^k$ and $W_h^k$ satisfy the following discrete inf-sup condition:*

$$\inf_{p_h \in W_h^k} \sup_{\mathbf{v}_h \in V_h^k} \frac{\int_\Omega \operatorname{div}(\mathbf{v}_h) p_h}{|p_h|_{0,q_\varepsilon,\Omega} |\mathbf{v}_h|_{1,q'_\varepsilon,\Omega}} \geqslant C|\Omega|^{-\frac{\varepsilon}{2}} h^\varepsilon,$$

*where $C = C(k, q'_\varepsilon, \Omega)$ is uniformly bounded with respect to $\varepsilon$ when $\varepsilon \to 0$.*

***Proof*** The proof is similar to the proof of Lemma 2. According to Assumption **B4**,

$$\inf_{p_h \in W_h^k} \sup_{\mathbf{v}_h \in V_h^k} \frac{\int_\Omega \operatorname{div}(\mathbf{v}_h) p_h}{|p_h|_{0,\Omega} |\mathbf{v}_h|_{1,\Omega}} \geqslant \beta > 0.$$

According to Proposition 2, for any $\mathbf{v}_h \in V_h^k$,

$$|\mathbf{v}_h|_{1,q'_\varepsilon,\Omega} \leqslant C(q'_\varepsilon) h^{-\varepsilon} |\mathbf{v}_h|_{1,2,\Omega}.$$

So, we have, for any $p_h \in W_h^k$ and $q_\varepsilon < 2$,

$$\begin{aligned} |p_h|_{0,q_\varepsilon,\Omega} &\leqslant |\Omega|^{\frac{\varepsilon}{2}} |p_h|_{0,\Omega} \leqslant \frac{|\Omega|^{\frac{\varepsilon}{2}}}{\beta} \sup_{\mathbf{v}_h \in V_h^k} \frac{\int_\Omega \operatorname{div}(\mathbf{v}_h) p_h}{|\mathbf{v}_h|_{1,2,\Omega}} \\ &\leqslant \frac{|\Omega|^{\frac{\varepsilon}{2}}}{C(q'_\varepsilon)\beta} h^{-\varepsilon} \sup_{\mathbf{v}_h \in V_h^k} \frac{\int_\Omega \operatorname{div}(\mathbf{v}_h) p_h}{|\mathbf{v}_h|_{1,q'_\varepsilon,\Omega}} \end{aligned}$$

Finally, we get

$$\inf_{p_h \in W_h^k} \sup_{\mathbf{v}_h \in V_h^k} \frac{\int_\Omega \operatorname{div}(\mathbf{v}_h) p_h}{|p_h|_{0,q_\varepsilon,\Omega} |\mathbf{v}_h|_{1,q'_\varepsilon,\Omega}} \geqslant C h^\varepsilon,$$

where $C = \beta C(q'_\varepsilon) |\Omega|^{-\frac{\varepsilon}{2}}$ is uniformly bounded with respect to $\varepsilon$ when $\varepsilon \to 0$. □

### 3.3 Estimates of $|\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega}$ and $|p - p_h|_{0,q_\varepsilon,\Omega}$

Following Corollary 1, the quantities $|\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega}$ and $|p - p_h|_{0,q_\varepsilon,\Omega}$ have to be estimated to prove Theorem 1. We will apply the last two results to bound them in terms of $|\mathbf{u}|_{1,q_\varepsilon,\Omega}$ and $|p|_{0,q_\varepsilon,\Omega}$.

**Lemma 4** *Let $(\mathbf{u}_h, p_h) \in V_h^k \times W_h^k$ be the Galerkin projection of the solution $(\mathbf{u}, p)$ of Problem (1), for any small enough real $\varepsilon > 0$, and for $q_\varepsilon = 2/(1+\varepsilon)$ and $q'_\varepsilon = 2/(1-\varepsilon)$,*

$$|\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega} \leqslant C h^{-\varepsilon} \left( |\mathbf{u}|_{1,q_\varepsilon,\Omega} + |p|_{0,q_\varepsilon,\Omega} \right),$$

*where $C = C(k, q'_\varepsilon, \Omega)$ is uniformly bounded with respect to $\varepsilon$ when $\varepsilon \to 0$.*

