Negative Norm Stabilization of Convection-Diffusion Problems *

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Abstract. We consider a model convection-diffusion problem in the convection-dominated regime. A functional setting is given for stabilized Galerkin approximations, in which the stabilizing terms are based on inner products of the type $H^{-1/2}$. These are explicitly computable via multiscale decompositions such as hierarchical finite elements or wavelets (while classical SUPG or Galerkin/least-squares methods mimic their effect through discrete element-by-element weighted L^2-inner products).

Key words. Singularly perturbed problems, convection-diffusion problems, stabilized Galerkin methods, multiscale decompositions, wavelets.

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1 Introduction

Convection-diffusion problems are simplified yet significant models of complex fluid dynamics phenomena, such as the flow of an incompressible fluid or a viscous gas. In the convection-dominated regime, i.e., when the diffusion coefficient is small compared to the characteristic velocity of the flow, the theoretical study of such models as well as their numerical approximation is quite challenging, due to the singular perturbation nature of the problem. On the discrete level, the use of numerical methods which do not properly take

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into account this aspect may lead to the onset of spurious numerical oscillations (*numerical instabilities*), often resulting in a dramatic loss of both qualitative and quantitative accuracy.

Many different approaches have been proposed to obtain stable and accurate approximation schemes for convection-diffusion problems (see, e.g., [11]). Among them, the use of *stabilization techniques* – such as the Streamline-Upwind Petrov Galerkin (SUPG) method by Brooks and Hughes [6] – has proven to be effective in a wide range of applications. The basic idea consists of modifying the classical variational formulation in a consistent way, by adding elementwise stabilization terms which measure the local (strong) residual of the equation in an L^2-type norm. Galerkin/least-squares methods [9] are a variant of this concept. These and other stabilization techniques can often be interpreted as the effect of augmenting the trial/test space by a suitable supplementary space [5].

From a mathematical point of view, the difficulties arising from convection-diffusion problems can be traced back to the fact that the bilinear form in the corresponding variational formulation has a coerciveness constant which is extremely small when compared to the continuity constant. On the discrete level, the problem essentially behaves as a non-coercive one. Adding stabilization terms may be viewed as a way of improving the coerciveness constant. Other classes of problems – such as the Stokes problem or domain decomposition methods – also lead to non-coercive bilinear forms, which are responsible for several numerical instabilities.

The idea of the stabilization of a non-coercive or weakly coercive problem has been formulated in an abstract setting ([1, 2]). Let us consider a problem of the form

$$Au = f,$$

where $A : V \to V'$, $f \in V'$, V being a Hilbert space. One can introduce the following stabilized variational formulation: find $u \in V$ such that, for all $v \in V$,

$$ {}_{V'}\langle Au, v\rangle_V + \delta \; {}_{V'}[Au, Av]_{V'} = \; {}_{V'}\langle f, v\rangle_V + \delta \; {}_{V'}[f, Av]_{V'}, \qquad \delta > 0, \tag{1.1}$$

(${}_{V'}\langle \cdot\,, \cdot\rangle_V$ being the duality pairing between V' and V, whereas ${}_{V'}[\,\cdot\,, \cdot\,]_{V'}$ is a suitable inner product in V'). It is possible to prove (see [1]) that if A is an non negative isomorphism between V and V', then the bilinear form on the left-hand side of (1.1) is continuous and coercive with respect to the norm of V, even in the case of a non-coercive A. Therefore, the standard Galerkin discretization of (1.1) inherits the good stability properties of the continuous problem. Obviously, the implementation of the method demands an easily computable expression for the inner product in V'.

In this paper, we apply similar ideas to the convection-diffusion problem. Its singular perturbation nature suggests us to introduce – in addition to the standard space V of the type H^1 – a new space X of the type $H^{1/2}$. Consequently, the stabilization terms in (1.1) are based on the inner product in X', which is of the type $H^{-1/2}$. Such an inner product can be numerically computed in an efficient way by resorting to a multiscale decomposition of V (see, e.g., [8, 4]). The proposed functional framework allows us to obtain, in the convection-dominated regime, a stabilized bilinear form which is continuous and coercive with respect to the norm of X. This provides an extra control on the streamline derivative of the solution in a $H^{-1/2}$-type norm, which inhibits the onset of numerical oscillations. Note that SUPG and other classical stabilization methods provide a control of the streamline derivative in certain weighted L^2-type norms: it is our opinion that the use of such norms is just a way of mimicking, on the discrete level, our $H^{-1/2}$-type norm.

