Substructuring Preconditioners for the Three Fields Domain Decomposition Method

Silvia Bertoluzza*

Abstract

We study a class of preconditioners based on substructuring, for the discrete Steklov-Poincaré operator arising in the three fields formulation of domain decomposition in two dimensions. Under extremely general assumptions on the discretization spaces involved, an upper bound is provided on the condition number of the preconditioned system, which is shown to grow at most as $\log(H/h)^2$ (H and h denoting respectively the diameter and the discretization mesh-size of the subdomains). Extensive numerical tests – performed on both a plain and a stabilized version of the method – confirm the optimality of such bound.

Key words: Three fields formulation, domain decomposition, stabilized methods, preconditioning.
AMS subject classification: 65N55, 65N22.

1 Introduction

The use of non conforming domain decomposition methods is becoming increasingly popular. This is due to several factors: the possibility of easily coupling different discretizations in different subdomains without need for imposing strong matching conditions allows to employ without any adaptation the technologies developed for treating the problem with a single-domain approach. As an example, in an adaptive strategy, refinement can be carried out in each subdomain independently. Moreover, it is possible to use different type of discretization (finite elements, spectral methods, wavelets) in the different subdomains, and therefore to employ, in each subdomain, the best suited method (spectral methods where the solution is expected to be very smooth, finite elements where a complicated geometry requires it, wavelets where isolated singularities in a regular background are expected).

Among the different non conforming domain decomposition method we consider here the *three fields formulation*, as proposed by F. Brezzi and L.D. Marini in [14]. Differently from other non-conforming formulations [3, 17] of domain decomposition, weak continuity

*Istituto di Analisi Numerica del C.N.R., v. Ferrata 1, 27100 Pavia (Italy), e-mail: aivlis@ian.pv.cnr.it, http://www.ian.pv.cnr.it/∼aivlis

is not imposed by requiring that the jump across the interface is orthogonal to some multiplier space, but by introducing a space Φ_h, approximating the traces of functions in $H^1(\Omega)$ on the interface Σ. The elements of Φ_h are to be used as a Dirichlet boundary condition — imposed by means of Lagrange multipliers — for the functions in the subdomains. As with other non conforming method great freedom is left in the choice of the discretization in the subdomains which, after applying if needed a suitable stabilization method [2, 7], can be chosen independently from each other. The disadvantage of introducing a third unknown field is compensated by the fact that all subdomains are treated exactly in the same way, which results in an easier implementation and possibly, when considering the parallelization of the method, in an easier balancing of the load between processors.

We deal, in this paper, with the problem of the efficient solution of the linear system arising when applying such a method. By applying a Schur complement technique, the solution of the resulting discrete problem can be reduced to the solution of an equation on the "trace" unknown, involving a *Steklov–Poincaré* type operator which can be shown to be a pseudo-differential operator of order one. The resulting linear system will then be solved by an iterative procedure, at each step of which there will be the need of computing the action of such operator on a given function. This reduces to solving independent Dirichlet problems on the subdomains, a task which can be carried out in parallel, subdomain by subdomain in a completely independent way.

In order to have an efficient scheme for solving the equation for the interface unknown we will then need two things:

- efficient solvers for the local Dirichlet problems on the subdomains;
- a preconditioner for the discrete Steklov-Poincaré operator so that the number of iterations is kept as low as possible.

We will concentrate here on the second issue, by analyzing and testing a class of preconditioners for the discrete Steklov-Poincaré operator. The approach that we will follow is the substructuring one, proposed by J.H. Bramble, J.E. Pasciak and A.H. Schatz [9] in the framework of conforming domain decomposition and already applied in the framework of non conforming methods to the mortar method in [1]. This consists in decomposing the trace space Φ_h in a direct sum of a coarse subspace (essentially linked to the geometrical decomposition of Ω in subdomains) and of suitable local subspaces, and then considering the related block-Jacobi type preconditioners.

From the theoretical point of view, we study the dependence of the condition number of the resulting preconditioned matrix on the mesh size of the discretizations, aiming at giving bounds which, under suitable assumptions, are independent of the number and size of the subdomains. In particular we provide an upper bound which applies to a wide variety of approximation spaces, including finite elements and wavelets. Though only sub-optimal in terms of the mesh size on the skeleton (the condition number of the preconditioned matrix grows like $(\log(H/h))^2$, H and h being respectively the diameter of the subdomains and the mesh size of the corresponding discretization), the estimate obtained is uniform with respect to the number and size of subdomains. The resulting method is then potentially almost perfectly scalable. Moreover this approach has over other asymptotically better preconditioners for the three fields formulation ([7]), the advantage of being itself naturally computable in parallel (each edge in the skeleton being treated independently of the others).

We conclude the paper by reporting a series of extensive numerical tests, performed on both the plain and on a stabilized formulation as proposed in [7]. The results of the tests are in agreement with the theory and confirm the polylogarithmic growth (with respect to the mesh size of the discretization) of the preconditioned system, as well as the independence (asymptotically) with respect to the number and size of the subdomains.

2 Domain decomposition method: the *three fields* formulation

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. We will consider the following simple model problem: given $f \in L^2(\Omega)$, find u satisfying

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial \boldsymbol{x}_j}\left(a_{i,j}(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{x}_i}\right) + a_0(\boldsymbol{x})u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1}$$

We assume that for almost all $\boldsymbol{x} \in \Omega$ we have $0 \le a_0(\boldsymbol{x}) \le R$ and that the matrix $(a_{ij}(\boldsymbol{x}))_{i,j=1,2}$ is symmetric positive definite, with smallest eigenvalue $\ge \alpha > 0$ and largest eigenvalue $\le \alpha'$, α, α' independent of $\boldsymbol{x}$.

In a domain decomposition framework, we will consider the *three fields* formulation ([14]) of such problem. Let $\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}_k$, Ω_k polygons, which, for simplicity, we will assume to be quadrangles. We set $\Gamma_k = \partial\Omega_k$, $\Sigma = \cup_k \Gamma_k$ being the skeleton of the decomposition, (which differently from [14] we define here so that it includes the external boundary of the domain Ω). We will make the following regularity assumptions on the subdomains Ω_k:

(A1) the subdomains are regular in shape and the geometrical decomposition is graded, that is

   (a) there exists a positive constant c_0 such that, for all k, Ω_k contains a ball of diameter $c_0 H_k$, it is contained in a ball of diameter H_k, and the length of each side is bounded from below by $c_0 H_k$; moreover any interior angle ω satisfies $0 < c_1 < \omega < c_2 < \pi$ (c_0, c_1, and c_2 independent of k);

   (b) there exists a positive constant c_3 such that, if ℓ, k are such that $|\partial\Omega_k \cap \partial\Omega_\ell| > 0$, then it holds

$$H_k / H_\ell \le c_3.$$

For technical reasons, we need to make also the following assumption (needed for the proof of (24) in the following), which in many cases can be shown to be actually a consequence of the previous one:

(A2) there exists a constant C_0 such that for all y, denoting by ℓ_y the line of equation $\boldsymbol{x}_2 = y$ and setting $K_y = \{k : \ \text{length}(\ell_y \cap \Omega_k) > 0\}$, it holds $\sum_{k \in K_y} H_k \leq C_0$.

We are interested here in explicitly studying the dependence of the estimates that we are going to prove on the number and size of the subdomains. To this end, in the following we will employ the notation $A \lesssim B$ (resp. $A \gtrsim B$) to say that the quantity A is bounded from above (resp. from below) by cB, with a constant c independent of k and of the H_k's, as well as of any mesh size parameter. The expression $A \simeq B$ will stand for $A \lesssim B \lesssim A$.

For each k we let the norm and seminorm of $H^1(\Omega_k)$ be defined as usual by $\|u\|^2_{H^1(\Omega_k)} = \int_{\Omega_k} |u|^2 \, d\boldsymbol{x} + \int_{\Omega_k} |\nabla u|^2 \, d\boldsymbol{x}$ and $|u|^2_{H^1(\Omega_k)} = \int_{\Omega_k} |\nabla u|^2 \, d\boldsymbol{x}$. On the boundary $\partial\Omega_k$ we let the norm and seminorm of $H^{1/2}(\partial\Omega_k)$ be defined respectively by

$$\|\varphi\|_{H^{1/2}(\partial\Omega_k)} = \inf_{\substack{u \in H^1(\Omega_k) \\ u=\varphi \text{ on } \partial\Omega_k}} \|u\|_{H^1(\Omega_k)},$$

and

$$|\varphi|^2_{H^{1/2}(\partial\Omega_k)} = \int_{\partial\Omega_k} \int_{\partial\Omega_k} \frac{(\varphi(x) - \varphi(y))^2}{|x-y|^2} \, ds(x) \, ds(y). \tag{2}$$

The space $H^{-1/2}(\partial\Omega_k)$ is defined as the dual of $H^{1/2}(\partial\Omega_k)$.

For $e \subset \partial\Omega_k$ segment of extrema a and b, we define the $H^{1/2}(e)$ seminorm as usual by the integral expression (2) where the integrals are taken over e. On e we will also consider the space $H^{1/2}_{00}(e)$ of function whose extension by zero is in $H^{1/2}(\partial\Omega_k)$, which we will equip with the norm

$$\|\varphi\|^2_{H^{1/2}_{00}(e)} = |\varphi|^2_{H^{1/2}(e)} + \int_e \frac{|\varphi(x)|^2}{|x-a|} \, ds(x) + \int_e \frac{|\varphi(x)|^2}{|x-b|} \, ds(x).$$

The functional setting for the three fields domain decomposition method is given by the following spaces:

$$V = \prod_{k=1}^{K} H^1(\Omega_k), \qquad \Lambda = \prod_{k=1}^{K} H^{-1/2}(\partial\Omega_k), \tag{3}$$

and

$$\Phi = \{\varphi \in L^2(\Sigma) : \text{there exists } u \in H^1_0(\Omega), \ u = \varphi \text{ on } \Sigma\} = H^1_0(\Omega)|_\Sigma, \tag{4}$$

respectively equipped with the norms:

$$\|u\|_V = \left(\sum_k \|u^k\|^2_{H^1(\Omega_k)} \right)^{1/2}, \qquad \|\lambda\|_\Lambda = \left(\sum_k \|\lambda^k\|^2_{H^{-1/2}(\partial\Omega_k)} \right)^{1/2}, \tag{5}$$

and (see [4])

$$\|\varphi\|_\Phi = \inf_{\substack{u\in H^1_0(\Omega)\\ u=\varphi \text{ on } \Sigma}} \|u\|_{H^1(\Omega)} \simeq \left(\sum_k |\varphi|^2_{H^{1/2}(\partial\Omega_k)}\right)^{1/2}. \tag{6}$$

Let $a^k : H^1(\Omega_k) \times H^1(\Omega_k) \to \mathbb{R}$ denote the bilinear form corresponding to the linear operator considered:

$$a^k(w,v) = \int_{\Omega_k} \left(\sum_{i,j=1}^{2} a_{ij}(\boldsymbol{x}) \frac{\partial w}{\partial \boldsymbol{x}_i} \frac{\partial v}{\partial \boldsymbol{x}_j} + a_0(\boldsymbol{x}) wv \right) d\boldsymbol{x}.$$

Under the assumptions made on the matrix $(a_{ij}(\boldsymbol{x}))_{i,j=1,2}$ and on a_0, the bilinear forms a^k are uniformly $H^1(\Omega_k)$-continuous, and semi-definite. More precisely, there exist positive constants α and L independent of k and H_k such that for all w, v in $H^1(\Omega_k)$

$$\alpha |w|^2_{H^1(\Omega_k)} \le a^k(w,w), \qquad |a^k(w,v)| \le L \|w\|_{H^1(\Omega_k)} \|v\|_{H^1(\Omega_k)}.$$

The three fields formulation of equation (1) is the following: find $(u, \lambda, \varphi) \in V \times \Lambda \times \Phi$ such that

$$\begin{cases} \forall k, \quad \forall v^k \in V^k, \quad \forall \mu^k \in \Lambda^k : \\ a^k(u^k, v^k) \quad - \displaystyle\int_{\partial\Omega_k} v^k \lambda^k \, ds \qquad\qquad = \displaystyle\int_{\Omega_k} f v^k \, d\boldsymbol{x}, \\ \displaystyle\int_{\partial\Omega_k} u^k \mu^k \, ds \qquad\qquad - \displaystyle\int_{\partial\Omega_k} \mu^k \varphi \, ds = 0, \\ \text{and } \forall \psi \in \Phi : \\ \qquad \displaystyle\sum_k \int_{\partial\Omega_k} \lambda^k \psi \, ds \qquad\qquad = 0. \end{cases} \tag{7}$$

It is known that this problem admits a unique solution (u, λ, φ) where u is indeed the solution of (1) and such that

$$\lambda^k = \frac{\partial u^k}{\partial \nu^k_a} \text{ on } \Gamma_k, \qquad \varphi = u \text{ on } \Sigma, \tag{8}$$

where ν^k_a denotes the outer conormal derivative to the subdomain Ω_k. Remark that in such a formulation the homogeneous Dirichlet boundary condition is embedded in the definition of the space Φ.

