Analysis of a stabilized three-fields domain decomposition method

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March 31, 2000

Abstract

In this paper we prove that, for suitable choices of the bilinear form involved in the stabilization procedure, the stabilized three fields domain decomposition method proposed in [8] is stable and convergent *uniformly in the number of subdomains and with respect to their sizes* under quite general assumptions on the decomposition and on the discretization spaces. The same is proven to hold for the resulting discrete Steklov-Poincaré operator.

Key words: Three fields formulation, domain decomposition, stabilized methods, scalability.
AMS subject classification: 65N12, 65N55, 65N30.

1 Introduction

Domain decomposition methods are becoming more and more popular also because they generate algorithms for the numerical solution of PDEs that have an important part which is intrinsically parallel, and then, also thanks to the increasing performances of parallel computers, they are very promising when considering large scale problems likewise the ones appearing in industrial simulations.

The recent technological development in the framework of large parallel computers are giving more and more importance to the issue of "scalability": doubling the number of processors should allow to halve the time needed for getting the solution of a given problem with a given accuracy. For this, it is necessary that the quality of the solution depends only on the degrees of freedom globally used, and not on how these are split between processors (less *d.o.f.* per processor with more processors or more *d.o.f.* per processor with less processors). In a domain decomposition strategy this requires also that the stability and convergence of the method should not depend on the number and size of the subdomains. While this automatically holds for conforming domain decomposition (for which different splittings of the *d.o.f.* between subdomains are in a way just different devices to parallelize *the same* functional problem in different ways), this is not clear for non conforming domain decomposition, where different decompositions of the computational domain correspond to different discrete solutions, even if the degrees of freedom are fundamentally the same.

We consider here the three fields formulation, introduced in [14]. Unlike other non-conforming formulations [4, 16] of domain decomposition, weak continuity is not imposed by requiring that the jump across the interface is orthogonal to a suitable multiplier space, but by introducing the space Φ of traces of functions in $H^1(\Omega)$ on the interface Σ. The elements of Φ are to be used as a Dirichlet boundary condition — imposed by means of Lagrange multipliers — for the functions in the subdomains. This method, which allows in principle to couple non-matching discretizations in the subdomains, has several advantages. The "local problems" in the subdomains are completely uncoupled and therefore their solution can be performed completely in parallel, with no communication between processors.

At the same time the "interface variable" does not in principle need to match the local subdomain discretization and therefore situations are allowed, where non uniform and/or unstructured grids in the subdomains (yielding a good agreement with the behavior of the solution) are coupled with uniform structured grids on the interface (allowing the design of optimal preconditioners). But in order to take full advantage of this flexibility some kind of stabilization is needed in order to circumvent one or both of the inf-sup conditions required for the stability of the method. This can be done in several ways: see for instance [2, 12, 8]. In this paper we deal with the issue of giving "scalable" (i.e. independent of the size and number of subdomains) bounds for the stabilized three fields formulation introduced in [8]. Under suitable assumptions on the geometrical decomposition $\Omega = \cup\Omega_k$ (which is assumed to be graded), on the subdomains (which must be "regular in shape") and on the local subdomain discretizations (which are assumed to satisfy local inf-sup conditions) and under *no assumption* on the interface discretization (which can be totally unrelated to the discretizations used in the subdomains) stability and optimal convergence bounds hold with constants independent of the size and number of subdomains. Moreover the corresponding discrete Steklov-Poincaré operator is continuous and coercive, again uniformly with respect to the size and number of subdomains.

A brief outline of the paper: in section 2, after recalling the stabilized three fields formulation of [8], we will state the assumptions under which uniform stability and convergence bounds will hold; in section 3 we will analyze the stabilized problem while keeping track of the influence of the size and number of subdomains; in section 4 we analyze the discrete Steklov-Poincaré operator obtained by reducing the global problem to the interface via a Schur complement technique; in section 5 we will give a set of alternative (standard) assumptions under which assumption (A3) (local inf-sup condition in Ω_k) holds; finally, in section 6 we will present a P1/P1/P1 non conforming discretization that falls in the framework here considered for which we will briefly derive an error estimate.