We believe that the main interest of our results is the definition of a natural functional framework for stabilized convection-diffusion problems. The methods for efficiently computing norms and inner products of fractional and/or negative order, which have been devised in recent years (by wavelets and multiscale analyses, hierarchical finite elements, incremental unknows, and so on), make our theoretical setting also practically appealing. In a forthcoming paper [3], we will discuss the details of the implementation and we will provide numerical results.

2 The exact problem

Let Ω be a bounded open set in $I\!R^n$ $(n \geq 1)$, whose boundary $\Gamma = \partial\Omega$ is a piecewise $\mathcal{C}^1$ manifold. Let $\nu > 0$ be a given constant, $\boldsymbol{a} \in (W^{1,\infty}(\Omega))^n$ be a given vector field, $b \in L^\infty(\Omega)$ be a given scalar function. For any $f \in L^2(\Omega)$, we are interested in solving the following model problem:

$$\begin{cases} -\nu\Delta u + \boldsymbol{a}\cdot\nabla u + bu = f\,, & \text{in } \Omega,\\ u = 0\,, & \text{on } \Gamma. \end{cases} \tag{2.1}$$

Existence and uniqueness of the solution in $H_0^1(\Omega)$ are assured for instance by the assumption: there exists a constant $\gamma > 0$ such that

$$-\frac{1}{2}\nabla\cdot\boldsymbol{a} + b \geq \gamma \qquad \text{a.e. in } \Omega. \tag{2.2}$$

Indeed, the bilinear form

$$\begin{aligned} &a \;:\; H_0^1(\Omega)\times H_0^1(\Omega) \to I\!R\\ &a(w,v) = \nu\textstyle\int_\Omega \nabla w\cdot\nabla v\,dx + \int_\Omega \boldsymbol{a}\cdot\nabla w\, v\,dx + \int_\Omega b\,w\,v\,dx \end{aligned} \tag{2.3}$$

is continuous and coercive in $H^1_0(\Omega)$, and $u \in H^1_0(\Omega)$ is the solution of the variational problem

$$a(u,v) = (f,v)_{L^2(\Omega)}\,, \qquad \forall v \in H^1_0(\Omega). \tag{2.4}$$

We are interested in the case in which, after a proper scaling of eq. (2.1), the diameter of Ω and both $\|\boldsymbol{a}\|_{L^\infty(\Omega)}$ and γ are of the order of unity, whereas $\nu \ll \|\boldsymbol{a}\|_{L^\infty(\Omega)}$. We will keep track of the influence of the constant ν: to this end, we shall use the notation $A \lesssim B$ to indicate $A \leq cB$ for a suitable constant c independent of ν; the symbol $A \asymp B$ means $A \lesssim B$ and $B \lesssim A$.

The standard energy estimate

$$\nu^{1/2}\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} \tag{2.5}$$

provides a weaker and weaker control on the H^1_0–seminorm of u, as ν become smaller and smaller. This is responsible for the onset of spurious numerical oscillations, in the pure Galerkin approximation of problem (2.1) at large Péclet numbers. The difficulty can be equivalently related to the higher and higher ratio between the continuity and coerciveness constants of the bilinear form a, which scales as ν^{-1}.

The motivation of our method precisely rises from a careful analysis of the bounds on a. For convenience, let us set

$$Dv = \boldsymbol{a}\cdot\nabla v \quad \text{ and } \quad Lv = -\nu\Delta v + Dv + bv\,.$$

Given $w, v \in H^1_0(\Omega)$, we have

$$\begin{aligned} |a(w,v)| &\lesssim \nu\|\nabla w\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|Dw\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\lesssim (\|w\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla w\|_{L^2(\Omega)} + \|Dw\|_{L^2(\Omega)})(\|v\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v\|_{L^2(\Omega)}). \end{aligned} \tag{2.6}$$

On the other hand, after an integration by parts, we get

$$a(w,v) = \nu\int_\Omega \nabla w\cdot\nabla v\,dx - \int_\Omega w\,Dv\,dx + \int_\Omega \tilde{b}\,w\,v\,dx,$$

with $\tilde{b} = b - \nabla\cdot\boldsymbol{a}$, whence

$$|a(w,v)| \lesssim (\|w\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla w\|_{L^2(\Omega)})(\|v\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v\|_{L^2(\Omega)} + \|Dv\|_{L^2(\Omega)}). \tag{2.7}$$

Recalling the interpolation theory of function spaces, the comparison of (2.6) and (2.7) suggests a symmetric (in w and v) bound for $a(w,v)$, of the form

$$|a(w,v)| \lesssim |||w|||\;|||v|||,$$

with $|||v||| = \|v\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v\|_{L^2(\Omega)} + \|Dv\|_{[L^2(\Omega),H^{-1}(\Omega)]_{1/2}}$. Since we aim at designing a consistent method, we shall actually replace Dv by Lv in the interpolation norm. We will show that the two definitions are equivalent on the discrete level.