Equation (7) can be discretized by a Galerkin scheme: after choosing discretization spaces $V_h = \prod_{k=1}^K V^k_h \subset \prod_{k=1}^K H^1(\Omega_k)$, $\Lambda_h = \prod_{k=1}^K \Lambda^k_h \subset \prod_{k=1}^K H^{-1/2}(\partial\Omega_k)$ and $\Phi_h \subset \Phi$, we consider the problem: find $(u_h, \lambda_h, \varphi_h) \in V_h \times \Lambda_h \times \Phi_h$, such that one has

$$
\begin{cases}
\forall k, \quad \forall v_h^k \in V_h^k, \quad \forall \mu_h^k \in \Lambda_h^k : \\
a^k(u_h^k, v_h^k) \quad - \int_{\partial\Omega_k} v_h^k \lambda_h^k \, ds \qquad\qquad = \int_{\Omega_k} f v_h^k \, d\boldsymbol{x}, \\
\int_{\partial\Omega_k} u_h^k \mu_h^k \, ds \qquad\qquad - \int_{\partial\Omega_k} \mu_h^k \varphi_h \, ds = 0, \\
\text{and } \forall \psi_h \in \Phi_h : \\
\qquad \sum_k \int_{\partial\Omega_k} \lambda_h^k \psi_h \, ds \qquad\qquad = 0.
\end{cases}
\tag{9}
$$

Existence, uniqueness and stability of the solution of (9) rely on the validity of suitable *inf-sup* conditions. More precisely we can make the following assumption:

(A3) the spaces V_h^k, Λ_h^k and Φ_h are such that

- (a) Λ_h^k contains the constant functions;
- (b) the following compatibility condition between V_h^k and Λ_h^k holds uniformly in k:

$$
\inf_{\lambda_h^k \in \Lambda_h^k} \sup_{u_h^k \in V_h^k} \frac{\displaystyle\int_{\partial\Omega_k} \lambda_h^k u_h^k \, ds}{\|u_h^k\|_{H^1(\Omega_k)} \|\lambda_h^k\|_{H^{-1/2}(\partial\Omega_k)}} \geq \beta > 0; \tag{10}
$$

- (c) the following compatibility condition between Λ_h and Φ_h holds:

$$
\inf_{\varphi_h \in \Phi_h} \sup_{\lambda_h \in \Lambda_h} \frac{\displaystyle\sum_k \int_{\partial\Omega_k} \lambda_h^k \varphi_h \, ds}{\|\lambda_h\|_\Lambda \|\varphi_h\|_\Phi} \geq \alpha > 0. \tag{11}
$$

Remark that (A3a) implies that for all $v_h^k \in \ker B_h^k = \{v_h^k \in V_h^k : \int_{\partial\Omega_k} v_h^k \mu_h^k = 0 \; \forall \mu_h^k \in \Lambda_h^k\}$ it holds $\int_{\partial\Omega_k} v_h \, ds = 0$ and then it is not difficult to see that a is elliptic on $\ker B_h^k$. Then, by a well known argument ([11]), there exists a unique solution to problem (9), which converges with an optimal rate to the solution of (7).

The linear system stemming from (9) takes the form

$$
\begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{u}_h \\ \underline{\lambda}_h \\ \underline{\varphi}_h \end{pmatrix} = \begin{pmatrix} \underline{f}_h \\ 0 \\ 0 \end{pmatrix}, \tag{12}
$$

($\underline{u}_h$, $\underline{\lambda}_h$, and $\underline{\varphi}_h$ being the vectors of the coefficients of u_h, λ_h and φ_h in the bases chosen for V_h, Λ_h and Φ_h respectively). By a Schur complement argument the solution of (12) can be reduced to a system in the unknown $\underline{\varphi}_h$ of the form

$$\mathbb{C}\mathbb{A}^{-1}\mathbb{C}^T\ \underline{\varphi}_h = -\mathbb{C}\mathbb{A}^{-1}\begin{pmatrix} \underline{f}_h \\ 0 \end{pmatrix}, \qquad \mathbb{C} = [\ 0 \quad C\ ], \qquad \mathbb{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}. \tag{13}$$

The matrix $S = \mathbb{C}\mathbb{A}^{-1}\mathbb{C}^T$ does not need to be assembled. The system (13) can be rather solved by an iterative technique (like for instance a conjugate gradient method) and therefore only the action of S on a given vector needs to be implemented. Multiplying by S implies the need for solving a linear system with matrix $\mathbb{A}$. This reduces, by a proper reordering of the unknowns, to independently solving K discrete Dirichlet problems with Lagrange multipliers in the K subdomains. Moreover under the above assumptions it is possible to prove that the bilinear form $s : \Phi_h \times \Phi_h \longrightarrow \mathbb{R}$,

$$s(\varphi_h, \psi_h) = \underline{\psi}_h^T S \underline{\varphi}_h, \qquad S = \mathbb{C}\mathbb{A}^{-1}\mathbb{C}^T, \tag{14}$$

corresponding to the Schur complement matrix S, satisfies the following continuity and coercivity assumptions:

Lemma 2.1 *If (A3) holds then the bilinear form s is continuous and coercive with respect to the Φ norm:*

$$s(\varphi_h, \psi_h) \lesssim \|\varphi_h\|_\Phi \|\psi_h\|_\Phi, \tag{15}$$
$$s(\varphi_h, \varphi_h) \gtrsim \|\varphi_h\|_\Phi^2. \tag{16}$$

Assumption (A3) is, in practice, quite restrictive. In particular the requirement that both (A3b) and (A3c) hold simultaneously poses some serious limitation on the choice of the discretization spaces for the three unknowns u, λ and φ. In order to be more free in such a choice, and more specifically in order to be allowed to choose the discretization space for the interface unknown φ independently of the choice of the other two discretization spaces, it is also possible to consider a stabilized version. Different ways have been proposed for stabilizing the three fields formulation ([2, 12, 13]). We consider here the approach of [7, 4]. For all k let a bilinear form $[\cdot,\cdot]_{k,\frac{1}{2}} : H^{1/2}(\partial\Omega_k) \times H^{1/2}(\partial\Omega_k) \to \mathbb{R}$ be given, satisfying the following bounds: for all $\eta, \xi \in H^{1/2}(\partial\Omega_k)$

$$[\eta, \eta]_{\frac{1}{2},k} \geq 0, \qquad [\eta, \xi]_{\frac{1}{2},k} \leq B|\eta|_{1/2,\partial\Omega_k}|\xi|_{1/2,\partial\Omega_k}, \tag{17}$$

for B positive constant independent of k and H_k.

We consider then the following discrete stabilized formulation: find $u_h \in V_h$, $\lambda_h \in \Lambda_h$,

$\varphi_h \in \Phi_h$ such that it holds that

$$
\left\{
\begin{array}{l}
\forall k, \quad \forall v_h^k \in V_h^k, \quad \forall \mu_h^k \in \Lambda_h^k : \\
a^k(u_h^k, v_h^k) + \gamma[u_h^k, v_h^k]_{\frac{1}{2},k} - \displaystyle\int_{\partial\Omega_k} v_h^k \lambda_h^k \, ds \quad - \gamma[\varphi_h, v_h^k]_{\frac{1}{2},k} \;=\; \displaystyle\int_{\Omega_k} f v_h^k \, d\boldsymbol{x}, \\
\displaystyle\int_{\partial\Omega_k} u_h^k \mu_h^k \, ds \qquad\qquad\qquad\qquad\qquad - \displaystyle\int_{\partial\Omega_k} \mu_h^k \varphi_h \, ds = 0, \\
\text{and } \forall \psi_h \in \Phi_h : \\
-\displaystyle\sum_k \gamma[u_h^k, \psi_h]_{k,\frac{1}{2}} + \displaystyle\sum_k \int_{\partial\Omega_k} \lambda_h^k \psi_h \, ds + \displaystyle\sum_k \gamma[\varphi_h, \psi_h]_{k,\frac{1}{2}} \;= 0,
\end{array}
\right. \tag{18}
$$

where $\gamma > 0$ is a parameter independent of the choice of the discretization spaces. Such formulation is consistent with the original continuous problem, *i.e.* by substituting the solution (u, λ, φ) of (7) at the place of $(u_h, \lambda_h, \varphi_h)$ in (18) we obtain an identity.

The assumptions needed to guarantee existence and uniqueness of the solution of the discrete problem, as well as an error estimate, can now be made on one hand on the spaces V_h and Λ_h, and on the other hand, in a completely independent way, on the action of the bilinear forms $[\cdot,\cdot]_{k,\frac{1}{2}}$ on $\Phi_h|_{\partial\Omega_k}$. For example we can replace assumption (A3c) with

(A3c') there exists a positive constant b such that for all $\varphi_h \in \Phi_h$ it holds for all k

$$[\varphi_h, \varphi_h]_{k,\frac{1}{2}} \geq b|\varphi_h|^2_{H^{1/2}(\partial\Omega_k)}.$$

It is then possible to show that the discrete problem (18) admits a unique solution satisfying an optimal error estimate ([4]). It is beyond the goals of this paper to throughly discuss the construction of suitable bilinear forms $[\cdot,\cdot]_{\frac{1}{2},k}$. We recall that this can be done with the aid of multiscale techniques and we refer to [7, 5, 10, 6] for a detailed analysis of possible solutions.

The linear system stemming from such a problem takes this time the following form:

$$
\begin{pmatrix} \breve{A} & B^T & -\gamma D^T \\ B & 0 & C^T \\ -\gamma D & C & \gamma E \end{pmatrix} \cdot \begin{pmatrix} \underline{u}_h \\ \underline{\lambda}_h \\ \underline{\varphi}_h \end{pmatrix} = \begin{pmatrix} \underline{f}_h \\ 0 \\ 0 \end{pmatrix}, \tag{19}
$$

with $\breve{A} = A + \gamma F$, (the matrices D, E and F deriving from the stabilizing terms).

Again, the solution of (19) can be reduced to a system in the unknown $\underline{\varphi}_h$, this time taking the form

$$
S\underline{\varphi}_h := \left(\mathbb{D}\breve{\mathbb{A}}^{-1}\mathbb{D}^T + \gamma E\right)\underline{\varphi}_h = -\mathbb{D}\mathbb{A}^{-1}\begin{pmatrix} \underline{f}_h \\ 0 \end{pmatrix} \tag{20}
$$

with

$$\breve{\mathbb{A}} = \begin{pmatrix} \breve{A} & B^T \\ B & 0 \end{pmatrix}, \qquad \mathbb{D} = [\ -\gamma D \quad C\ ]. \tag{21}$$

Once again we let $s : \Phi_h \times \Phi_h \longrightarrow \mathbb{R}$ be the bilinear form corresponding to the Schur complement matrix S

$$s(\varphi_h, \psi_h) = \underline{\psi}_h^T S \underline{\varphi}_h, \tag{22}$$

and also for the stabilized formulation it holds ([4]):

Lemma 2.2 *Under assumptions (A3a,b,c') the bilinear form s is continuous and coercive with respect to the Φ norm:*

$$s(\varphi_h, \psi_h) \ \lesssim\ \|\varphi_h\|_\Phi \|\psi_h\|_\Phi, \tag{23}$$

$$s(\varphi_h, \varphi_h) \ \gtrsim\ \|\varphi_h\|_\Phi^2. \tag{24}$$

3 Preconditioning by Substructuring

For both original and stabilized three fields formulation, we end up with an equation on the interface space Φ_h of the form

$$S\underline{\varphi}_h = \underline{f}_h$$

corresponding, through a relation of the form (22), to a bilinear form $s : \Phi_h \times \Phi_h \to \mathbb{R}$ verifying for all $\varphi_h, \psi_h \in \Phi_h$

$$|s(\varphi_h, \psi_h)| \lesssim \|\varphi_h\|_\Phi \|\psi_h\|_\Phi, \qquad s(\varphi_h, \varphi_h) \gtrsim \|\varphi_h\|_\Phi^2. \tag{25}$$

Our aim is to provide a preconditioner for the matrix S. The approach that we will follow is the one proposed, in the framework of conforming domain decomposition, by J.H. Bramble, J.E. Pasciak and A.H. Schatz in [9] and already studied in [1] for the mortar method case.