2 Domain decomposition method: the stabilized three fields formulation

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. We will consider the following problem: given $f \in L^2(\Omega)$, find u satisfying

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial \boldsymbol{x}_j}\left(a_{i,j}(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{x}_i}\right) + a_0(\boldsymbol{x})u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1}$$

We assume that for almost all $\boldsymbol{x} \in \Omega$ we have $0 \le a_0(\boldsymbol{x}) \le R$ and that the matrix $(a_{ij}(\boldsymbol{x}))_{i,j=1,2}$ is symmetric positive definite, with smallest eigenvalue $\ge \alpha > 0$ and largest eigenvalue $\le \alpha'$, α, α' independent of $\boldsymbol{x}$.

We will consider here a domain decomposition method for the numerical solution of such a problem, by a three fields approach ([14]). More precisely, let $\overline{\Omega} = \cup_{k=1}^{K}\overline{\Omega}_k$, with $diam(\Omega_k) \sim H_k$. We let $\Gamma_k = \partial\Omega_k$, $\Sigma = \cup_k \Gamma_k$ being the skeleton of the decomposition (which differently from [14, 8] we define here so that it includes the external boundary of the domain Ω). For simplicity we will make the assumption that the Ω_k's are quadrilaterals, but it is not difficult to see that the results that will be obtained are easily extended to more general situations (with constants possibly depending on the shape of the subdomains).

In the following for $r \ge 0$ and for $\hat{\Omega}$ being either Ω, Ω_k or Γ_k we will denote by $\|\cdot\|_{r,\hat{\Omega}}$ (resp. $|\cdot|_{r,\hat{\Omega}}$) the $H^r(\hat{\Omega})$ norm (resp. seminorm). The functional setting for the three fields domain decomposition method is given by the following spaces:

$$V = \prod_{k=1}^{K} H^1(\Omega_k), \qquad \Lambda = \prod_{k=1}^{K} H^{-1/2}(\Gamma_k), \tag{2}$$

and

$$\Phi = \{\varphi \in L^2(\Sigma) : \text{there exists } u \in H_0^1(\Omega),\ u = \varphi \text{ on } \Sigma\} \tag{3}$$

respectively equipped with the norms:

$$\|u\|_V = \left(\sum_k \|u^k\|^2_{1,\Omega_k}\right)^{1/2}, \qquad \|\lambda\|_\Lambda = \left(\sum_k \|\lambda^k\|^2_{\Lambda^k}\right)^{1/2}, \tag{4}$$

with

$$\|\lambda^k\|_{\Lambda^k} = \sup_{u^k \in V^k} \frac{\int_{\Gamma_k} \lambda^k u^k \, ds}{\|u^k\|_{1,\Omega_k}},$$

while

$$\|\varphi\| = \inf_{\substack{u \in H_0^1(\Omega) \\ u = \varphi \text{ on } \Sigma}} \|u\|_{1,\Omega}. \tag{5}$$

In order to define the stabilized three fields formulation we start by considering the bilinear forms $a^k : H^1(\Omega_k) \times H^1(\Omega_k) \to \mathbb{R}$ defined by

$$a^k(w,v) = \int_{\Omega_k} \left(\sum_{i,j=1}^{2} a_{ij}(\boldsymbol{x}) \frac{\partial w}{\partial \boldsymbol{x}_i} \frac{\partial v}{\partial \boldsymbol{x}_j} + a_0 wv \right) d\boldsymbol{x}.$$

Under the assumptions made on a_{ij} and on a_0, the bilinear forms a^k are uniformly $H^1(\Omega_k)$ continuous, and semi-definite. More precisely, there exists positive constants α and M independent of k and H_k such that for all w, v in $H^1(\Omega_k)$

$$\alpha |w|^2_{1,\Omega_k} \leq a^k(w,w), \qquad |a^k(w,v)| \leq M \|w\|_{1,\Omega_k} \|v\|_{1,\Omega_k}.$$

In order to define the stabilization term, we also assume to have symmetric positive semidefinite bilinear forms $[\cdot,\cdot]_{\frac{1}{2},k} : H^{1/2}(\Gamma_k) \times H^{1/2}(\Gamma_k) \to \mathbb{R}$ satisfying for all $\eta, \xi \in H^{1/2}(\Gamma_k)$

$$[\eta,\eta]_{\frac{1}{2},k} \geq b|\eta|^2_{1/2,\Gamma_k}, \qquad [\eta,\xi]_{\frac{1}{2},k} \leq B|\eta|_{1/2,\Gamma_k} |\xi|_{1/2,\Gamma_k}, \tag{6}$$

for b, B positive constants independent of k and H_k. It is beyond the goals of this paper to consider how such bilinear forms can be numerically realized in practice. See [8, 6, 7, 10] for possible realization in terms of multilevel decomposition and/or wavelets.