***Proof*** First, we will estimate $|\mathbf{u}_h|_{1,q_\varepsilon,\Omega}$ in terms of $|\mathbf{u}|_{1,q_\varepsilon,\Omega}$. As we have $\operatorname{div}(\mathbf{u}) = 0$ on $\Omega$, by (2) we have

$$\int_\Omega \operatorname{div}(\mathbf{u}_h) q_h = 0, \quad \forall q_h \in W_h^k,$$

and so, $\mathbf{u}_h \in \mathring{V}_h^k$. According to Lemma 2, there exists $\mathbf{v}_h \in \mathring{V}_h^k$ such as $|\mathbf{v}_h|_{1,q'_\varepsilon,\Omega} = 1$, and

$$|\mathbf{u}_h|_{1,q_\varepsilon,\Omega} \leqslant Ch^{-\varepsilon} \int_\Omega \nabla \mathbf{u}_h :: \nabla \mathbf{v}_h,$$

where $C = C(k, q'_\varepsilon, \Omega) > 0$ and uniformly bounded with respect to $\varepsilon$ when $\varepsilon \to 0$. Moreover, equality (2) gives

$$\int_\Omega \nabla \mathbf{u}_h :: \nabla \mathbf{v}_h = \int_\Omega \nabla \mathbf{u} :: \nabla \mathbf{v}_h - \int_\Omega \operatorname{div}(\mathbf{v}_h)(p - p_h).$$

Now, $\mathbf{v}_h \in \mathring{V}_h^k$, so

$$\int_\Omega \operatorname{div}(\mathbf{v}_h) p_h = 0.$$

Finally, as $|\mathbf{v}_h|_{1,q'_\varepsilon,\Omega} = 1$, we get

$$\begin{aligned}|\mathbf{u}_h|_{1,q_\varepsilon,\Omega} &\leqslant Ch^{-\varepsilon} \left( \int_\Omega \nabla \mathbf{u} :: \nabla \mathbf{v}_h - \int_\Omega \operatorname{div}(\mathbf{v}_h) p \right), \\ &\leqslant Ch^{-\varepsilon} \left( |\mathbf{u}|_{1,q_\varepsilon,\Omega} + |p|_{0,q_\varepsilon,\Omega} \right).\end{aligned}$$

We conclude with the triangulary inequality:

$$|\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega} \leqslant |\mathbf{u}|_{1,q_\varepsilon,\Omega} + |\mathbf{u}_h|_{1,q_\varepsilon,\Omega} \leqslant Ch^{-\varepsilon} \left( (1 + h^\varepsilon)|\mathbf{u}|_{1,q_\varepsilon,\Omega} + |p|_{0,q_\varepsilon,\Omega} \right).$$

And thus for $\varepsilon$ sufficiently small:

$$|\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega} \leqslant Ch^{-\varepsilon} \left( |\mathbf{u}|_{1,q_\varepsilon,\Omega} + |p|_{0,q_\varepsilon,\Omega} \right).$$

□

We can now estimate $|p - p_h|_{0,q_\varepsilon,\Omega}$.

**Lemma 5** *Let $(\mathbf{u}_h, p_h) \in V_h^k \times W_h^k$ be the Galerkin projection of the solution $(\mathbf{u}, p)$ of Problem (1), for any small enough real $\varepsilon > 0$, and for $q_\varepsilon = 2/(1+\varepsilon)$ and $q'_\varepsilon = 2/(1-\varepsilon)$,*

$$|p - p_h|_{0,q_\varepsilon,\Omega} \leqslant Ch^{-2\varepsilon} \left( |\mathbf{u}|_{1,q_\varepsilon,\Omega} + (1 + |\Omega|^{\frac{\varepsilon}{2}})|p|_{0,q_\varepsilon,\Omega} \right),$$

*where $C = C(k, q'_\varepsilon, \Omega)$ is uniformly bounded with respect to $\varepsilon$ when $\varepsilon \to 0$.*

***Proof*** The proof is similar to the velocity case: according to Lemma 3, there exists $\mathbf{v}_h \in V_h^k$ such as $|\mathbf{v}_h|_{1,q'_\varepsilon,\Omega} = 1$ and

$$|p_h|_{0,q_\varepsilon,\Omega} \leqslant Ch^{-\varepsilon} \int_\Omega \operatorname{div}(\mathbf{v}_h) p_h,$$

where $C = C(k, q'_\varepsilon, \Omega)$ is uniformly bounded with respect to $\varepsilon$ when $\varepsilon \to 0$. By (2), we have

$$\int_\Omega \operatorname{div}(\mathbf{v}_h) p_h = -\int_\Omega \nabla(\mathbf{u} - \mathbf{u}_h) :: \nabla \mathbf{v}_h + \int_\Omega \operatorname{div}(\mathbf{v}_h) p.$$