Let us consider a splitting of the skeleton $\Sigma \setminus \Omega$ as the disjoint union of M segments e_i of extrema a_i and b_i

$$\Sigma \setminus \Omega = \cup_{i=1}^M e_i,$$

(corresponding, for instance, to the splitting of Σ as the union of segments of the form $\partial\Omega_k \cap \partial\Omega_\ell$). For each k let E_k individuate those segments that, together with $\partial\Omega \cap \partial\Omega_k$, build up the boundary of the subdomain Ω_k:

$$\partial\Omega_k \setminus \partial\Omega = \cup_{i \in E_k} e_i, \qquad \text{with} \quad E_k = \{i : \ |e_i \cap \partial\Omega_k| > 0\}.$$

We make the following assumption:

(A4) there exists a constant c_4 such that it holds

$$\sup_k \sup_{i \in E_k} \frac{|\partial\Omega_k|}{|e_i|} \leq c_4.$$

Remark that (A4) implies that $\#(E_k) \leq c_4$ for all k. This will allow us several times to write inequalities of the form $\| \sum_{i \in E_k} q_i \|^2 \lesssim \sum_{i \in E_k} \|q_i\|^2$, where $\| \cdot \|$ stands for various norms and seminorms, (the constant in the inequality depending on c_4).

Following [9], the main idea for constructing a preconditioner for the matrix S is to decompose Φ_h as the direct sum of a coarse space $\mathcal{L}_H$ and some local spaces $\Phi^0_{h,i}$ (one for each edge e_i of the decomposition of Σ), and then considering the corresponding block Jacobi type preconditioners. More precisely, let $\mathcal{L}_H \subset \Phi$,

$$\mathcal{L}_H = \{\varphi \in C^0(\Sigma) : \quad \forall i = 1, \ldots, M, \quad \varphi|_{e_i} \in \mathbb{P}_1(e_i), \ \varphi = 0 \text{ on } \partial\Omega\},$$

denote the subset of those functions of Φ whose restriction to each segment e_i is linear. In order to construct the preconditioner we make the following assumptions on the approximation space Φ_h:

(A5) the two following conditions hold:

(a) $\mathcal{L}_H \subset \Phi_h$;

(b) for all $\varphi_h \in \Phi_h$ such that at the extrema a_i and b_i of some edge e_i it holds $\varphi_h(a_i) = \varphi_h(b_i) = 0$, the function $\varphi^0_{h,i}$ defined as

$$\varphi^0_{h,i} = \varphi_h \quad \text{on } e_i, \qquad \varphi^0_{h,i} = 0 \quad \text{on } \Sigma \setminus e_i$$

verifies

$$\varphi^0_{h,i} \in \Phi_h;$$

(A6) $\Phi_h \subset H^1(\Sigma)$ and for all $\varphi_h \in \Phi_h$ the two following inverse inequalities hold for all k, $k = 1, \ldots, K$, and for all r, t, $0 \leq r < t \leq 1$:

$$|\varphi_h|_{H^t(\partial\Omega_k)} \lesssim h_k^{r-t} |\varphi_h|_{H^r(\partial\Omega_k)}, \tag{26}$$

$$|\varphi_h|_{H^t(e_i)} \lesssim h_k^{r-t} |\varphi_h|_{H^r(e_i)}, \qquad \forall i \in E_k. \tag{27}$$

Remark 3.1 *Assumption (A5) is not at all restrictive. In a finite element framework it reduces to asking that the extrema a_i and b_i of the segments e_i into which the skeleton is split, are nodes of the finite element grid for Φ_h.*

Remark 3.2 *The parameter h_k appearing in the inverse inequality (26) has here the meaning of the smallest mesh size of the restriction to $\partial\Omega_k$ of Φ_h.*

Under assumption (A5) it is easy to see that Φ_h can be split as

$$\Phi_h = \mathcal{L}_H \oplus \Phi^0_h, \qquad \varphi_h = \varphi_H + \varphi^0_h,$$

with Φ^0_h defined as

$$\Phi^0_h = \{\varphi_h \in \Phi_h, \ \varphi_h = 0 \text{ at the extrema } a_i \text{ and } b_i \text{ of each edge } e_i \text{ of } \Sigma\}.$$

The subspace Φ_h^0 verifies

$$\Phi_h^0 = \oplus_{i=1}^M \Phi_{h,i}^0, \qquad \Phi_{h,i}^0 = \{\varphi_h \in \Phi_h^0 : \quad \varphi_h = 0 \text{ on } \Sigma \setminus e_i\} \sim \Phi_h^0|_{e_i}.$$

Remark that all $\varphi_h \in \Phi_h^0$ verify $\varphi_h|_{e_i} \in H_{00}^{1/2}(e_i)$ for all i, and in the following we will write $\|\varphi_h\|_{H_{00}^{1/2}(e_i)}$ for $\|\varphi_h|_{e_i}\|_{H_{00}^{1/2}(e_i)}$ (and analogously for other norms defined on portions of Σ).

For each $i = 1, \cdots, M$, we assume we are given a bilinear form $\check{s}_i : \Phi_h^0|_{e_i} \times \Phi_h^0|_{e_i} \to \mathbb{R}$, satisfying the following continuity and positivity bounds uniformly in h: for all $\eta, \xi \in \Phi_h^0|_{e_i}$:

$$\check{s}_i(\xi, \eta) \lesssim \|\xi\|_{H_{00}^{1/2}(e_i)} \|\eta\|_{H_{00}^{1/2}(e_i)}, \qquad \check{s}_i(\xi, \xi) \gtrsim \|\xi\|^2_{H_{00}^{1/2}(e_i)}. \tag{B1}$$

Moreover we assume that we are given a bilinear form $\check{s}_H : \mathcal{L}_H \times \mathcal{L}_H \to \mathbb{R}$ such that for all ξ_H, $\zeta_H \in \mathcal{L}_H$ it holds that

$$\check{s}_H(\xi_H, \eta_H) \lesssim \|\xi_H\|_\Phi \|\zeta_H\|_\Phi, \qquad \check{s}_H(\xi_H, \xi_H) \gtrsim \|\xi_H\|^2_\Phi. \tag{B3}$$

It is beyond the goals of this paper to fully discuss the different ways in which the bilinear forms $\check{s}_i$ and $\check{s}_H$ can be constructed. In the following section we will briefly recall a possible solution and we refer to [9, 10, 16] for further discussions on this topic.

Following [9], the preconditioner will be finally constructed with the aid of a bilinear form $\check{s} : \Phi_h \times \Phi_h \to \mathbb{R}$ assembled by blocks: for $\eta_h, \zeta_h \in \Phi_h$, with $\eta_h = \eta_H + \eta_h^0$, $\zeta_h = \zeta_H + \zeta_h^0$, $\eta_H, \zeta_H \in \mathcal{L}_H$ and $\eta_h^0, \zeta_h^0 \in \Phi_h^0$ we set

$$\check{s}(\eta_h, \zeta_h) = \check{s}_H(\eta_H, \zeta_H) + \sum_{i=1}^M \check{s}_i(\eta_h^0|_{e_i}, \zeta_h^0|_{e_i}). \tag{28}$$

The main result of this paper is the following theorem.

Theorem 3.1 *For all $\varphi_h \in \Phi_h$ it holds*

$$s(\varphi_h, \varphi_h) \lesssim \check{s}(\varphi_h, \varphi_h) \lesssim \max_k \left(1 + \log \frac{H_k}{h_k}\right)^2 s(\varphi_h, \varphi_h). \tag{29}$$

By a well known argument, Theorem 3.1 implies that we can derive the following corollary, where we denote by S and $\check{S}$ respectively the matrices corresponding to the Galerkin discretization of the bilinear forms s and $\check{s}$.

Corollary 3.1 *It holds*

$$cond(\check{S}^{-1} S) \lesssim \max_k \left(1 + \log \frac{H_k}{h_k}\right)^2. \tag{30}$$

We recall that by the definition of the bilinear form s, the matrix S is indeed the Schur complement matrix defined by either (13) on (20).

In order to prove Theorem 3.1 we will basically follow the guidelines of the proof of the analogous theorems of [9, 1]. The main difference is that here we consider a general discretization space, not necessarily consisting in piecewise linear functions. Moreover we will be working with functions and operators defined directly on the interface.

In view of the definition of $\check{s}_i$ and $\check{s}_H$ on one hand, and of the property (25) of s on the other hand, proving (29) is equivalent to prove that $\forall \varphi_h \in \Phi_h$, $\varphi_h = \varphi_H + \varphi_h^0$, $\varphi_H \in \mathcal{L}_H$ and φ_h^0 in Φ_h^0, it holds that

$$\|\varphi_h\|_\Phi^2 \lesssim \|\varphi_H\|_\Phi^2 + \sum_{i=1}^M \|\varphi_h^0\|^2_{H^{1/2}_{00}(e_i)} \lesssim \max_k \left(1 + \log \frac{H_k}{h_k}\right)^2 \|\varphi_h\|_\Phi^2.$$

We start by observing that it holds

$$\|\varphi_h\|_\Phi^2 \lesssim \|\varphi_H\|_\Phi^2 + \|\varphi_h^0\|_\Phi^2 \lesssim \|\varphi_H\|_\Phi^2 + \sum_{i=1}^M \|\varphi_h^0\|^2_{H^{1/2}_{00}(e_i)},$$

where the last bound descends by splitting $\varphi_h^0 \in \Phi_h^0$ as $\varphi_h^0 = \sum_{i=1}^M \varphi_{h,i}^0$ which implies (since $\#(E_k) \leq c_4$ and since each i belongs to E_k for at most two k's)

$$\|\varphi_h^0\|_\Phi^2 \lesssim \sum_k |\varphi_h^0|^2_{H^{1/2}(\partial\Omega_k)} \lesssim \sum_k \sum_{i \in E_k} |\varphi_{h,i}^0|^2_{H^{1/2}(\partial\Omega_k)} \lesssim \sum_{i=1}^M \|\varphi_h^0\|^2_{H^{1/2}_{00}(e_i)},$$

the last inequality deriving from the $H^{1/2}_{00}(e_i)$ to $H^{1/2}(\partial\Omega_k)$ boundedness of the trivial extension-by-zero operator.

Moreover, by direct computation, using the linearity of φ_H one can see that it holds

$$\|\varphi_H\|_\Phi^2 \lesssim \sum_{k=1}^K |\varphi_H|^2_{H^{1/2}(\partial\Omega_k)} \lesssim \sum_{i=1}^M (\varphi_H(a_i) - \varphi_H(b_i))^2, \tag{31}$$

a_i and b_i being the extrema of e_i. Then, proving Theorem 3.1 reduces to proving the following result:

Theorem 3.2 *For all $\varphi_h \in \Phi_h$ with $\varphi_h = \varphi_H + \varphi_h^0$, ($\varphi_H \in \mathcal{L}_H$ and $\varphi_h^0 \in \Phi_h^0$) it holds*

$$\sum_{i=1}^M (\varphi_H(a_i) - \varphi_H(b_i))^2 + \sum_{i=1}^M \|\varphi_h^0\|^2_{H^{1/2}_{00}(e_i)} \lesssim \max_k \left(1 + \log \frac{H_k}{h_k}\right)^2 \|\varphi_h\|_\Phi^2. \tag{32}$$

In order to prove Theorem 3.2 we will need several bounds. We start by proving the following Lemma.