For a fixed parameter $\gamma > 0$, whose choice will be made later, we can now consider the following class of discrete problems: given closed finite dimensional subspaces $V_h = \prod_{k=1}^K V_h^k \subset V$, $\Lambda_h = \prod_{k=1}^K \Lambda_h^k \subset \Lambda$ and $\Phi_h \subset \Phi$ find $u_h \in V_h$, $\lambda_h \in \Lambda_h$, $\varphi_h \in \Phi_h$ such that it holds

$$\begin{cases} \forall k, \quad \forall v_h^k \in V_h^k, \quad \forall \lambda_h^k \in \Lambda_h^k : \\ a^k(u_h^k, v_h^k) + \gamma[u_h^k, v_h^k]_{\frac{1}{2},k} - \int_{\Gamma_k} v_h^k \lambda_h^k \, ds \quad - \gamma[\varphi_h, v_h^k]_{\frac{1}{2},k} = \int_{\Omega_k} f v_h^k \, d\boldsymbol{x}, \\ \int_{\Gamma_k} u_h^k \mu_h^k \, ds \qquad\qquad - \int_{\Gamma_k} \mu_h^k \varphi_h \, ds = 0, \\ \text{and } \forall \psi_h \in \Phi_h : \\ -\sum_k \gamma[u_h^k, \psi_h]_{\frac{1}{2},k} + \sum_k \int_{\Gamma_k} \lambda_h^k \psi_h \, ds + \sum_k \gamma[\varphi_h, \psi_h]_{\frac{1}{2},k} = 0. \end{cases} \tag{7}$$

We recall (see [14]) that for u solution of (1), $u^k = u|_{\Omega_k}$, $\lambda^k = \frac{\partial u}{\partial \nu_k^a}$ (ν_k^a denoting the outer conormal derivative to the subdomain Ω_k) and $\varphi = u|_\Sigma$ satisfy (7), which is therefore a consistent discretization of (1).

Such a stabilized version of the *three fields* formulation was introduced and analyzed (in a slightly different form) in [8] where, in the case of a fixed decomposition, existence and uniqueness of the discrete solution as well as an estimate on the error $\|u - u_h\|_V$ were proven under suitable assumptions on the spaces V_h and Λ_h. The constants appearing in the stability and convergence bounds were there implicitly depending on the decomposition chosen (and therefore on the shape, but also on the size and the number of subdomains). We aim here at giving uniform (in terms of size and number of subdomains) estimates. In order to do so we make some regularity assumption on the decomposition:

(A1) the subdomains are regular in shape and the decomposition is graded, that is

- (a) there exists a positive constant C_0 such that, for all k, Ω_k contains a ball of diameter $C_0 H_k$, it is contained in a ball of diameter H_k, and the length of each side is bounded from below by $C_0 H_k$; moreover any interior angle ω satisfies $0 < C_1 < \omega < C_2 < \pi$ (C_0, C_1, and C_2 independent of k);
- (b) there exists a positive constant C_3 such that, if ℓ, k are such that $|\Gamma_k \cap \Gamma_\ell| > 0$, then it holds

$$H_k / H_\ell \leq C_3;$$

(A2) there exists a constant c_0 such that for all constants c, denoting by l_c the line of equation $\boldsymbol{x}_2 = c$ and setting $K_c = \{k : \ \text{length}(l_c \cap \Omega_k) > 0\}$, it holds

$$\sum_{k \in K_c} H_k \leq c_0.$$

Moreover we assume that the spaces V_h^k and Λ_h^k are chosen in such a way that local inf-sup conditions hold with a lower bound independent of k and of H_k. More precisely