We now come to the precise definition of our stabilized formulation; to this end, we describe the functional setting in which we will work. Let us introduce the family $\mathcal{H}^r(\Omega)$ for $r \geq 0$, setting $\mathcal{H}^0(\Omega) = L^2(\Omega)$, $\mathcal{H}^r(\Omega) = H^r(\Omega) \cap H_0^1(\Omega)$ for an integer $r \geq 1$, and extending the definition to any real $r > 0$ by interpolation; in particular, for $0 \leq r \leq 1$, one has

$$\mathcal{H}^r(\Omega) = \begin{cases} H^r(\Omega), & 0 \leq r < 1/2, \\ H_{00}^{1/2}(\Omega), & r = 1/2, \\ H_0^r(\Omega), & 1/2 < r \leq 1. \end{cases}$$

For $r < 0$, we set $\mathcal{H}^r(\Omega) = (\mathcal{H}^{-r}(\Omega))'$. We choose an inner product on $\mathcal{H}^r(\Omega)$; it will be denoted by $[\xi, \eta]_{\mathcal{H}^r(\Omega)}$, $\forall \xi, \eta \in \mathcal{H}^r(\Omega)$. Examples of 'computable' inner products of fractional order will be provided in Section 4. Next, for $0 \leq r \leq 1$, we define the space

$$X^r(\Omega) = \{v \in H_0^1(\Omega) : Lv \in \mathcal{H}^{r-1}(\Omega)\};$$

recalling the estimate (2.5), it is natural to equip it with the norm

$$\|v\|_{X^r(\Omega)} = \|v\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v\|_{L^2(\Omega)} + \delta^{1/2}\|Lv\|_{\mathcal{H}^{r-1}(\Omega)},$$

where $\delta > 0$ is a suitable parameter independent of ν, whose definition will be made precise later on.

We are now ready to introduce the stabilized bilinear form

$$\begin{aligned} &A \ : \ X^{1/2}(\Omega) \times X^{1/2}(\Omega) \to I\!R \\ &A(w,v) = a(w,v) + \delta[Lw, Lv]_{\mathcal{H}^{-1/2}(\Omega)} \end{aligned} \tag{2.8}$$

and the corresponding linear form

$$\begin{aligned} &F \ : \ X^{1/2}(\Omega) \to I\!R \\ &F(v) = (f,v)_{L^2(\Omega)} + \delta[f, Lv]_{\mathcal{H}^{-1/2}(\Omega)}. \end{aligned} \tag{2.9}$$

Obviously, the solution u of (2.1) is also the unique solution of the variational problem

$$\begin{cases} u \in X^{1/2}(\Omega) \\ A(u,v) = F(v), \quad \forall v \in X^{1/2}(\Omega). \end{cases} \tag{2.10}$$

Uniqueness follows from the coerciveness of the form A in $X^{1/2}(\Omega)$; indeed, we have, for a suitable $\bar{\alpha} > 0$ independent of ν,

$$\begin{aligned} A(w,w) = \ & a(w,w) + \delta\|Lw\|^2_{\mathcal{H}^{-1/2}(\Omega)} \\ & \geq \nu\|\nabla w\|^2_{L^2(\Omega)} + \gamma\|w\|^2_{L^2(\Omega)} + \delta\|Lw\|^2_{\mathcal{H}^{-1/2}(\Omega)} \\ & \geq \bar{\alpha}\|w\|^2_{X^{1/2}(\Omega)}, \quad \forall w \in X^{1/2}(\Omega). \end{aligned} \tag{2.11}$$