Lemma 3.1 *Assume that Φ_h satisfies assumptions (A5) and (A6). Then the following bounds hold:*

(i) for all $\varphi_h \in \Phi_h$ such that $\varphi_h(p) = 0$ for some $p \in \bar{e}_i$ with $i \in E_k$, we have

$$\|\varphi_h\|^2_{L^\infty(e_i)} \lesssim \left(1 + \log \frac{H_k}{h_k}\right) |\varphi_h|^2_{H^{1/2}(e_i)}; \tag{33}$$

(ii) for all $\varphi_h \in \Phi_h$ it holds

$$(\varphi_h(a_i) - \varphi_h(b_i))^2 \lesssim \left(1 + \log \frac{H_k}{h_k}\right) |\varphi_h|^2_{H^{1/2}(e_i)}, \tag{34}$$

a_i and b_i being the extrema of e_i, $i \in E_k$;

(iii) for all $\varphi_h^0 \in \Phi_h^0$ it holds

$$\sum_{i \in E_k} \|\varphi_h^0\|^2_{H^{1/2}_{00}(e_i)} \lesssim \left(1 + \log \frac{H_k}{h_k}\right)^2 |\varphi_h^0|^2_{H^{1/2}(\partial\Omega_k)}. \tag{35}$$

Proof: Let $i \in E_k$. By assumptions (A1a) and (A4) we have that $|e_i| \simeq H_k$. We start by observing that for any $\zeta \in H^{1/2+\varepsilon}(e_i)$ it holds that

$$\|\zeta\|^2_{L^\infty(e_i)} \lesssim H_k^{-1} \|\zeta\|^2_{L^2(e_i)} + \frac{H_k^{2\varepsilon}}{\varepsilon} |\zeta|^2_{H^{1/2+\varepsilon}(e_i)}. \tag{36}$$

This is easily seen by observing that on the reference segment $\hat{e} =]0,1[$ the continuity bound of the injection $H^{1/2+\varepsilon}(\hat{e}) \subset L^\infty(\hat{e})$ depends on ε as follows: for all $\zeta \in H^{1/2+\varepsilon}(\hat{e})$

$$\|\zeta\|^2_{L^\infty(\hat{e})} \lesssim \|\zeta\|^2_{L^2(\hat{e})} + \frac{1}{\varepsilon} |\zeta|^2_{H^{1/2+\varepsilon}(\hat{e})}.$$

A change of variable then yields (36). Let now $\varphi_h \in \Phi_h$ and $\alpha \in \mathbb{R}$. By assumption (A6) it holds

$$\frac{H_k^{2\varepsilon}}{\varepsilon} |\varphi_h - \alpha|^2_{H^{1/2+\varepsilon}(e_i)} = \frac{H_k^{2\varepsilon}}{\varepsilon} |\varphi_h|^2_{H^{1/2+\varepsilon}(e_i)} \lesssim \frac{H_k^{2\varepsilon} h_k^{-2\varepsilon}}{\varepsilon} |\varphi_h|^2_{H^{1/2}(e_i)}. \tag{37}$$

After choosing $\varepsilon = 1/\log(H_k/h_k)$, inequality (36) for $\zeta = \varphi_h - \alpha$, combined with (37), yields:

$$\|\varphi_h - \alpha\|^2_{L^\infty(e_i)} \lesssim H_k^{-1} \|\varphi_h - \alpha\|^2_{L^2(e_i)} + \log \frac{H_k}{h_k} |\varphi_h|^2_{H^{1/2}(e_i)}. \tag{38}$$

In view of (38), (33) can then be proven by applying exactly the arguments of [9], that for completeness we repeat here: let now choose α to be the average of φ_h on e_i. Applying Poincaré inequality gives

$$H_k^{-1/2} \|\varphi_h - \alpha\|_{L^2(e_i)} \lesssim |\varphi_h|_{H^{1/2}(e_i)} \tag{39}$$

which yields

$$\|\varphi_h - \alpha\|^2_{L^\infty(e_i)} \lesssim \left(1 + \log \frac{H_k}{h_k}\right) |\varphi_h|^2_{H^{1/2}(e_i)}. \tag{40}$$

Now we remark that $\varphi_h(p) = 0$ for some $p \in \bar{e}_i$ implies

$$|\alpha| \lesssim \|\varphi_h - \alpha\|_{L^\infty(e_i)} \tag{41}$$

and we get (33) by triangular inequality.

(ii) then follows by applying (i) to the function $\varphi(x) - \varphi(a_i)$ (which vanishes in $p = a_i$).

As far as (iii) is concerned, it is not difficult to check that for $\varphi_h^0 \in \Phi_h^0$, the inverse inequality (26) with $i \in E_k$ implies

$$\|\varphi_h^0\|_{H^{1/2}_{00}(e_i)} \lesssim h_k^{-\varepsilon} |\varphi_h^0|_{H_0^{1/2-\varepsilon}(e_i)}.$$

Again we bound the $H_0^{1/2-\varepsilon}(e_i)$ seminorm of φ_h^0 by a change of variable, using the corresponding bound on the reference segment $\hat{e} =]0,1[$. We recall that $H_0^{1/2-\varepsilon}(\hat{e})$ and $H^{1/2-\varepsilon}(\hat{e})$ coincide, with equivalent norms, the constant in the equivalence depending on ε. In particular it is possible to prove that the following bound holds for all $\zeta \in H^{1/2}(\hat{e})$ and for all $\alpha \in \mathbb{R}$:

$$|\zeta|_{H_0^{1/2-\varepsilon}(\hat{e})} \lesssim \frac{1}{\varepsilon} \|\zeta - \alpha\|_{H^{1/2}(\hat{e})} + \frac{1}{\sqrt{\varepsilon}} |\alpha|,$$

which rescales as

$$|\zeta|_{H_0^{1/2-\varepsilon}(e_i)} \lesssim \frac{H_k^\varepsilon}{\varepsilon} \left(H_k^{-1/2} \|\zeta - \alpha\|_{L^2(e_i)} + |\zeta - \alpha|_{H^{1/2}(e_i)}\right) + \frac{H_k^\varepsilon}{\sqrt{\varepsilon}} |\alpha|.$$

Again, taking $\zeta = \varphi_h^0$ and α as its average on e_i, we can apply on one hand Poincaré inequality (39), and on the other hand the bound (41), which holds since $\varphi_h^0(a_i) = 0$, and we get

$$\|\varphi_h^0\|_{H^{1/2}_{00}(e_i)} \lesssim \frac{H^\varepsilon h_k^{-\varepsilon}}{\varepsilon} |\varphi_h^0|_{H^{1/2}(e_i)} + \frac{H^\varepsilon h_k^{-\varepsilon}}{\sqrt{\varepsilon}} \|\varphi_h^0 - \alpha\|_{L^\infty(e_i)}.$$

We then take once again $\varepsilon = 1/\log(H_k/h_k)$, and, thanks to bound (40) we obtain

$$\|\varphi_h^0\|_{H^{1/2}_{00}(e_i)} \lesssim \left(1 + \log \frac{H_k}{h_k}\right) |\varphi_h^0|_{H^{1/2}(e_i)}.$$

Since $\sum_{i \in E_k} |\cdot|^2_{H^{1/2}(e_i)} \lesssim |\cdot|^2_{H^{1/2}(\partial\Omega_k)}$, by squaring and then taking the sum over $i \in E_k$, we obtain (35). $\square$

Analogously to what happens in [9], the main ingredient of the proof of Theorem 3.2 is then the following Lemma.

Lemma 3.2 *Let $\varphi_h^0 \in \Phi_h^0$ and $\zeta_H \in \mathcal{L}_H$. Then for all k it holds*

$$\sum_{i \in E_k} \|\varphi_h^0\|^2_{H^{1/2}_{00}(e_i)} \lesssim \left(1 + \log \frac{H_k}{h_k}\right)^2 |\varphi_h^0 + \zeta_H|^2_{H^{1/2}(\partial\Omega_k)}. \tag{42}$$

Provided such Lemma holds, (32) is obtained thanks to (ii) of Lemma 3.1, by applying (42) for $\zeta_H = \varphi_H$. In fact, for $\varphi_h = \varphi_H + \varphi_h^0$, $\varphi_H \in \mathcal{L}_H$, $\varphi_h^0 \in \Phi_h^0$, we can write

$$\begin{aligned}
\sum_{i=1}^{M} \|\varphi_h^0\|^2_{H^{1/2}_{00}(e_i)} &= \frac{1}{2}\sum_{k=1}^{K}\sum_{i \in E_k} \|\varphi_h^0\|^2_{H^{1/2}_{00}(e_i)} \\
&\lesssim \sum_k \left(1 + \log \frac{H_k}{h_k}\right)^2 |\varphi_h|^2_{H^{1/2}(\partial\Omega_k)} \\
&\lesssim \max_k \left(1 + \log \frac{H_k}{h_k}\right)^2 \|\varphi_h\|^2_{\Phi},
\end{aligned}$$

which, combined with (34), yields (32). The only thing that we need to do is then to prove Lemma 3.2.

Proof of Lemma 3.2: Let k, $1 \leq k \leq K$. We start by introducing a function $\zeta^0 \in \Phi_h^{k,0}$

$$\Phi_h^{k,0} = \{\varphi_h \in \Phi_h|_{\partial\Omega_k} :\ \varphi_h = 0 \text{ at the cross points of } \partial\Omega_k\} = \Phi_h^0|_{\partial\Omega_k}, \tag{43}$$

ζ^0 depending on ζ_H, defined as the unique element of $\Phi_h^{k,0}$ satisfying

$$(\zeta^0, \xi)_{H^{1/2}(\partial\Omega_k)} = (\zeta_H, \xi)_{H^{1/2}(\partial\Omega_k)}, \qquad \forall \xi \in \Phi_h^{k,0},$$

where $(\cdot,\cdot)_{H^{1/2}(\partial\Omega_k)}$ is the bilinear form

$$(\zeta, \xi)_{H^{1/2}(\partial\Omega_k)} = \int_{\partial\Omega_k}\int_{\partial\Omega_k} \frac{(\zeta(x) - \zeta(y))(\xi(x) - \xi(y))}{|x-y|^2}\, dx\, dy$$

inducing the $H^{1/2}(\partial\Omega_k)$ seminorm. It is easy to check that $|\cdot|_{H^{1/2}(\partial\Omega_k)}$ is a norm on $\Phi_h^{k,0}$ and then we can conclude by standard arguments that

$$|\zeta^0|_{H^{1/2}(\partial\Omega_k)} \lesssim |\zeta_H|_{H^{1/2}(\partial\Omega_k)}. \tag{44}$$

Now, adding and subtracting ζ^0 at the left hand side of (42), we can write:

$$\sum_{i \in E_k} \|\varphi_h^0\|^2_{H^{1/2}_{00}(e_i)} \lesssim \sum_{i \in E_k} \|\varphi_h^0 + \zeta^0\|^2_{H^{1/2}_{00}(e_i)} + \sum_{i \in E_k} \|\zeta^0\|^2_{H^{1/2}_{00}(e_i)}. \tag{45}$$

Since $\varphi_h^0 + \zeta^0 \in \Phi_h^{k,0}$, by the definition of ζ^0 we have that

$$(\zeta_H - \zeta^0, \varphi_h^0 + \zeta^0)_{H^{1/2}(\partial\Omega_k)} = 0, \tag{46}$$

which yields

$$|\varphi_h^0 + \zeta_H|^2_{H^{1/2}(\partial\Omega_k)} \geq |\varphi_h^0 + \zeta^0|^2_{H^{1/2}(\partial\Omega_k)}.$$

The first sum on the right hand side of (45) can then be bound thanks to Lemma 3.1, (iii): we have

$$\sum_{i\in E_k} \|\varphi_h^0 + \zeta^0\|^2_{H^{1/2}_{00}(e_i)} \lesssim \left(1 + \log\frac{H_k}{h_k}\right)^2 |\varphi_h^0 + \zeta^0|^2_{H^{1/2}(\partial\Omega_k)} \tag{47}$$

$$\lesssim \left(1 + \log\frac{H_k}{h_k}\right)^2 |\varphi_h^0 + \zeta_H|^2_{H^{1/2}(\partial\Omega_k)}. \tag{48}$$