(A3) the two following conditions hold

- (a) Λ_h^k contains the global constants;
- (b) there exists a positive constant β such that the following inf-sup condition holds for all k

$$\inf_{\lambda_h^k \in \Lambda_h^k} \sup_{v_h^k \in V_h^k} \frac{\int_{\Gamma_k} \lambda_h^k v_h^k \, ds}{\|\lambda_h^k\|_{\Lambda^k} \|v_h^k\|_{1,\Omega_k}} \geq \beta > 0.$$

Remark 2.1 *Assumption (A1b) does not imply that the decomposition is uniform (it does not hold $H_k / H_\ell \leq C_3$ for all k, ℓ).*

Remark 2.2 *Assumption (A2) is not particularly restrictive. In many cases it can be shown that it is actually a consequence of (A1). This holds for instance for the so called "geometrically conforming" decompositions (see the Appendix), or for decompositions where all subdomain edges are parallel to the axes. It is our opinion that this is true in general, though we were not able to prove it.*

Remark 2.3 *Assumption (A3) is weaker than the assumptions on Λ_h^k and on V_h^k under which the analysis of Problem (7) was carried out in [8].*

We make *no assumption* on the subspace Φ_h.

Under such assumptions it will be possible (see the next section) to prove the main result of our paper.

Theorem 2.1 *Under assumptions (A1–3), there exists $\gamma_0 > 0$ (γ_0 independent of K and of the H_k's) such that for all γ with $0 < \gamma < \gamma_0$, Problem (7) admits a unique solution $u_h, \lambda_h, \varphi_h$ satisfying*

$$\|u_h\|_V + \|\lambda_h\|_\Lambda + \|\varphi_h\|_\Phi \leq c\|f\|_{V'},$$

and the following error estimate holds:

$$\|u - u_h\|_V \ + \ \left\| \frac{\partial u}{\partial \nu_k^a} - \lambda_h \right\|_\Lambda \ + \ \|u - \varphi_h\|_\Phi \leq \tilde{c}\big(\inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{\mu_h \in \Lambda_h} \left\| \frac{\partial u}{\partial \nu_k^a} - \mu_h \right\|_\Lambda + \inf_{\psi_h \in \Phi} \|u - \psi_h\|_\Phi \big)$$

with $c = c(\gamma)$ and $\tilde{c} = \tilde{c}(\gamma)$ independent of K and of the H_k's.

3 Stability and convergence analysis

In the following we will keep track of the influence of the size of the subdomains and of their number on the stability and convergence estimates: to this end, we shall use the notation $A \lesssim B$ to indicate $A \leq cB$ for a suitable constant c independent of the number of subdomains and their diameter H_k, as well as of the mesh size of the discretizations in the subdomains. By $A \simeq B$ we will mean $A \lesssim B \lesssim A$.

We start by recalling that the classical trace theorems as well as other classical bounds on the solution of elliptic PDEs hold in Ω_k with constants depending on H_k. In order to get bounds independent of H_k we need to rescale Ω_k to a reference domain of unitary diameter. Applying some classical bounds [17], it is then easy to check that we have the following inequalities.

Proposition 3.1 *The following bounds hold:*

- *for all $u \in H^1(\Omega_k)$ it holds*

$$H_k^{-1/2}\|u\|_{0,\Gamma_k} + |u|_{1/2,\Gamma_k} \lesssim H_k^{-1}\|u\|_{0,\Omega_k} + |u|_{1,\Omega_k} \tag{8}$$

and

$$|u|_{1/2,\Gamma_k} \lesssim |u|_{1,\Omega_k}; \tag{9}$$

- *for all $u \in H^2(\Omega_k)$ it holds*

$$\left| \frac{\partial u}{\partial \nu_k^a} \right|_{1/2,\Gamma_k} \lesssim |u|_{2,\Omega_k}, \qquad |u|_{3/2,\Gamma_k} \lesssim |u|_{2,\Omega_k}; \tag{10}$$