By applying Lemma 4, as $|\mathbf{v}_h|_{1,q'_\varepsilon,\Omega} = 1$, we get

$$\begin{aligned} |p_h|_{0,q_\varepsilon,\Omega} &\leqslant Ch^{-\varepsilon} \left( -\int_\Omega \nabla(\mathbf{u} - \mathbf{u}_h) :: \nabla \mathbf{v}_h + \int_\Omega \operatorname{div}(\mathbf{v}_h) p \right), \\ &\leqslant Ch^{-\varepsilon} \left( |\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega} + |p|_{0,q_\varepsilon,\Omega} \right), \\ &\leqslant Ch^{-2\varepsilon} \left( |\mathbf{u}|_{1,q_\varepsilon,\Omega} + (1 + |\Omega|^{\frac{\varepsilon}{2}}) |p|_{0,q_\varepsilon,\Omega} \right). \end{aligned}$$

□

### 3.4 Proof of Theorem 1

We can now prove Theorem 1.

***Proof*** The functions $\mathbf{u}$ and $p$ are analytic on $\overline{\Omega}_1$, so the quantities $\|\mathbf{u}\|_{k+1,\Omega_1}$ and $\|p\|_{k,\Omega_1}$ are bounded. Let us note that in this case $(\mathbf{u}, p) \notin H_0^1(\Omega)^2 \times \mathbb{L}_0^2(\Omega)$, but Remark 1 allows us to apply Arnold and Liu Theorem. For $k_1 = k$ and

$$k_2 = \begin{cases} 1 & \text{if} \quad k = 1, \\ k - 1 & \text{if} \quad k \geqslant 2, \end{cases}$$

and $l = k + 1$ and $t = k - 1$, we have

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega_0} + \|p - p_h\|_{0,\Omega_0} \leqslant C(h^k + \|\mathbf{u} - \mathbf{u}_h\|_{-k+1,\Omega_1} + \|p - p_h\|_{-k,\Omega_1}).$$

By combining Corollary 1, Lemmas 4 and 5, and inequalities (4) and (5), we get

$$\|\mathbf{u} - \mathbf{u}_h\|_{-k+1,\Omega_1} + \|p - p_h\|_{-k,\Omega_1} \leqslant Ch^k \frac{h^{-3\varepsilon}}{\sqrt{2 - q_\varepsilon}},$$

with $C = C(k, q'_\varepsilon, \Omega)$ uniformly bounded with respect to $\varepsilon$ when $\varepsilon \to 0$. Since $q_\varepsilon = 2/(1 + \varepsilon)$, with $\varepsilon < 1$,

$$\frac{1}{\sqrt{2 - q_\varepsilon}} = \frac{\sqrt{1 + \varepsilon}}{\sqrt{2\varepsilon}} \leqslant \frac{1}{\sqrt{\varepsilon}},$$

therefore, taking $\varepsilon = |\ln h|^{-1}$,

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega_0} + \|p - p_h\|_{0,\Omega_0} \leqslant Ch^k \sqrt{|\ln h|},$$

which ends the proof of Theorem 1. □

## 4 General case

Theorem 1 and its proof have been written in the particular case of the $\mathbb{P}_k/\mathbb{P}_{k-1}$ finite element method, $k \geqslant 2$, and the $\mathbb{P}_1\text{b}/\mathbb{P}_1$ elements (which corresponds to the case $k = 1$). But we can state a more general result.