Let us bound the second sum on the right hand side of (45). We recall that we have

$$\|\zeta^0\|^2_{H^{1/2}_{00}(e_i)} = |\zeta^0|^2_{H^{1/2}(e_i)} + I_1(\zeta^0) + I_2(\zeta^0), \tag{49}$$

with

$$I_1(\zeta^0) = \int_{a_i}^{b_i} \frac{|\zeta^0(x)|^2}{|x-a_i|}\,dx, \qquad I_2(\zeta^0) = \int_{a_i}^{b_i} \frac{|\zeta^0(x)|^2}{|x-b_i|}\,dx, \tag{50}$$

(we recall that a_i and b_i denote the two vertices of the edge e_i). Now, using equations (44), (31) and (34), since φ_h^0 vanishes at a_i and b_i, we can write

$$\sum_{i\in E_k} |\zeta^0|^2_{H^{1/2}(e_i)} \lesssim |\zeta^0|^2_{H^{1/2}(\partial\Omega_k)} \lesssim |\zeta_H|^2_{H^{1/2}(\partial\Omega_k)} \tag{51}$$

$$\lesssim \sum_{i\in E_k} (\zeta_H(a_i) - \zeta_H(b_i))^2 \tag{52}$$

$$= \sum_{i\in E_k} ([\varphi_h^0 + \zeta_H](a_i) - [\varphi_h^0 + \zeta_H](b_i))^2 \tag{53}$$

$$\lesssim \left(1 + \log\frac{H_k}{h_k}\right) |\varphi_h^0 + \zeta_H|^2_{H^{1/2}(\partial\Omega_k)}. \tag{54}$$

Let us now bound $I_1(\zeta^0)$. For notational simplicity let us identify $e_i =]0, T[$ with $a_i = 0$ and $b_i = T$. We have

$$I_1(\zeta^0) = \int_0^T \frac{|\zeta^0(x)|^2}{|x|}\,dx \tag{55}$$

$$= \int_0^T \frac{|\zeta^0(x) + \zeta_H(x) - \zeta_H(x) + \zeta_H(0) - \zeta_H(0)|^2}{|x|}\,dx \tag{56}$$

$$\lesssim \int_0^T \frac{|\zeta^0(x) - \zeta_H(x) + \zeta_H(0)|^2}{|x|}\,dx \tag{57}$$

$$+ \int_0^T \frac{|\zeta_H(x) - \zeta_H(0)|^2}{|x|}\,dx. \tag{58}$$

Let us set $\zeta^\perp = \zeta^0 - \zeta_H$. Remarking that $\zeta^\perp(0) = \zeta_H(0)$ we have

$$\begin{aligned}\int_0^T \frac{|\zeta^0(x) - \zeta_H(x) + \zeta_H(0)|^2}{|x|}\,dx &= \int_0^T \frac{|\zeta^\perp(x) - \zeta^\perp(0)|^2}{|x|}\,dx \\ &= \int_0^{h_k} \frac{|\zeta^\perp(x) - \zeta^\perp(0)|^2}{|x|}\,dx + \\ &\quad \int_{h_k}^T \frac{|\zeta^\perp(x) - \zeta^\perp(0)|^2}{|x|}\,dx.\end{aligned}$$

The first term is bound, using Schwarz Hölder's inequality and inequality (27), by

$$\begin{aligned}\int_0^{h_k} \frac{|\zeta^\perp(x) - \zeta^\perp(0)|^2}{|x|}\,dx &= \int_0^{h_k} \frac{1}{|x|}\left|\int_0^x \zeta_x^\perp(\tau)\,d\tau\right|^2 dx \\ &\lesssim \int_0^{h_k}\int_0^x |\zeta_x^\perp(\tau)|^2\,d\tau\,dx \\ &\lesssim h_k|\zeta^\perp|^2_{H^1(e_i)} \lesssim |\zeta^\perp|^2_{H^{1/2}(e_i)},\end{aligned}$$

while, using (33) we bound the second by

$$\begin{aligned}\int_{h_k}^T \frac{|\zeta^\perp(x) - \zeta^\perp(0)|^2}{|x|}\,dx &\lesssim \|\zeta^\perp - \zeta^\perp(0)\|^2_{L^\infty(e_i)}\int_{h_k}^T \frac{1}{x}\,dx \\ &\lesssim \left(1 + \log\frac{H_k}{h_k}\right)^2 |\zeta^\perp|^2_{H^{1/2}(e_i)},\end{aligned}$$

where we used $T \simeq H_k$. The second integral in (57) can be bound directly using the linearity of ζ_H.

$$\int_0^T \frac{|\zeta_H(x) - \zeta_H(0)|^2}{|x|}\,dx \lesssim (\zeta_H(b_i) - \zeta_H(a_i))^2 \lesssim \left(1 + \log\frac{H_k}{h_k}\right)|\zeta_H + \varphi_h^0|^2_{H^{1/2}(e_i)}$$

where again we used (34) and the fact that φ_h^0 vanishes at a_i and b_i. Then, adding up the different contributions, we conclude that

$$I_1(\zeta^0) \lesssim \left(1 + \log\frac{H_k}{h_k}\right)^2 |\zeta^0 - \zeta_H|^2_{H^{1/2}(e_i)} + \left(1 + \log\frac{H_k}{h_k}\right)|\zeta_H + \varphi_h^0|^2_{H^{1/2}(e_i)}. \tag{59}$$

By using the same arguments we get a similar bound for I_2

$$I_2(\zeta^0) \lesssim \left(1 + \log\frac{H_k}{h_k}\right)^2 |\zeta^0 - \zeta_H|^2_{H^{1/2}(e_i)} + \left(1 + \log\frac{H_k}{h_k}\right)|\zeta_H + \varphi_h^0|^2_{H^{1/2}(e_i)}. \tag{60}$$

We can now insert bounds (51), (59) and (60) back into (49) and sum over $i \in E_k$. By observing that for any $w \in H^{1/2}(\partial\Omega_k)$ it holds $\sum_{i\in E_k}|w|^2_{H^{1/2}(e_i)} \lesssim |w|^2_{H^{1/2}(\partial\Omega_k)}$ we obtain the bound

$$\sum_{i\in E_k}\|\zeta^0\|^2_{H^{1/2}_{00}(e_i)} \lesssim \left(1 + \log\frac{H_k}{h_k}\right)|\varphi_h^0 + \zeta_H|^2_{H^{1/2}(\partial\Omega_k)} \tag{61}$$

$$+ \left(1 + \log\frac{H_k}{h_k}\right)^2 |\zeta^0 - \zeta_H|^2_{H^{1/2}(\partial\Omega_k)}. \tag{62}$$

We now observe once again that equation (46) implies that $|\zeta_H - \zeta^0|^2_{H^{1/2}(\partial\Omega_k)} \lesssim |\zeta_H + \varphi_h^0|^2_{H^{1/2}(\partial\Omega_k)}$ and we get the thesis. $\square$

4 Numerical Tests

We will test the preconditioner we propose on the following problem: find u such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega =]0,1[\times]0,1[, \\ \quad u = 0, & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{63}$$

We consider an uniform, geometrically conforming, decomposition of $\Omega =]0,1[\times]0,1[$ in $K = N \times N$ equal square subdomains of size $H \times H$, $H = 1/N$.

In each subdomain Ω_k we take an uniform mesh $\mathcal{T}^k$ composed by $n_k \times n_k$ equal square elements of size $\delta_k \times \delta_k$, $\delta_k = H/n_k = 1/(Nn_k)$. We then define V_h^k to be the space of $\mathbb{Q}_1$ finite elements on the mesh $\mathcal{T}^k$,

$$V_h^k = \{u_h \in C^0(\Omega) : \ u_h|_\tau \in \mathbb{Q}_1(\tau), \ \forall \tau \in \mathcal{T}^k\},$$

where $\mathbb{Q}_1(\tau)$ denotes the space of polynomials of degree less or equal than one in each variable. The multiplier space Λ_h^k is then defined as the trace on $\partial\Omega_k$ of V_h^k. With such a choice it is possible to prove that (A3) holds (see [4]).

The skeleton $\Sigma \setminus \partial\Omega$ is split as $\Sigma \setminus \partial\Omega = \cup_{i=1}^{2(N-1)N} e_i$ with $e_i = \Gamma_k \cap \Gamma_\ell$ for some $k = k(i)$ and $\ell = \ell(i)$. We remark that $|e_i| = H$ for all i. We consider a grid $\mathcal{G}$ on Σ obtained by uniformly splitting each e_i into L_i elements of size h_i and we define Φ_h to be the space of $\mathbb{P}_1$ finite elements on such a grid:

$$\Phi_h = \{\varphi_h \in C^0(\Sigma) : \varphi_h|_\kappa \in \mathbb{P}_1(\kappa), \ \forall \kappa \in \mathcal{G}, \quad \varphi_h|_{\partial\Omega} = 0\}.$$

With this choice, assumption (A6) holds for $h_k = \min_{i \in E_k} h_i$.

Letting $\underline{\varphi}_{h,i}^0$ (resp. $\underline{\psi}_{h,i}^0$) denote the vector of the values of φ_h^0 (resp ψ_h^0) $\in \Phi_h^0$ at the interior nodes of the grid $\mathcal{G}|_{e_i}$, we define the bilinear form $\check{s}_i : \Phi_{h,i}^0|_{e_i} \times \Phi_{h,i}^0|_{e_i}$ as

$$\check{s}_i(\varphi_h^0|_{e_i}, \psi_h^0|_{e_i}) = (\underline{\psi}_{h,i}^0)^T (R_i)^{1/2} \underline{\varphi}_{h,i}^0,$$

where R_i is the stiffness matrix associated to the discretization of the operator $-d^2/dx^2$ by $\mathbb{P}_1$ finite element on e_i with homogeneous Dirichlet boundary conditions at the extrema a_i and b_i. R_i being positive definite, its square root is well defined and it is computed and stored at the start.

As far as the coarse preconditioner $\check{s}_H$ is concerned, we considered two choices:

- (BPS) following [9] we set

$$\check{s}_H(\eta_H,\zeta_H) := 2\sum_{i=1}^{M}(\eta_H(a_i)-\eta_H(b_i))(\zeta_H(a_i)-\zeta_H(b_i)) \tag{64}$$

- (Laplace) denoting by $\tilde{\eta}_H$ and $\tilde{\zeta}_H$ the $H^1_0(\Omega)$ functions obtained by lifting η_H and ζ_H harmonically in each subdomain, we set

$$\check{s}_H(\eta_H,\zeta_H) := \int_\Omega \nabla\tilde{\eta}_H\nabla\tilde{\zeta}_H\,d\boldsymbol{x}; \tag{65}$$

  remark that the set $\{\tilde{\eta}_H,\ \ \eta_H\in\mathcal{L}_H\}$ coincides with the set of $\mathbb{Q}_1$ finite elements on the coarse grid corresponding to the decomposition $\Omega=\cup\Omega_k$, and then applying the coarse preconditioner reduces to numerically solve the Laplace equation on such space.

For all tests we set $f=1$. We solved the linear system by a preconditioned Conjugate Gradient method, using $\varphi_h=0$ as initial guess. We report the number of iterations needed to reduce the residual of a factor 10^{-5}.

4.1 Plain Formulation

4.1.1 Conforming Decomposition

We start by considering the case of a conforming domain decomposition, that is, in the framework described above we set $n_k=n$ for all k and $L_i=n$ for all i. In this case $\delta_k=h_i=h=H/n$. With such a choice, it is not difficult to check that the inf-sup condition (11) is satisfied uniformly in h and then we are in the range of Lemma 2.1. It is easy to realize that the discrete solution verifies $u_h^k=\varphi_h$ on $\partial\Omega_k$, and then the function u_h, defined by $u_h|_{\Omega_k}=u_h^k$, verifies $u_h\in H^1_0(\Omega)$. In such a case we obtain the same solution that we would get using a conforming finite element method on a regular grid of $Nn\times Nn$ square elements of dimension $h\times h$. The use of a non conforming method is therefore, in this case, only a device to implement a solver for the discrete problem in an easily parallelizable way.