- *for all $\varphi \in H^{1/2}(\Gamma_k)$ let u^φ be the function such that*
$$-\Delta u^\varphi = 0 \quad in\ \Omega_k, \qquad u^\varphi = \varphi \quad on\ \Gamma_k; \tag{11}$$
then it holds
$$H_k^{-1}\|u^\varphi\|_{0,\Omega_k} + |u^\varphi|_{1,\Omega_k} \lesssim H_k^{-1/2}\|\varphi\|_{0,\Gamma_k} + |\varphi|_{1/2,\Gamma_k} \tag{12}$$
and
$$|u^\varphi|_{1,\Omega_k} \lesssim |\varphi|_{1/2,\Gamma_k}; \tag{13}$$
- *for any $\varphi \in H^{1/2}(\Gamma_k)$ with $\int_{\Gamma_k} \varphi\, ds = 0$ it holds*
$$H_k^{-1/2}\|\varphi\|_{0,\Gamma_k} \lesssim |\varphi|_{1/2,\Gamma_k}. \tag{14}$$

We can then give a characterization of the norm $\|\cdot\|_\Phi$ in terms of Sobolev seminorms of type $H^{1/2}$.

Proposition 3.2 *The following norm equivalence holds for all $\varphi \in \Phi$:*
$$\|\varphi\|_\Phi \simeq \left(\sum_k |\varphi|^2_{1/2,\Gamma_k}\right)^{1/2}.$$

Proof: Let $\varphi \in \Phi$. For any $u \in H^1_0(\Omega)$ such that $u|_\Sigma = \varphi$ it holds, by (9),
$$|\varphi|_{1/2,\Gamma_k} \lesssim |u|_{1,\Omega_k}.$$
This implies
$$\sum_k |\varphi|^2_{1/2,\Gamma_k} \lesssim \sum_k |u|^2_{1,\Omega_k} = |u|^2_{1,\Omega} \lesssim \|u\|^2_{1,\Omega}.$$
By the arbitrariness of u we get
$$\left(\sum_k |\varphi|^2_{1/2,\Gamma_k}\right)^{1/2} \lesssim \inf_{\substack{u\in H^1_0(\Omega)\\ u=\varphi \text{ on } \Sigma}} \|u\|_{1,\Omega} = \|\varphi\|_\Phi.$$

Let now $u^\varphi \in L^2(\Omega)$ be the function satisfying (11) in each Ω_k with data $\varphi|_{\Gamma_k}$. Since $\varphi \in H^1_0(\Omega)|_\Sigma$, it is easy to check that $u^\varphi \in H^1_0(\Omega)$ and then, using Poincaré inequality in $H^1_0(\Omega)$, we have
$$\begin{aligned}\|\varphi\|_\Phi &= \inf_{\substack{u\in H^1_0(\Omega)\\ u=\varphi \text{ on } \Sigma}} \|u\|_{1,\Omega} \le \|u^\varphi\|_{1,\Omega} \lesssim |u^\varphi|_{1,\Omega} \\ &= \left(\sum_k |u^\varphi|^2_{1,\Omega_k}\right)^{1/2} \lesssim \left(\sum_k |\varphi|^2_{1/2,\Gamma_k}\right)^{1/2}\end{aligned}$$
where the last inequality derives from (13). □

In order to prove Theorem 2.1 we first briefly observe that, in view of (9) and of Proposition 3.2, all the bilinear terms appearing on the left hand side of (7) are continuous with respect to the norm $\|\cdot\|_V + \|\cdot\|_\Lambda + \|\cdot\|_\Phi$. Let us prove it for the only term for which this might not be immediate, that is $\sum_k \int_{\Gamma_k} \varphi\lambda^k\,ds$; for all $u \in H_0^1(\Omega)$ such that $\varphi = u|_\Sigma$ we have $\int_{\Gamma_k} \varphi\lambda^k\,ds \le \|u\|_{1,\Omega_k}\|\lambda^k\|_{\Lambda^k}$ (by definition of $\|\cdot\|_{\Lambda^k}$), whence

$$\sum_k \int_{\Gamma_k} \varphi\lambda^k\,ds \lesssim \sum_k \|u\|_{1,\Omega_k}\|\lambda^k\|_{\Lambda^k} \lesssim \left(\sum_k \|u\|_{1,\Omega_k}^2\right)^{1/2} \left(\sum_k \|\lambda^k\|_{\Lambda^k}^2\right)^{1/2}.$$

From the arbitrariness of u we obtain

$$\sum_k \int_{\Gamma_k} \varphi_h\lambda^k \le \|\lambda\|_\Lambda \inf_{\substack{u\in H_0^1(\Omega)\\ u=\varphi \text{ on } \Sigma}} \|u\|_{1,\Omega} = \|\lambda\|_\Lambda\|\varphi\|_\Phi.$$