First, let us focus on the assumptions: let $\mathcal{T}_h$ be a family of quasi-uniform simplicial triangulations of $\Omega$, let $V_h^{k_1}$ and $W_h^{k_2}$ be two approximation spaces satisfying Assumption 2. We will also assume that $V_h^{k_1} \in \mathscr{C}(\Omega)$: this assumption ensures that the finite element solution is well-defined. Moreover, we will need two more assumptions, they will play the role of Propositions 1 and 2:

**Assumption 3** Given $B \subset \Omega$, consider $q' \geqslant 2$, there exists an $h_0$ such that for all $0 < h \leqslant h_0$, we have for some positive integers $k_1$ and $k_2$:

$\widetilde{\mathbf{B}}\mathbf{1}$ For any $1 \leqslant \ell$, for each $\mathbf{v} \in H^\ell(B)^2$, there exists $\eta \in V_h^{k_1}$ such that, for any mesh element $T \subset B$,

$$|\mathbf{v} - \eta|_{1,q',T} \leqslant Ch^{d(1/q'-1/2)} h^{r_1 - 1} |\mathbf{v}|_{\ell,2,T}, \quad r_1 = \min(k_1 + 1, \ell).$$

For any $0 \leqslant s$, for each $\pi \in H^s(B)$, there exists $\xi \in W_h^{k_2}$ such that, for any mesh element $T \subset B$,

$$|\pi - \xi|_{0,q',T} \leqslant Ch^{d(1/q'-1/2)} h^{r_2} |\pi|_{s,2,T}, \quad r_2 = \min(k_2 + 1, s).$$

$\widetilde{\mathbf{B}}\mathbf{3}$ For all $\mathbf{v}_h \in V_h^{k_1}$, for any mesh element $T \in \mathcal{T}_h$, we have

$$\|\mathbf{v}_h\|_{1,q',T} \leqslant Ch^{2(1/q'-1/2)} \|\mathbf{v}_h\|_{1,2,T}.$$

Assumptions $\widetilde{\mathbf{B}}\mathbf{1}$ and $\widetilde{\mathbf{B}}\mathbf{3}$ are also satisfied by a wide class of finite element spaces, including all finite element spaces defined on quasi-uniform meshes [10]. They are actually common generalisations of Assumptions **B1** and **B3**.

We can now state the following result:

**Theorem 2** *Consider* $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ *satisfying Assumption* 1, $1 \leqslant q < 2$, *let* $(\mathbf{u}, p) \in W_0^{1,q}(\Omega) \times \mathbb{L}_0^q(\Omega)$ *be the solution of Problem* (1) *and* $(\mathbf{u}_h, p_h)$ *its Galerkin projection onto* $V_h^{k_1} \times W_h^{k_2}$ *satisfying* $\int_\Omega p_h = 0$ *and*

$$\int_\Omega \nabla(\mathbf{u} - \mathbf{u}_h) \colon\colon \nabla\eta - \int_\Omega (p - p_h)\operatorname{div}(\eta) = 0 \text{ for all } \eta \in V_h^{k_1},$$
$$\int_\Omega \operatorname{div}(\mathbf{u} - \mathbf{u}_h)\xi = 0 \text{ for all } \xi \in W_h^{k_2}.$$

*Under the assumption that* $(\mathbf{u}, p) \in H^{k_0+1}(\Omega_1)^2 \times H^{k_0}(\Omega_1)$, *there exists* $h_1$ *such that if* $0 < h \leqslant h_1$, *we have,*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega_0} + \|p - p_h\|_{0,\Omega_0} \leqslant C(\Omega_0, \Omega_1, \Omega) h^{k_0} \sqrt{|\ln h|},$$

*where* $k_0 = \min(k_1, k_2 + 1)$.

***Proof*** We will not develop the complete proof here because it is essentially the same as the proof of Theorem 1 (see Sect. 3). But we will explain two differences between the both proofs:

- the result of Lemma 1 holds in this case, but for $0 \leqslant t \leqslant \min(k_1 - 1, k_2)$.
- the result of Corollary 1 becomes

$$\|\mathbf{u}-\mathbf{u}_h\|_{-k_0+1,\Omega} + \|p - p_h\|_{-k_0,\Omega} \leqslant Ch^{-\varepsilon}h^{k_0}\left(|\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega} + |p - p_h|_{0,q_\varepsilon,\Omega}\right),$$

where $k_0 = \min(k_1, k_2 + 1)$.

The end of the proof is the same. □

## 5 Numerical illustrations

In this section, we present some computations which illustrate the theoretical results proved in this paper.