In order to study both the dependence on H (size of the subdomains) and on h we tested the preconditioner for different values of n in the range $[5,40]$ and N in the range $[4,32]$. We tested both coarse preconditioners for all combinations of N and n. The results are summarized in Tables 1 and 2. As one can see they are in close agreement with the theory (remark that in this case $\max_k h_k/H_k=h/H=n$).

4.1.2 Nonconforming decomposition

We consider then the case of a truly non conforming decomposition, without stabilization (formulation (9)). In our test for $e_i=\partial\Omega_k\cap\partial\Omega_\ell$, we set $L_i=\min\{n_k,n_\ell\}$. This discretization does not in general satisfy the assumptions required for the uniform stability

$K = N^2$	$n = 5$ # Iter.	$n = 10$ # Iter.	$n = 20$ # Iter.	$n = 40$ # Iter.
16	10	12	14	15
64	14	17	20	23
144	15	18	22	26
256	15	19	23	27
400	16	20	23	27
576	16	20	24	27
784	16	20	24	27
1024	16	20	24	27

Table 1: Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5}, with coarse preconditioner given by (65) for different combinations of the number $K = N^2$ of subdomains and n of elements per edge (n^2 elements per subdomain).

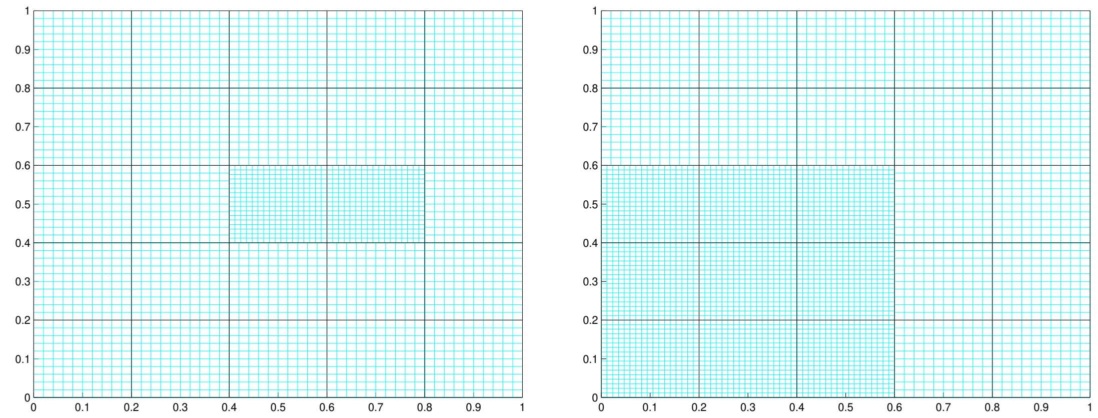
$K = N^2$	$n = 5$ # Iter.	$n = 10$ # Iter.	$n = 20$ # Iter.	$n = 40$ # Iter.
16	8	9	11	13
64	10	13	17	22
144	11	15	20	26
256	13	16	22	30
400	13	17	23	32
576	13	17	24	33
784	14	18	25	33
1024	14	18	25	33

Table 2: Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5}, with coarse preconditioner given by (64) for different combinations of the number $K = N^2$ of subdomains and n of elements per edge (n^2 elements per subdomain).

of the discrete problem. Nevertheless using a standard technique it is not difficult to prove that (11) holds provided that for all ℓ, k such that $|\partial\Omega_\ell \cap \partial\Omega_k| > 0$ it either holds $n_\ell = n_k$ or $\max\{n_\ell, n_k\}/\min\{n_\ell, n_k\} \geq C$ for some constant $C > 1$.

We considered two test configurations, both on a decomposition of Ω into 5×5 subdomains. In the first case (see figure 1, left), for all subdomains except two, we set $n_k = 10$, while on the remaining two subdomains (subdomain n. 13 and 14, if we number all subdomains row-wise) we set $n_k = n$, where n varies in the set $[10, \ldots, 52]$. With the above criterion used to fix L_i, we have that $L_i = 10$ for all edges e_i, except the common edge $\Gamma_{13} \cap \Gamma_{14}$ (the two subdomains are adjacent), where $L_i = \max\{n_{13}, n_{14}\} = n$ varies.

In the second configuration (see figure 1, right) for all subdomains we set again $n_k = 10$, this time except that on a three by three block of subdomains where $n_k = n$, with n varying again between 10 and 52. Here, for 12 edges of Σ out of 40, we have that $L_i = n$. For both configurations we tested both coarse preconditioners. The results are provided in Table 3.



(1) (2)

Figure 1: The two configurations on which the preconditioner is tested for the plain formulation (7) in the non conforming case.

As one can easily see, though the theoretical estimate on the condition number only depends on $\max_k H_k/h_k$, which takes the same value for both configurations, in the first case in which the overall discretization of the skeleton is sensibly coarser – the mesh being fine only in a small portion (in our case one) of the edges composing Σ – the influence of refining the mesh is weaker than in the second case, in which a relevant portion of the skeleton is refined.

4.2 Stabilized Formulation

We now consider the stabilized formulation (18). Since the $H^{1/2}(\partial\Omega_k)$ seminorm is invariant under changes of scale, we can describe the stabilizing form for subdomains scaled in

	Laplace # It.	BPS # It
$n = 10$	12	8
$n = 17$	18	11
$n = 24$	18	13
$n = 31$	18	13
$n = 38$	18	14
$n = 45$	19	14
$n = 52$	19	14

(a) Configuration 1.

	Laplace # It.	BPS # It
$n = 10$	12	8
$n = 17$	20	14
$n = 24$	23	16
$n = 31$	24	18
$n = 38$	25	19
$n = 45$	26	20
$n = 52$	27	20

(b) Configuration 2.

Table 3: Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} for the two configurations 1. and 2. (corresponding respectively to the left and to the right decompositions illustrated in figure 1). Both coarse preconditioners (64) (BPS) and (65) (Laplace) are tested for different values of the number n of elements per edge in the refined part of the skeleton (one edge for configuration 1. and twelve edges for configuration 2.). The number of elements per edge in the remaining part of the skeleton is set to 10.

such a way that $|\partial\Omega_k| = 1$ (that is $H = 1/4$). We can then identify $\partial\Omega_k$ with the circle T of unitary length. Following the proposal of [7], the bilinear forms $[\cdot,\cdot]_{k,\frac{1}{2}}$ are designed by means of a wavelet decomposition. More precisely, we restrict the number L_i of elements of the i-th edge e_i to be a power of two,

$$L_i = 2^{j_i} \qquad \text{for some } j_i \geq 1,$$

so that for all k, $\Phi_h|_{\partial\Omega_k} \subset V_{j_k+2}$ where for $j > 0$, V_j denotes the space of $\mathbb{P}_1$ finite elements on the uniform grid $\mathcal{G}_j$ of T with mesh size $1/2^j$.

The sequence $\{V_j\}_{j\geq 0}$ forms a so called *multiresolution analysis* of $L^2(T)$ and it is well known (see for example [15]) that there exists several wavelet bases associated with such a multiresolution analysis. More precisely there exist several $\mathbb{P}_1$ compactly supported functions $\vartheta \in C^0(\mathbb{R})$ defined on the uniform grid of mesh size 1 and integer nodes, such that, setting

$$\vartheta_{m\ell} = \sum_{n=-\infty}^{+\infty} 2^{m/2}\vartheta(2^m(x-n)-\ell),$$

all functions $\eta_j \in V_j$ can be written as

$$\eta_j = \eta_0 + \sum_{m=0}^{j-1}\sum_{\ell=1}^{2^m} \eta_{m\ell}\vartheta_{m\ell}, \qquad \eta_0 \text{ constant},$$

and such that

$$|\eta_j|^2_{H^{1/2}(\partial\Omega_k)} \simeq \sum_{m=0}^{j-1}\sum_{\ell=1}^{2^m} 2^m|\eta_{m\ell}|^2.$$

Given the values of η_j at the nodes of the uniform grid the coefficients $(\eta_{m\ell})_{m,l}$ can be retrieved by a fast ($O(2^j)$) wavelet transform (FWT).

If we denote by $\mathbf{P}_k : L^2(T) \to V_{j_k+2}$ the $L^2(T)$ orthogonal projection, for $\zeta, \xi \in L^2(T)$ we can define corresponding wavelet coefficients $\check{\zeta}_{m\ell}$ and $\check{\xi}_{m\ell}$ such that

$$\mathbf{P}_k(\zeta) = \zeta_0 + \sum_{m=0}^{j_k} \sum_{\ell=1}^{2^m} \check{\zeta}_{m\ell} \vartheta_{m\ell}, \qquad \mathbf{P}_k(\xi) = \xi_0 + \sum_{m=0}^{j_k} \sum_{\ell=1}^{2^m} \check{\xi}_{m\ell} \vartheta_{m\ell}.$$

Then we define the bilinear form $[\cdot,\cdot]_{k,\frac{1}{2}}$ as

$$[\zeta, \xi]_{k,\frac{1}{2}} = \sum_{m=0}^{j_k} \sum_{\ell=1}^{2^m} 2^m \check{\zeta}_{m\ell} \check{\xi}_{m\ell}.$$

It is possible to prove ([7, 14]) that the bilinear forms thus defined satisfies assumptions (A4) and (17).

Remark 4.1 *Remark that the coefficients $\check{\zeta}_{m\ell}$ and $\check{\xi}_{m\ell}$ are not the classical wavelet coefficients of the functions ζ and ξ. These would in fact be defined as the scalar product of respectively ζ and ξ with suitable* dual functions $\tilde{\vartheta}_{m\ell} \in L^2(T)$. *The coefficients $\check{\zeta}_{m\ell}$ and $\check{\xi}_{m\ell}$ are rather the scalar products of the dual functions $\tilde{\vartheta}_{m\ell}$ with the projections of ζ and ξ onto V_{j_k+2} respectively. Naturally, such coefficients coincide with the classical ones, provided $f \in V_{j_k+2}$.*

With this definition, the computation of $[u_h^k - \varphi_h, v_h^k - \psi_h]_{k,\frac{1}{2}}$ essentially reduces to first computing the nodal values of $\mathbf{P}_k(u_h^k)$, $\mathbf{P}_k(v_h^k)$, $\mathbf{P}_k(\varphi_h)$ and $\mathbf{P}_k(\psi_h)$ respectively and then applying a FWT. As far as φ_h and ψ_h are concerned, the choice of j_k implies that $\varphi_h, \psi_h \in V_{j_k+2}$ and then their nodal values are simply computed by interpolation. As far as u_h^k and v_h^k are concerned, we need to compute their L^2 projection on V_{j_k+2}. Observing that the mass matrix in V_{j_k+2} has a circulant structure with bandwidth equals to three, this can also be done in $O(2^{j_k})$ operations by a modification of the Croute reduction algorithm for solving linear systems with tridiagonal matrices. For further details on the implementation of the stabilized formulation we refer to [8].

We tested the preconditioner (with both form (64) and (65) of the coarse bilinear form) on four different splittings of Ω into respectively 4×4, 8×8, 12×12 and 16×16 subdomains. In each case we randomly assigned the values of n_k in such a way that for $\sim 1/3$ of the subdomains $n_k = 5$, for about another third $n_k = 10$, and for the remaining subdomains $n_k = 15$. The four configurations considered are shown in picture 2. For approximating the interface function φ we used a uniform discretization of the interface ($L_i = n$ for all i). For each configuration we tested the preconditioner for different values of n and of the stabilization parameter γ. The function ϑ appearing in the definition of the stabilizing bilinear form is chosen to be the piecewise linear wavelet corresponding to the *2.2 B-spline* biorthogonal setting ([15]). Remark (see Table 4), that for $\gamma = 0$ (no stabilization!) if n is big the CG algorithm does not converge within the maximum number of iterations (which was set equals to 100), confirming the need for using some kind of stabilization.