3 The approximate problem

Let $h > 0$ be a discretization parameter, and let V_h be a finite dimensional subspace of $H_0^1(\Omega)$. For the sake of simplicity, we make the regularity hypothesis

$$V_h \subset \mathcal{H}^{3/2}(\Omega)\,; \tag{3.1}$$

this implies $Lv_h \in \mathcal{H}^{-1/2}(\Omega)$, $\forall v_h \in V_h$, hence, $V_h \subset X^{1/2}(\Omega)$. Condition (3.1) is satisfied, e.g., for a space of $\mathcal{C}^1$–finite elements or splines of order ≥ 2 or sufficiently smooth wavelets. At the end of the section, we will briefly sketch the necessary modification in our method in order to deal with the case of a $\mathcal{C}^0$–approximation (i.e., $V_h \not\subset \mathcal{H}^{3/2}(\Omega)$, but $V_h \subset \mathcal{H}^{3/2-\varepsilon}$, $\forall \varepsilon > 0$). Throughout this section, the symbol $\lesssim$ will indicate a bound independent of ν and h. We assume that, for some $\sigma \geq 3/2$, there exists a linear operator $\Pi_h : \mathcal{H}^\sigma(\Omega) \to V_h$ such that

$$\|v - \Pi_h v\|_{H^s(\Omega)} \lesssim h^{\sigma - s}|v|_{\mathcal{H}^\sigma(\Omega)} \qquad \forall v \in \mathcal{H}^\sigma(\Omega),\ 0 \leq s \leq 3/2. \tag{3.2}$$

Finally, we assume the following inverse inequality:

$$\|v_h\|_{\mathcal{H}^{3/2}(\Omega)} \lesssim h^{-1/2}\|\nabla v_h\|_{L^2(\Omega)}, \qquad \forall v_h \in V_h. \tag{3.3}$$

Within this framework, we are led to consider the following Galerkin approximation of problem (2.10):

$$\begin{cases} u_h \in V_h \\ A(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h. \end{cases} \tag{3.4}$$

Existence and uniqueness of the solution of this problem, as well as a stability estimate in the $X^{1/2}(\Omega)$–norm, follow from the coerciveness (2.11) of A.

The following results will be crucial in the sequel.

Lemma 3.1 *Let $L^*v = -\nu\Delta v - Dv + (b - \nabla\cdot\boldsymbol{a})v$ be the adjoint of L. Then,*

$$\delta^{1/2}\|L^* v_h\|_{\mathcal{H}^{-1/2}(\Omega)} \lesssim \Big(1 + (\frac{\nu\delta}{h})^{\frac{1}{2}}\Big)\|v_h\|_{X^{1/2}(\Omega)}, \qquad \forall v_h \in V_h.$$

Proof. We observe that $L^*v_h = -Lv_h - 2\nu\Delta v_h + (2b - \nabla\cdot\boldsymbol{a})v_h$. Thus,

$$\begin{aligned} \delta^{1/2}\|L^* v_h\|_{\mathcal{H}^{-1/2}(\Omega)} \leq\ & \delta^{1/2}\|Lv_h\|_{\mathcal{H}^{-1/2}(\Omega)} + 2\nu\delta^{1/2}\|\Delta v_h\|_{\mathcal{H}^{-1/2}(\Omega)} \\ & + \delta^{1/2}(\|a\|_{W^{1,1}(\Omega)} + 2\|b\|_{L^\infty(\Omega)})\|v_h\|_{L^2(\Omega)}\,. \end{aligned}$$

Using (3.3), we get

$$\|\Delta v_h\|_{\mathcal{H}^{-1/2}(\Omega)} \lesssim \|v_h\|_{\mathcal{H}^{3/2}(\Omega)} \lesssim h^{-1/2}\|\nabla v_h\|_{L^2(\Omega)}\,,$$

whence the estimate. □

A similar argument yields

Lemma 3.2 *The following estimate holds:*

$$\delta^{1/2}\|Dv_h\|_{\mathcal{H}^{-1/2}(\Omega)} \lesssim \Big(1+(\frac{\nu\delta}{h})^{\frac{1}{2}}\Big)\|v_h\|_{X^{1/2}(\Omega)}, \qquad \forall v_h \in V_h. \ \Box$$

From now on, we make the assumption

$$\frac{\nu\delta}{h} \lesssim 1. \tag{3.5}$$

This means that we place ourselves in the so-called 'convection-dominated' regime. Indeed, by choosing δ of the order of $\|\boldsymbol{a}\|^{-1}_{L^\infty(\Omega)}$, the left-hand side of (3.5) can be viewed as the inverse of a mesh Péclet number.