**Concentration of the error around the singularity.** First, we define $\Omega$ as the unit square,

$$\Omega = [0, 1]^2.$$

and solve the following Stokes problem with $\mathbf{F} = {}^t[1, 1]$ and $\mathbf{x}_0 = (0.5, 0.5)$,

$$\begin{cases} -\triangle\mathbf{u} + \nabla p = \delta_{\mathbf{x}_0}\mathbf{F} & \text{in} \quad \Omega, \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{u}_\delta & \text{on} \quad \partial\Omega, \end{cases} \tag{19}$$

where $\mathbf{u}_\delta$ is the 2d Stokeslet defined in (3).

***Remark 2*** Unlike Problem (1), Problem (19) has non homogeneous Dirichlet boundary conditions, but in this case, the exact solution is known: $\mathbf{u}_\delta$. Thus, it is easier to get some information on the error.

Figures 2, 3, 4 and 5 show the repartition of the error on the velocity with a $\mathbb{P}_1\text{b}/\mathbb{P}_1$ method for respectively $1/h \simeq 5, 10, 20$ and 30. Figures 6, 7, 8 and 9 show the repartition of the error on the pressure for the same values of $h$. In both cases, they illustrate the fact that the error concentrates around the singularity. These simulations made us think that the convergence could be optimal on a subdomain which does not contain the singularity: quasi-optimality has been proved in this paper (Theorem 1).