As one can see from the results, the value of the parameter γ has an influence on the conditioning of the matrix $\check{S}^{-1}S$. If γ is too small, see Table 5, the action of the stabilization term is not sufficient to compensate the lack of validity of the inf-sup condition, and therefore the system behaves almost like the one obtained using the plain formulation. By increasing γ, see Tables 6, 7, the situation improves dramatically and one can see that, for instance for $\gamma = .05$ in the case of 16×16 subdomains with $n = 32$ (which corresponds to 15360 degrees of freedom for φ_h) the residual is reduced of a factor 10^{-5} in about 35 iterations, which is only slightly worse than the result obtained for the analogous values in the conforming case. However, when γ is further increased (see Table 8), the performances deteriorate. This (as well as the difference with respect to the conforming case) is probably due to fact that the the constants in the estimates (25) – which have an influence on the bound on the condition number – depend on the ratio B/b (B and b being the constants in (17) and in assumption (A3c') respectively) *amplified by the factor* γ. If such a ratio (which is always greater or equal to one) in not close enough to 1, the choice of the right balance between the original bilinear form and the stabilizing term can become important from the point of view of the spectral properties of the operator, which are essential for preconditioning.

	(a)	(b)	(c)	(d)
n	# It.	# It.	# It.	# It.
4	14	17	18	19
8	47	57	60	69
16	—	—	—	—
32	—	—	—	—

Laplace

	(a)	(b)	(c)	(d)
n	# It.	# It.	# It.	# It.
4	11	13	14	15
8	40	52	54	64
16	—	—	—	—
32	—	—	—	—

BPS

Table 4: Number of conjugate gradient iterations needed to reduce the residual of a factor 10^{-5} on the four configurations depicted in figure 2. with the plain formulation (9). Both coarse preconditioners are tested for different values of the number n of elements per edge. As one can see for $n \geq 16$ the CG method does not converge within the maximum number of iteration (= 100), confirming the instability of the plain formulation for the discretization considered.

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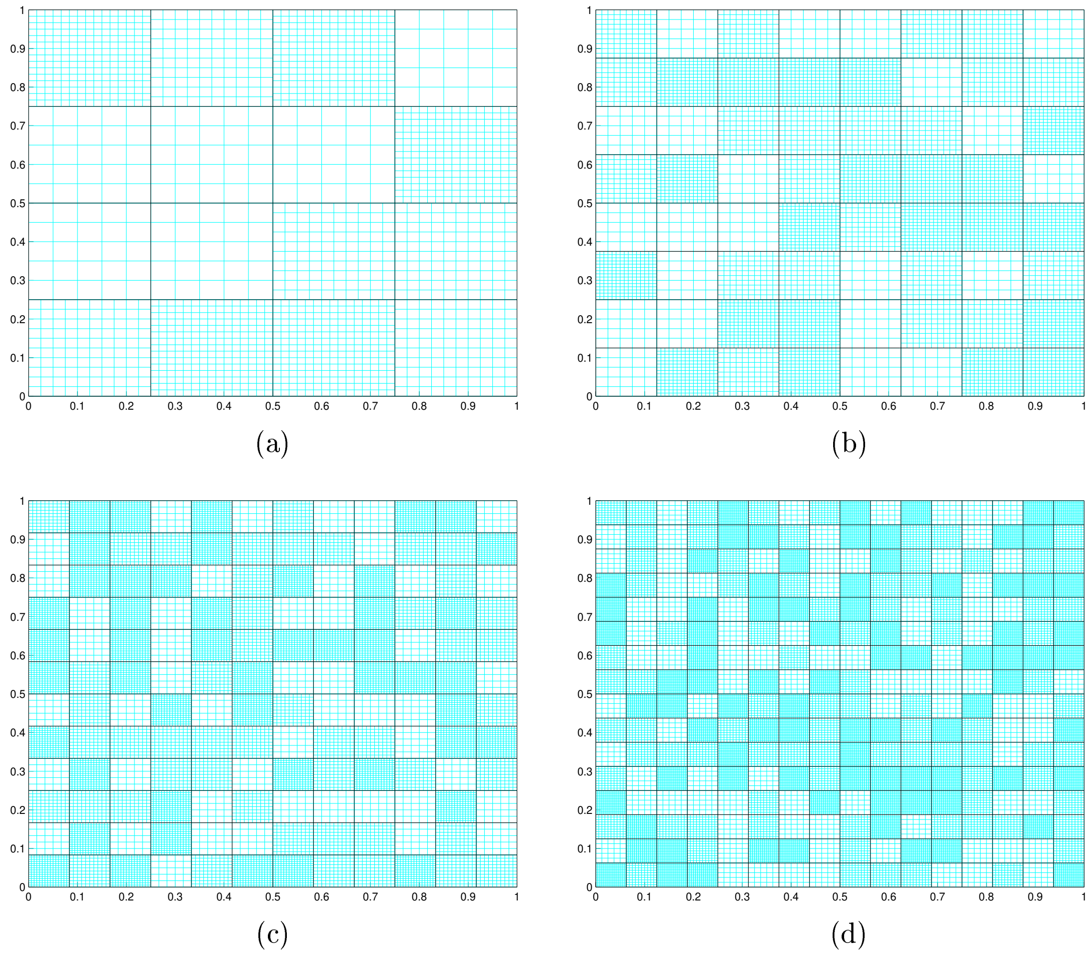

Figure 2: The four configurations on which the preconditioner is tested for the stabilized formulation (9)

n	(a) # It.	(b) # It.	(c) # It.	(d) # It.
4	14	17	18	19
8	53	61	65	85
16	71	87	94	—
32	67	85	92	98

Laplace

n	(a) # It.	(b) # It.	(c) # It.	(d) # It.
4	11	13	14	15
8	44	52	59	79
16	57	78	91	—
32	59	82	96	—

BPS

Table 5: Number of conjugate gradient iterations needed to reduce the residual of a factor 10^{-5} on the four configurations depicted in figure 2. with the stabilized formulation (18) for $\gamma = .00125$. Both coarse preconditioners are tested for different values of the number n of elements per edge. One can clearly see that the method converges very badly. This is due to the fact that, since the stabilization parameter is very small, the action of the stabilization term is not sufficient to compensate the lack of validity of the inf-sup condition, and therefore the system behaves like the one obtained using the plain formulation.

n	(a) # It.	(b) # It.	(c) # It.	(d) # It.
4	14	17	18	19
8	29	34	37	41
16	31	37	39	41
32	29	35	36	38

Laplace

n	(a) # It.	(b) # It.	(c) # It.	(d) # It.
4	11	13	14	15
8	24	28	32	37
16	25	33	37	41
32	25	34	38	42

BPS

Table 6: Number of conjugate gradient iterations needed to reduce the residual of a factor 10^{-5} on the four configurations depicted in figure 2. with the stabilized formulation (18) for $\gamma = .0125$. Both coarse preconditioners are tested for different values of the number n of elements per edge.

n	(a) # It.	(b) # It.	(c) # It.	(d) # It.
4	14	17	18	19
8	20	24	25	26
16	22	27	28	30
32	25	33	34	36

Laplace

n	(a) # It.	(b) # It.	(c) # It.	(d) # It.
4	11	13	14	15
8	15	19	21	24
16	17	23	26	30
32	21	28	30	35

BPS

Table 7: Number of conjugate gradient iterations needed to reduce the residual of a factor 10^{-5} on the four configurations depicted in figure 2. with the stabilized formulation (18) for $\gamma = .05$. Both coarse preconditioners are tested for different values of the number n of elements per edge.

n	(a) # It.	(b) # It.	(c) # It.	(d) # It.
4	14	17	18	19
8	20	24	25	27
16	28	35	37	40
32	42	50	49	53

Laplace

n	(a) # It.	(b) # It.	(c) # It.	(d) # It.
4	11	13	14	15
8	16	17	19	22
16	25	31	31	33
32	41	50	48	51

BPS

Table 8: Number of conjugate gradient iterations needed to reduce the residual of a factor 10^{-5} on the four configurations depicted in figure 2. with the stabilized formulation (18) for $\gamma = .25$. Both coarse preconditioners are tested for different values of the number n of elements per edge.

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5 Appendix: some Sobolev space injection bounds

The aim of this section is to recall some known bounds for which we provide a new proof based on the use of equivalent norms expressed through wavelet coefficients. Such proof will easily reveal the dependence of the constants in the bounds on the smoothness of the spaces considered and the way these constants explode as the smoothness index approaches the limit of validity of the bounds.

We introduce two orthonormal wavelet bases $\{\rho_k\}_k \cup \{\vartheta_{j,k}\}_{j,k}$ and $\{\rho_k^0\}_k \cup \{\vartheta_{j,k}^0\}_{j,k}$ for $L^2(0,1)$, which we assume to satisfy the following properties:

- all $u \in L^2(0,1)$ can be written as

$$\begin{aligned} u &= \sum_{k=0}^{2^{j_0}} < u, \rho_k > \rho_k + \sum_{j=j_0}^{\infty} \sum_{k=1}^{2^j} < u, \vartheta_{j,k} > \vartheta_{j,k} \\ u &= \sum_{k=0}^{2^{j_0}} < u, \rho_k^0 > \rho_k^0 + \sum_{j=j_0}^{\infty} \sum_{k=1}^{2^j} < u, \vartheta_{j,k}^0 > \vartheta_{j,k}^0 \end{aligned}$$

  and the following norm equivalences hold for all r with $0 \leq r \leq 1$:

$$\|u\|_{H^r(0,1)} \simeq \left(\sum_{k=0}^{2^{j_0}} | < u, \rho_k > |^2 + \sum_{j=j_0}^{\infty} \sum_{k=1}^{2^j} 2^{2jr} | < u, \vartheta_{j,k} > |^2 \right)^{1/2}, \tag{66}$$

$$\|u\|_{H_0^r(0,1)} \simeq \left(\sum_{k=0}^{2^{j_0}} | < u, \rho_k^0 > |^2 + \sum_{j=j_0}^{\infty} \sum_{k=1}^{2^j} 2^{2jr} | < u, \vartheta_{j,k}^0 > |^2 \right)^{1/2}, \tag{67}$$

  with constants independent of r;

- for all $j > j_0$ and for all $k = 1, \ldots, 2^j$ the functions $\vartheta_{j,k}$ and $\vartheta_{j,k}^0$ verify the cancellation properties

$$< \vartheta_{j,k}, p >= 0, \quad \forall p \in \mathcal{P}_M, \qquad < \vartheta_{j,k}^0, p >= 0, \quad \forall p \in \mathcal{P}_M \cap H_0^1,$$

  ($\mathcal{P}_M$ denoting the space of polynomials of degree lower or equal than M);

- the supports of the basis functions are properly scaled with respect to j:

  $\operatorname{supp} \vartheta_{j,k} \subset [(k-N)2^{-j}, (k+N)2^{-j}] \cap]0,1[, \qquad \operatorname{supp} \vartheta_{j,k}^0 \subset [(k-N)2^{-j}, (k+N)2^{-j}] \cap]0,1[;$

- the interior functions of the two bases coincide:

$$\vartheta_{j,k} = \vartheta_{j,k}^0, \qquad \forall k = N, \ldots, 2^j - N;$$

- the normalization of the basis functions in $L^2(0,1)$ implies

$$|\rho_k(x)| \leq C, \qquad |\rho_k^0(x)| \leq C, \qquad |\vartheta_{j,k}(x)| \lesssim 2^{j/2}, \qquad |\vartheta_{j,k}^0(x)| \lesssim 2^{j/2};$$

Remark that the constants N and j_0 (and therefore 2^{j_0}) are fixed once for and depend only on the choice of the wavelet basis. We will therefore consider them as an $O(1)$ as far as the estimates that we are going to prove are concerned.