Remark 3.3 Under the assumption (3.5), the previous lemmas easily imply that the norms

$$\|v_h\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v_h\|_{L^2(\Omega)} + \delta^{1/2}\|\mathcal{L}v_h\|_{\mathcal{H}^{-1/2}(\Omega)}\,,$$

where $\mathcal{L}=L$ (i.e., the $X^{1/2}(\Omega)$–norm), or $\mathcal{L}=L^*$, or $\mathcal{L}=D$, are all uniformly equivalent on V_h. $\Box$

We now establish a continuity estimate for the bilinear form A on $X^{1/2}(\Omega)\times V_h$.

Proposition 3.4 *Assume (3.5). For any $w \in X^{1/2}(\Omega)$ and for any $v_h \in V_h$,*

$$|A(w,v_h)| \lesssim (\delta^{-1/2}\|w\|_{\mathcal{H}^{1/2}(\Omega)} + \delta^{1/2}\|Lw\|_{\mathcal{H}^{-1/2}})\|v_h\|_{X^{1/2}(\Omega)}\,.$$

Proof. We have

$$a(w,v_h) = {}_{H^{-1}(\Omega)}\langle Lw, v_h\rangle_{H^1_0(\Omega)} = {}_{H^1_0(\Omega)}\langle w, L^*v_h\rangle_{H^{-1}(\Omega)}.$$

By (3.1), we get $L^*v_h \in \mathcal{H}^{-1/2}(\Omega)$; thus $a(w,v_h) = {}_{\mathcal{H}^{1/2}(\Omega)}\langle w, L^*v_h\rangle_{\mathcal{H}^{-1/2}(\Omega)}$, so that

$$|a(w,v_h)| \le \|w\|_{\mathcal{H}^{1/2}(\Omega)}\|L^*v_h\|_{\mathcal{H}^{-1/2}(\Omega)}\,.$$

Then, the result follows from Lemma 3.1 and the definition of A. $\Box$

Theorem 3.5 *Assume (3.5). Then*

$$\|u-u_h\|_{X^{1/2}(\Omega)} \lesssim \inf_{w_h\in V_h}\Big(\|u-w_h\|_{X^{1/2}(\Omega)} + \delta^{-1/2}\|u-w_h\|_{\mathcal{H}^{1/2}(\Omega)}\Big)\,. \tag{3.6}$$

Proof. By (2.10) and (3.4), we get

$$A(u - u_h, v_h) = 0\,, \qquad \forall v_h \in V_h\,.$$

Let w_h be any element in V_h. Using (2.11) and the previous proposition, we have

$$\begin{aligned} \|w_h - u_h\|^2_{X^{1/2}(\Omega)} &\lesssim A(w_h - u_h, w_h - u_h) = A(w_h - u, w_h - u_h) \\ &\lesssim \Big(\delta^{-1/2}\|u - w_h\|_{\mathcal{H}^{1/2}(\Omega)} + \delta^{1/2}\|L(u - w_h)\|_{\mathcal{H}^{-1/2}}\Big)\|w_h - u_h\|_{X^{1/2}(\Omega)}. \end{aligned} \tag{3.7}$$

Thus, recalling the definition of the norm in $X^{1/2}(\Omega)$, we get

$$\|w_h - u_h\|_{X^{1/2}(\Omega)} \lesssim \delta^{-1/2}\|u - w_h\|_{\mathcal{H}^{1/2}(\Omega)} + \|u - w_h\|_{X^{1/2}(\Omega)}\,; \tag{3.8}$$

using the triangle inequality and taking the infimum over all $w_h \in V_h$, the theorem is proven. □

Corollary 3.6 *Assume (3.5) and let* $u \in H^\sigma(\Omega)$. *Then*

$$\|u - u_h\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla(u - u_h)\|_{L^2(\Omega)} + \delta^{1/2}\|Du - Du_h\|_{\mathcal{H}^{-1/2}(\Omega)} \lesssim h^{\sigma-1/2}|u|_{\mathcal{H}^\sigma(\Omega)}.$$

Proof. We take $w_h = \Pi_h u$ and we use (3.2) to estimate the right–hand side of (3.6). Then, we conclude by Remark 3.3. □

We end this section by briefly discussing how to deal with a locally smooth, globally $\mathcal{C}^0$–approximation, i.e., $V_h \subset \mathcal{H}^{3/2-\varepsilon}(\Omega)$ $\forall\varepsilon > 0$, but $V_h \not\subset \mathcal{H}^{3/2}(\Omega)$. Obviously, the bound for s in (3.2) is now $0 \le s < 3/2$, and the inverse inequality (3.3) has to be replaced by $\|v_h\|_{\mathcal{H}^{3/2-\varepsilon}(\Omega)} \lesssim h^{-1/2+\varepsilon}\|\nabla v_h\|_{L^2(\Omega)}$, $\forall v_h \in V_h$.