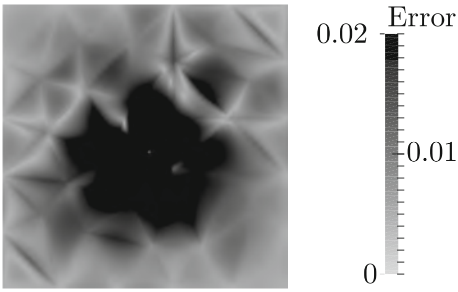


**Fig. 2** Error in velocity, $1/h \simeq 5$

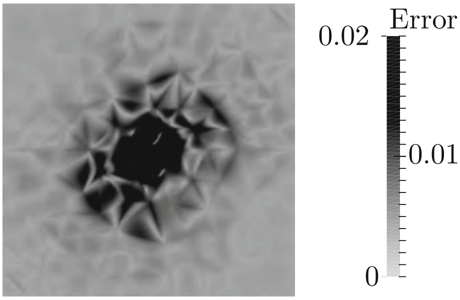


**Fig. 3** Error in velocity, $1/h \simeq 10$

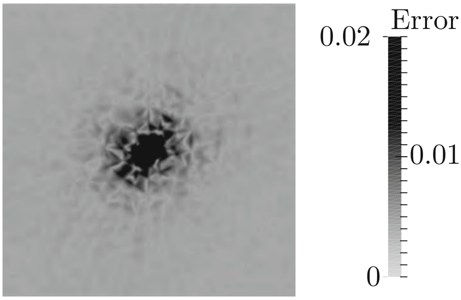


**Fig. 4** Error in velocity, $1/h \simeq 20$

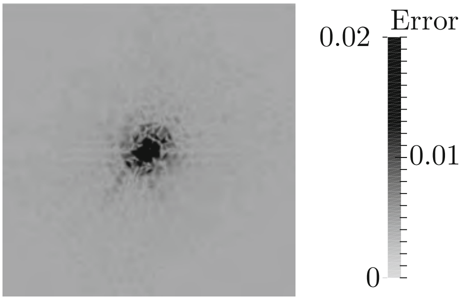


**Fig. 5** Error in velocity $1/h \simeq 30$

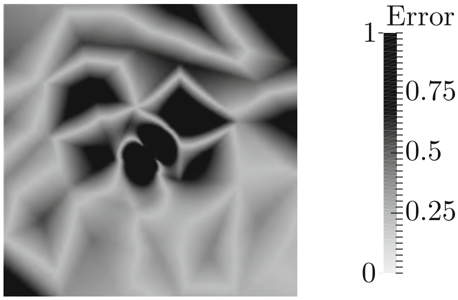


**Fig. 6** Error in pressure, $1/h \simeq 5$

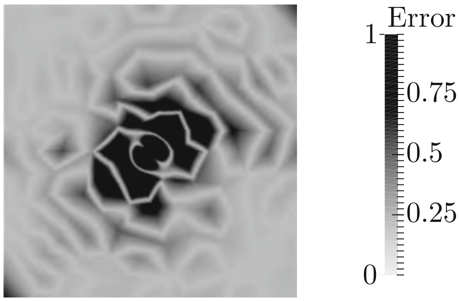


**Fig. 7** Error in pressure, $1/h \simeq 10$

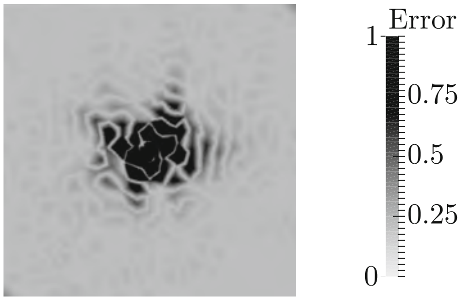


**Fig. 8** Error in pressure, $1/h \simeq 20$

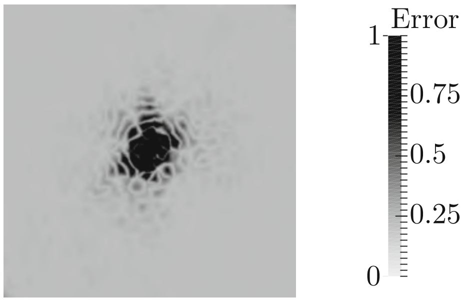


**Fig. 9** Error in pressure $1/h \simeq 30$

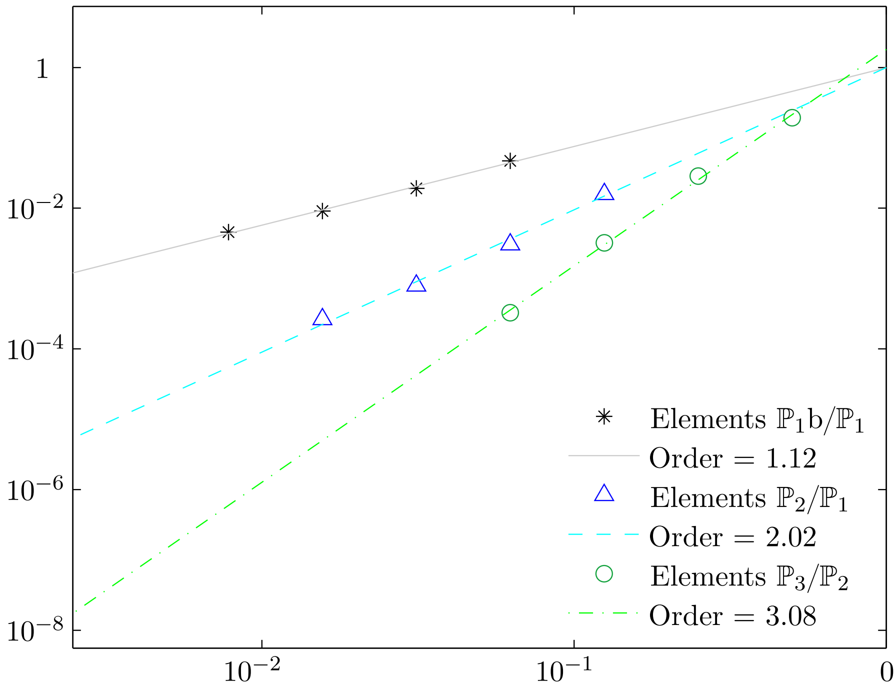


**Fig. 10** Estimated order of convergence for the $H^1(\Omega_0)$-norm of the velocity

**Estimated orders of convergence.** For this second example, the domain $\Omega$ is still the unit square, and $\Omega_0$ is defined as the following portion of $\Omega$,

$$\Omega_0 = \{\mathbf{x} \in \Omega : \|\mathbf{x} - \mathbf{x}_0\|_2 > 0.4\},$$

where $\mathbf{x}_0 = (0.5, 0.5)$. We fix $\mathbf{F} = {}^t[1, 1]$ and solve Problem 1 for different mesh sizes $h$ with the $\mathbb{P}_1\text{b}/\mathbb{P}_1$, $\mathbb{P}_2/\mathbb{P}_1$ and $\mathbb{P}_3/\mathbb{P}_2$ finite element methods.