We recall the following bound on the sum of a geometric series, which we will need in the following:

$$\sum_{j \geq 1} 2^{-aj} \lesssim \frac{1}{a}.$$

We start by recalling Poincaré's inequality:

Lemma 5.1 *(***Poincaré inequality***). Let $u \in H^s(0,1)$, $0 < s < 1$ such that $\int_0^1 u\,dx = 0$. Then it holds*

$$\|u\|_{L^2(0,1)} \lesssim |u|_{H^s(0,1)}.$$

Proof: We have:

$$\begin{aligned} \|u\|^2_{L^2(0,1)} &= \int_0^1 \left(u(x) - \int_0^1 u(y)\,dy\right)^2 dx = \int_0^1 \left(\int_0^1 (u(x) - u(y))\,dy\right)^2 dx \\ &\leq \int_0^1 \int_0^1 (u(x) - u(y))^2\,dy\,dx = \int_0^1 \int_0^1 \frac{(u(x) - u(y))^2}{|x-y|^{2s+1}} |x-y|^{2s+1}\,dy\,dx \\ &\leq \int_0^1 \int_0^1 \frac{(u(x) - u(y))^2}{|x-y|^{2s+1}}\,dy\,dx \end{aligned}$$

where we used that for $x, y \in (0,1)$ it holds that $|x - y| \leq 1$. □

The first bound that we consider regards the injection $H^s(0,1) \subset L^\infty(0,1)$, which is known to hold for all $s > 1/2$. As far as the dependence of the constant in the bound on s is concerned, we have the following lemma.

Lemma 5.2 *The following bound hold for all $u \in H^s(0,1)$ with $s > 1/2$:*

$$\|u\|_{L^\infty(0,1)} \lesssim \|u\|_{L^2(0,1)} + \frac{1}{\sqrt{s - 1/2}} |u|_{H^s(0,1)}.$$

Proof: Let $u = \sum_{k=0}^{2^{j_0}} < u, \rho_k > \rho_k + \sum_{j=0}^{\infty} \sum_{k=1}^{2^j} < u, \vartheta_{j,k} > \vartheta_{j,k}$. We have (with

$\check{u} = \int_0^1 u\, ds$)

$$
\begin{aligned}
\sup_x |u(x)| &\lesssim \sup_x |\sum_{k=1}^{2^{j_0}} < u, \rho_k > \rho_k(x) + \sum_{j=j_0}^{\infty} \sum_{k=1}^{2^j} < u, \vartheta_{j,k} > \vartheta_{j,k}(x)| \\
&\lesssim \max_x |\sum_{k=0}^{2^{j_0}} < u, \rho_k > \rho_k(x)| + \sum_j \max_x |\sum_{k=1}^{2^j} < u, \vartheta_{j,k} > \vartheta_{j,k}(x)| \\
&\lesssim \max_k | < u, \rho_k > | + \sum_j \max_k | < u, \vartheta_{j,k} > | 2^{sj} 2^{(1/2-s)j} \\
&\leq \left(\sum_k | < u, \rho_k > |^2 \right)^{1/2} + \left(\sum_j 4^{(1/2-s)j} \right)^{1/2} \left(\sum_j \sum_k | < u - \check{u}, \vartheta_{j,k} > |^2 2^{2sj} \right)^{1/2} \\
&\lesssim \|u\|_{L^2(0,1)} + \frac{1}{\sqrt{s-1/2}} \|u - \check{u}\|_{H^s(0,1)}.
\end{aligned}
$$

We conclude by applying Poincaré inequality which yields $\|u - \check{u}\|_{H^s(0,1)} \lesssim |u - \check{u}|_{H^s(0,1)} = |u|_{H^s(0,1)}$ □

The second issue that we consider is the equivalence between the spaces $H^s(0,1)$ and $H^s_0(0,1)$. This is known to hold for all $s \in [0, 1/2[$. The dependence of the constant in the bound on s is the object of the following lemma.

Lemma 5.3 *The following bound holds for all $\eta \in H^s$, $0 \leq s < 1/2$*

$$
\|\eta\|_{H^s(0,1)} \lesssim \|\eta\|_{H^s_0(0,1)} \lesssim \frac{1}{1/2 - s} \|\eta\|_{H^s(0,1)}. \tag{68}
$$

Proof: The first inequality immediately descends by the definition of $H^s_0(0,1)$. Let us then concentrate on the second inequality. Let $\eta \in H^s(0,1)$,

$$
\eta = \sum_{k=0}^{2^{j_0}} \zeta_k \rho_k + \sum_j \sum_k \eta_{j,k} \vartheta_{j,k}, \qquad \text{so that } \|\eta\|^2_{H^s(0,1)} \simeq \sum_{k=0}^{2^{j_0}} |\zeta_k|^2_{L^2} + \sum_{j \geq j_0} 2^j \sum_{k=1}^{2^j} |\eta_{j,k}|^2.
$$

We can decompose η as

$$
\eta = \eta_0 + \eta^{left} + \eta^{center} + \eta^{right}
$$

with

$$
\begin{aligned}
\eta_0 &= \sum_{k=0}^{2^{j_0}} \zeta_k \rho_k, \\
\eta^{left} &= \sum_{j\geq j_0 \sum_{k=1}^{N-1} \eta_{j,k}\vartheta_{j,k},} \\
\eta^{center} &= \sum_{j\geq j_0 \sum_{k=N}^{2^j-N} \eta_{j,k}\vartheta_{j,k} = \sum_{j\geq j_0 \sum_{k=N}^{2^j-N} \eta_{j,k}\vartheta^0_{j,k},}} \\
\eta^{right} &= \sum_{j\geq j_0 \sum_{k=2^j-N+1}^{2^j} \eta_{j,k}\vartheta_{j,k}.}
\end{aligned}
$$

and we have

$$
\|\eta\|_{H^s_0(0,1)} \leq \|\eta_0\|_{H^s_0(0,1)} + \|\eta^{left}\|_{H^s_0(0,1)} + \|\eta^{center}\|_{H^s_0(0,1)} + \|\eta^{right}\|_{H^s_0(0,1)}.
$$

We start by observing that for all j', k' such that $\vartheta_{j',k'} = \vartheta^0_{j',k'}$ it holds $< \eta^{center}, \vartheta^0_{j',k'} >= \eta_{j',k'}$, while for all other j', k', $< \eta^{center}, \vartheta^0_{j',k'} >= 0$ and this yields

$$
\|\eta^{center}\|^2_{H^s_0(0,1)} \lesssim \sum_j 2^{2sj} \sum_{k=N}^{2^j-N} |\eta_{j,k}|^2.
$$

Let us bound the $H^s_0(0,1)$ norm of η^{left}. Considering that for all $k' \geq N$, $k \leq N$ it holds that $< \vartheta_{j,k}, \vartheta^0_{j',k'} >= 0$, we have

$$
\|\eta^{left}\|^2_{H^s_0(0,1)} \lesssim \sum_{k'=0}^{2^{j_0}} | < \eta^{left}, \rho_{k'} > |^2 + \sum_{j'=j_0}^{\infty} 2^{2sj} \sum_{k'=1}^{N-1} | < \eta^{left}, \vartheta^0_{j',k'} > |^2.
$$

It is not difficult to realize that the first sum on the right hand side can be bound as follows:

$$
\sum_{k'} | < \eta^{left}, \rho_{k'} > |^2 \lesssim \|\eta^{left}\|^2_{L^2(0,1)} \leq \|\eta\|_{H^s(0,1)}.
$$

In order to bound the second sum, let us now bound $| < \eta^{left}, \vartheta^0_{j',k'} > |$. Observing that for all j, k, j', k' it holds that $| < \vartheta_{j,k}, \vartheta^0_{j',k'} > | \lesssim 2^{-|j-j'|/2}$, we have

$$
| < \eta^{left}, \vartheta^0_{j',k'} > | \leq \sum_{j=j_0}^{\infty} \sum_{k=1}^{N-1} |\eta_{j,k}| \cdot | < \vartheta_{j,k}, \vartheta^0_{j',k'} > | \lesssim 2^{|j-j'|/2} \sum_{j=j_0}^{\infty} \sum_{k=1}^{N-1} |\eta_{j,k}|.
$$

This yields

$$
\begin{aligned}
\sum_{j'=j_0}^{\infty} 2^{2j's} \sum_{k'=1}^{N-1} | < \eta^{left}, \vartheta^0_{j',k'} > |^2 &\lesssim \sum_{j'=j_0}^{\infty} 2^{2j's} \sum_{k'=1}^{N-1} | \sum_{j=j_0}^{\infty} 2^{|j-j'|/2} \sum_{k=1}^{N-1} |\eta_{j,k}||^2 \\
&\lesssim (N-1) \sum_{j'=j_0}^{\infty} 2^{2j's} \left(\sum_{j=j_0}^{\infty} 2^{|j-j'|/2} \sum_{k=1}^{N-1} |\eta_{j,k}| \right)^2 \\
&\lesssim \sum_{j'=j_0}^{\infty} \left(\sum_{j=j_0}^{\infty} 2^{(1/2-s)|j-j'|} \sum_{k=1}^{N-1} 2^{js} |\eta_{j,k}| \right)^2 \\
&\lesssim \left(\sum_{j=j_0}^{\infty} 2^{(1/2-s)j} \right)^2 \left(\sum_{j=j_0}^{\infty} 2^{2js} \sum_{k=1}^{N-1} |\eta_{j,k}|^2 \right) \\
&\lesssim \frac{1}{(1/2-s)^2} \left(\sum_{j=j_0}^{\infty} 2^{2js} \sum_{k=1}^{N-1} |\eta_{j,k}|^2 \right)
\end{aligned}
$$

where we applied the well known inequality (consequence of the Schur lemma)

$$
\sum_{j'} | \sum_{j} a_{j'-j} b_j |^2 \lesssim \left(\sum_j |a_j| \right)^2 \left(\sum_j |b_j|^2 \right),
$$

as well as

$$
\left(\sum_{k=1}^{N} a_k \right) \leq C(N) \sum_{k=1}^{N} |a_k|^2.
$$

In summary we get

$$
\|\eta^{left}\|^2_{H^s_0(0,1)} \lesssim \frac{1}{(1/2-s)^2} \left(\sum_{j=0}^{\infty} 2^{2js} \sum_{k=1}^{N} |\eta_{j,k}|^2 \right).
$$

Analogously, we can prove

$$
\|\eta^{right}\|^2_{H^s_0(0,1)} \lesssim \frac{1}{(1/2-s)^2} \left(\sum_{j=0}^{\infty} 2^{2js} \sum_{k=2^j-N+1}^{2^j} |\eta_{j,k}|^2 \right).
$$

Adding the bounds for $\|\eta_0\|_{H^s_0(0,1)}$, $\|\eta^{left}\|_{H^s_0(0,1)}$, $\|\eta^{center}\|_{H^s_0(0,1)}$ and $\|\eta^{right}\|_{H^s_0(0,1)}$, since, for s in $[0,1/2[$ it holds that $1 \leq 1/(1/2-s)$, in view of the norm equivalence (66) we obtain the thesis. □

Corollary 5.1 *For all $\eta \in H^{1/2}(0,1)$ and for all $\alpha \in \mathbb{R}$ it holds, for all $s \in]0,1[$,*

$$
\|\eta\|_{H^s_0(0,1)} \leq \frac{1}{1/2-s} \|\eta - \alpha\|_{H^{1/2}(0,1)} + \frac{|\alpha|}{\sqrt{s} \cdot \sqrt{1/2-s}}.
$$

Proof: Trivially it holds that

$$\|\eta\|_{H_0^s(0,1)} \leq \|\eta - \alpha\|_{H_0^s(0,1)} + \|\alpha\|_{H_0^s(0,1)}.$$

In view of bound (68) the only thing that needs to be proven is a bound on $\|\alpha\|_{H_0^s(0,1)}$.

It holds

$$\|\alpha\|^2_{H_0^s(0,1)} = |\alpha|^2 + \int_{-\infty}^0 dx \int_0^1 dy \frac{|\alpha|^2}{|x-y|^{2s+1}} + \int_0^1 dx \int_{-\infty}^0 dy \frac{|\alpha|^2}{|x-y|^{2s+1}}.$$

Let us bound the first integral (the second is bound analogously).

$$\begin{aligned}
\int_{-\infty}^0 dx \int_0^1 dy (y-x)^{-(2s+1)} &= \int_0^1 dy \left[\frac{1}{2s}(y-x)^{-2s}\right]_{-\infty}^0 \\
&= -\int_0^1 \frac{1}{2s} y^{-2s}\, dy = \left[\frac{1}{2s}\frac{1}{1-2s} y^{1-2s}\right]_0^1 \\
&= \frac{1}{2s} \cdot \frac{1}{1-2s} \lesssim \frac{1}{1-2s}
\end{aligned}$$

$\square$