In this situation, let us fix a (sufficiently small) $\varepsilon > 0$, and let us replace, in the definitions of the forms A and F, the space $X^{1/2}(\Omega)$ by $X^{1/2-\varepsilon}(\Omega)$, the space $\mathcal{H}^{-1/2}(\Omega)$ by $\mathcal{H}^{-1/2-\varepsilon}(\Omega)$, and the parameter δ by $\bar\delta h^{-2\varepsilon}$, with $\bar\delta > 0$ independent of h. Note that $V_h \subset X^{1/2-\varepsilon}(\Omega)$; hence, we consider again the stabilized approximation (3.4). Under the assumption $\nu\bar\delta/h \lesssim 1$ (which is the analog of (3.5)), one can prove that the norms $\|v_h\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla v_h\|_{L^2(\Omega)} + \bar\delta^{1/2}h^{-\varepsilon}\|\mathcal{L}v_h\|_{\mathcal{H}^{-1/2-\varepsilon}(\Omega)}$ are all uniformly equivalent on V_h, for $\mathcal{L} = L,\ L^*$ and D. Then, (3.6) takes the form

$$\|u - u_h\|_{X^{1/2-\varepsilon}(\Omega)} \lesssim \inf_{w_h \in V_h}\Big(\|u - w_h\|_{X^{1/2-\varepsilon}(\Omega)} + \bar\delta^{-1/2}h^{\varepsilon}\|u - w_h\|_{\mathcal{H}^{1/2+\varepsilon}(\Omega)}\Big)\,,$$

whence we get the final estimate

$$\|u - u_h\|_{L^2(\Omega)} + \nu^{1/2}\|\nabla(u - u_h)\|_{L^2(\Omega)} + \bar\delta^{1/2}h^{-\varepsilon}\|Du - Du_h\|_{\mathcal{H}^{-1/2-\varepsilon}(\Omega)} \lesssim h^{\sigma-1/2}|u|_{\mathcal{H}^\sigma(\Omega)}.$$

4 Computing the negative-order inner product

The effective implementation of the method relies on the possibility of computing the negative-order inner product appearing in (2.8) and (2.9) in a cheap and efficient way. In recent years, multilevel decomposition ideas have been developed, which enable us to perform this task. Let $V_{j_0} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots \subset H_0^1(\Omega)$ be a nested sequence of subspaces of $H_0^1(\Omega)$, and let $P_j : L^2(\Omega) \to V_j$ be approximation operators. Then, under suitable assumptions (see [8, 10]), an increasing sequence of positive parameters θ_j exists, such that

$$\|v\|^2_{\mathcal{H}^r(\Omega)} \asymp \sum_{j=j_0}^{+\infty} \theta_j^{2r} \|(P_j - P_{j-1})v\|^2_{L^2(\Omega)}\,, \qquad \forall v \in \mathcal{H}^r(\Omega)\,, \quad 0 \le r \le 1,$$

(with $P_{j_0-1} \equiv 0$). Furthermore, this norm representation extends to negative values of r in the interval $[-1, 0)$, provided each P_j is replaced by its adjoint $\widetilde{P}_j : H^{-1}(\Omega) \to L^2(\Omega)$. Thus, it is natural to define the inner product

$$[f, g]_{\mathcal{H}^{-1/2}(\Omega)} = \sum_{j=j_0}^{+\infty} \theta_j^{-1}((\widetilde{P}_j - \widetilde{P}_{j-1})f, (\widetilde{P}_j - \widetilde{P}_{j-1})g)_{L^2(\Omega)}. \tag{4.1}$$

Inner product of this form can be constructed, e.g., in the framework of wavelet elements [7] or hierarchical finite elements [4].

It is possible to prove [2, 3] that if the trial space V_h satisfies, for some $r > -1/2$, an inverse inequality of the form

$$\|Dv_h\|_{\mathcal{H}^r(\Omega)} \;\lesssim\; h^{-r-\frac{1}{2}} \|Dv_h\|_{\mathcal{H}^{-1/2}(\Omega)}\,, \qquad \forall v_h \in V_h,$$

then in computing the $\mathcal{H}^{-1/2}$-inner product in the stabilized bilinear and linear forms (2.8) and (2.9), the infinite sum (4.1) can be replaced by a finite (computable) sum – the retained addends being dependent on h – and still obtain the same stability and accuracy estimates given in the previous section.

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