Figure 10 (respectively Fig. 11) presents the estimated orders of convergence for the $H^1(\Omega_0)$-norm of the velocity (respectively the $\mathbb{L}^2(\Omega_0)$-norm of the pressure) for these three methods. The convergence far from the singularity (*i.e.* on $\Omega_0$) is the same as in the regular case: the $\mathbb{P}_k/\mathbb{P}_{k-1}$ method (or the $\mathbb{P}_1\text{b}/\mathbb{P}_1$ method if $k = 1$) converges at the order $k$ on $\Omega_0$ in $H^1$-norm for the velocity and in $\mathbb{L}^2$-norm for the pressure, as proved in this paper. Let us just note that there is an over-convergence in pressure for the $\mathbb{P}_1\text{b}/\mathbb{P}_1$ elements: the estimated order of convergence is approximately 2, greater than the convergence expected by Theorem 1.

When focusing on the error in $\mathbb{L}^2(\Omega_0)$-norm for the velocity, Fig. 12 suggests that the $\mathbb{P}_k/\mathbb{P}_{k-1}$ finite element method (or $\mathbb{P}_1\text{b}/\mathbb{P}_1$ if $k = 1$) converges at the order $k + 1$ on $\Omega_0$. This result has only been observed numerically but it is still an open question.

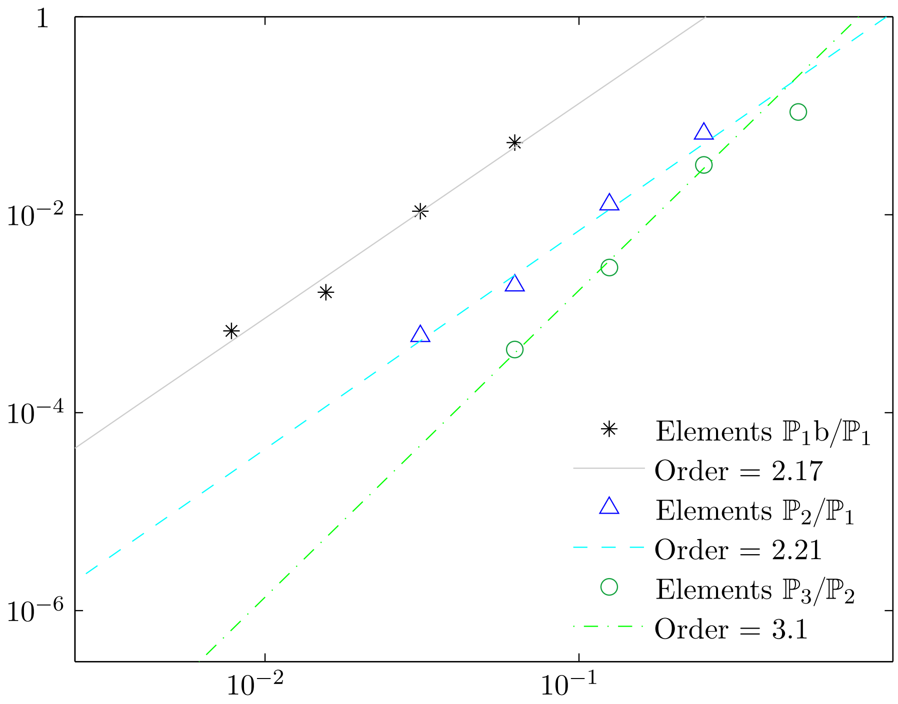


**Fig. 11** Estimated order of convergence for the $\mathbb{L}^2(\Omega_0)$-norm of the pressure

## 6 A concluding remark: the 3d case

The approach presented in this paper can be extended in order to obtain a local estimate of the numerical solution to problem $(P_\delta)$ in dimension 3, that is to the Stokes problem with as source term a Dirac measure concentrated at one point. However straightforward adaptations of the proof lead to a suboptimal result. In fact, the solution $u_\delta$ to problem $(P_\delta)$ in 3d belongs to $W_0^{1,q}(\Omega)$ for all $q$ in $[1, \frac{3}{2}[$. As a consequence the couple $(q_\varepsilon, q'_\varepsilon)$ defined in Corollary (1) and its related proof has to be taken near to $(\frac{3}{2}, 3)$. For instance,

$$q_\varepsilon = \frac{3}{2+\varepsilon} \quad \text{and} \quad q'_\varepsilon = \frac{3}{1-\varepsilon},$$

so that, with the same notations, the result of Corollary 1 becomes

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}_h\|_{-k+1,\Omega} + \|p - p_h\|_{-k,\Omega} \\ &\quad \leqslant Ch^{-\varepsilon-\frac{1}{2}} h^k \left(|\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega} + |p - p_h|_{0,q_\varepsilon,\Omega}\right). \end{aligned}$$