Let us now analyze the stability of the discrete problem. To this end we reformulate it as a saddle point problem as follows: letting $\mathcal{Z}_h = V_h \times \Phi_h$, find $Z_h \in \mathcal{Z}_h$ and $\lambda_h \in \Lambda_h$ such that

$$\begin{cases} \forall X_h \in \mathcal{Z}_h,\ \forall \mu_h \in \Lambda_h : \\ \mathfrak{A}_\gamma(Z_h, X_h) - \mathfrak{B}(\lambda_h, X_h) \quad = \quad < F, X_h >, \\ \mathfrak{B}(\mu_h, Z_h) \quad = \quad 0, \end{cases} \tag{15}$$

where for $Z_h = (u_h, \varphi_h)$ and $X_h = (v_h, \psi_h)$ we define

$$\begin{aligned} \mathfrak{A}_\gamma(Z_h, X_h) &= \sum_k a^k(u_h^k, v_h^k) + \gamma[u_h^k - \varphi_h, v_h^k - \psi_h]_{\frac12,k}, \\ \mathfrak{B}(\mu_h, Z_h) &= \sum_k \int_{\Gamma_k} (u_h^k - \varphi_h)\mu_h^k\,ds, \\ < F, X_h > &= \int_\Omega f v_h\,d\boldsymbol{x}. \end{aligned}$$

It is well known (see [11]), that the uniform stability of such a problem relies on the validity of an inf-sup condition on $\mathfrak{B}$ and on the coercivity of $\mathfrak{A}_\gamma$ on the kernel of $\mathfrak{B}$. As far as the inf-sup condition on $\mathfrak{B}$ is concerned, under assumption (A3) (local inf-sup condition in the subdomains) it holds, setting $\|(u,\varphi)\|_\mathcal{Z} = \|u\|_V + \|\varphi\|_\Phi$,

$$\inf_{\lambda_h\in\Lambda_h} \sup_{X_h\in\mathcal{Z}_h} \frac{\mathfrak{B}(\lambda_h, X_h)}{\|X_h\|_\mathcal{Z}\|\lambda_h\|_\Lambda} \ge \inf_{\lambda_h\in\Lambda_h} \sup_{v_h\in V_h} \frac{\mathfrak{B}(\lambda_h, (v_h, 0))}{\|\lambda_h\|_\Lambda\|v_h\|_V} \gtrsim 1, \tag{16}$$

where the last inequality descends from the following Proposition (see, for instance [8]).

Proposition 3.3 *If the spaces V_h^k and Λ_h^k satisfy assumption (A3b) then it holds*

$$\inf_{\lambda_h\in\Lambda_h} \sup_{v_h\in V_h} \frac{\sum_k \int_{\Gamma_k} \lambda_h^k v_h^k\,ds}{\|\lambda_h\|_\Lambda\|v_h\|_V} \ge \beta.$$

In order to prove coercivity of $\mathfrak{A}_\gamma$ on $\ker \mathfrak{B}$ we start by proving a discrete Poincaré type inequality.

Lemma 3.1 *Under assumption (A1–2) and (A3a), if $Z = (u, \varphi) \in \mathcal{Z} = V \times \Phi$ satisfies*

$$\mathfrak{B}(\lambda_h, Z) = 0, \qquad \forall \lambda_h \in \Lambda_h \tag{17}$$

then it holds

$$\|u\|_{0,\Omega}^2 \lesssim \sum_k \left(|u^k|_{1,\Omega_k}^2 + |\varphi|_{1/2,\Gamma_k}^2 \right). \tag{18}$$

Proof: We follow the approache used in [1, 3] for proving a similar inequality in the case of the mortar method. Let a, a', b, b' be such that $\Omega \subset \Omega^* =]a, a'[\times]b, b'[$ with $|a' - a| \simeq |b' - b| \simeq 1$. Let $Z = (u, \varphi) \in \mathcal{Z}$ satisfy (17) and assume that for all k, $u^k \in C^\infty(\bar{\Omega}_k)$. By abuse of notation we will also denote by u the $L^2(\Omega^*)$ function obtained by extending u to $\Omega_0 = \Omega^* \setminus \Omega$ with value $u^0 = 0$. Remark that, since $\varphi \in H_0^1(\Omega)|_\Sigma$ it holds $u^0 - \varphi \equiv 0$ on $\Gamma_0 = \partial\Omega$.