Moreover, the discrete inf-sup condition in dimension 3 is

$$\inf_{\mathbf{u}_h \in \mathring{V}_h^k} \sup_{\mathbf{v}_h \in \mathring{V}_h^k} \frac{\int_\Omega \nabla \mathbf{u}_h :: \nabla \mathbf{v}_h}{|\mathbf{u}_h|_{1,q_\varepsilon,\Omega} |\mathbf{v}_h|_{1,q'_\varepsilon,\Omega}} \geqslant Ch^{\varepsilon+\frac{1}{2}}.$$

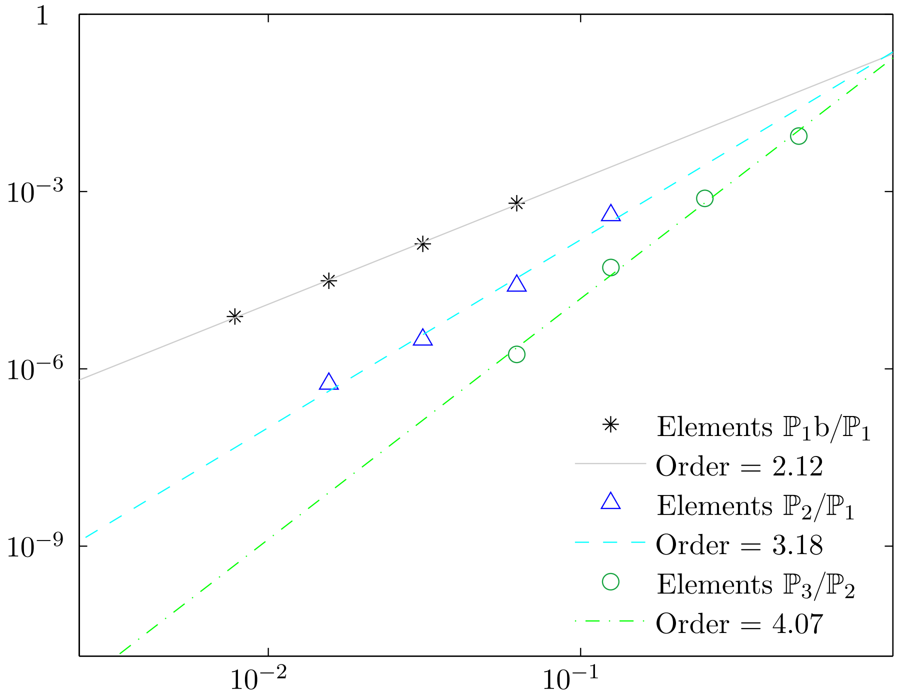


**Fig. 12** Estimated order of convergence for the $\mathbb{L}^2(\Omega_0)$-norm of the velocity

Thus when dealing with the estimate for $|\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega}$, we get

$$|\mathbf{u} - \mathbf{u}_h|_{1,q_\varepsilon,\Omega} \leqslant Ch^{-\varepsilon-\frac{1}{2}} \left(|\mathbf{u}|_{1,q_\varepsilon,\Omega} + |p|_{0,q_\varepsilon,\Omega}\right).$$

Finally, the Stokeslet in 3d writes:

$$\mathbf{u}_\delta(\mathbf{x}) = \frac{1}{8\pi}\left(\frac{\mathbf{I}_3}{\|\mathbf{x}\|} + \frac{\mathbf{x}\otimes\mathbf{x}}{\|\mathbf{x}\|^3}\right)\mathbf{F}, \qquad p_\delta(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{F}}{4\pi\,\|\mathbf{x}\|^3},$$

which leads to the estimate, for $1 \leq q < \frac{3}{2}$,

$$\|\mathbf{u}\|_{1,q,\Omega} \leq \frac{C}{(3-2q)^{2/3},} \qquad \|p\|_{0,q,\Omega} \leq \frac{C}{(3-2q)^{2/3}},$$

where $C > 0$ is a constant independent of $q$. Combining the previous properties, we get the estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega_0} + \|p - p_h\|_{0,\Omega_0} \leqslant C(\Omega_0, \Omega_1, \Omega)h^{k-1}(|\ln h|)^{\frac{2}{3}},$$

which is clearly suboptimal.

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