For almost all $y \in (b, b')$ the line of fixed y coordinate crosses the skeleton Σ in a finite number $N(y)$ of points $(a_1(y), y)$, $(a_2(y), y)$, ... , $(a_{N(y)}(y), y)$ with $a_1(y) < a_2(y) < \ldots < a_{N(y)}(y)$. It is not difficult to see that for all x in the interval $]a_i(y), a_{i+1}(y)[$ it holds

$$u(x, y) = \sum_{n=1}^{i} \int_{a_{n-1}(y)}^{a_n(y)} \partial_x u(t, y)\, dt + \sum_{n=1}^{i} [u](a_n(y), y) + \int_{a_i(y)}^{x} \partial_x u(t, y)\, dt$$

where $[\cdot]$ denotes the jump across the interface and where we use the convention $a_0(y) = a$, $a_{N(y)+1}(y) = a'$. This implies for almost all $(x, y) \in \Omega^*$

$$|u(x, y)| \le \sum_{n=1}^{N(y)+1} \left| \int_{a_{n-1}(y)}^{a_n(y)} \partial_x u(t, y)\, dt \right| + \left| \int_{a_{i_x}(y)}^{x} \partial_x u(t, y)\, dt \right| + \sum_{n=1}^{N(y)} |[u](a_n(y), y)|,$$

with $i_x = i_x(y) = \max\{i : a_i(y) < x\}$. Then it holds

$$\begin{aligned} \|u\|_{0,\Omega}^2 \lesssim & \int_{\Omega^*} \left(\sum_{n=1}^{N(y)+1} \left| \int_{a_{n-1}(y)}^{a_n(y)} \partial_x u(t, y)\, dt \right| \right)^2 dx\, dy \\ & + \int_{\Omega^*} \left| \int_{a_{i_x}(y)}^{x} \partial_x u(t, y)\, dt \right|^2 dx\, dy + \int_{\Omega^*} \left(\sum_{n=1}^{N(y)} |[u](a_n(y), y)| \right)^2 dx\, dy. \end{aligned} \tag{19}$$

Following [1, 3] it is possible to show that

$$\int_{\Omega^*} \left(\sum_{n=1}^{N(y)+1} \left| \int_{a_{n-1}(y)}^{a_n(y)} \partial_x u(t, y)\, dt \right| \right)^2 dx\, dy \lesssim \sum_k |u^k|_{1,\Omega_k}^2, \tag{20}$$

and then we only need to bound the $L^2(\Omega^*)$ norm of

$$G_1(x,y) = \left| \int_{a_{i_x}(y)}^{x} \partial_x u(t,y)\, dt \right| \qquad \text{and of} \qquad G_2(x,y) = g_2(y) = \sum_{n=1}^{N(y)} |[u](a_n(y),y)| \,.$$

As far as $G_1(x,y)$ is concerned, using a Schwartz-Hölder inequality, we have:

$$\begin{aligned} \|G_1\|_{0,\Omega^*}^2 &\lesssim \int_a^{a'} \int_b^{b'} (x - a_{i_x}(y)) \int_{a_{i_x}(y)}^{x} |\partial_x u(t,y)|^2 \, dt\, dx\, dy \\ &\lesssim \int_a^{a'} \int_b^{b'} (a_{i_x+1}(y) - a_{i_x}(y)) \int_{a_{i_x}(y)}^{a_{i_x+1}(y)} |\partial_x u(t,y)|^2 \, dt\, dx\, dy. \end{aligned}$$

For fixed y the integrand function is a piecewise constant, and therefore it is not difficult to check that setting $\Delta_i = a_{i+1}(y) - a_i(y)$

$$\begin{aligned} \int_a^{a'} \Delta_{i_x} \int_{a_{i_x}(y)}^{a_{i_x+1}(y)} |\partial_x u(t,y)|^2 \, dt\, dx &= \sum_{i=1}^{N(y)} \Delta_i^2 \int_{a_i(y)}^{a_{i+1}(y)} |\partial_x u(t,y)|^2 \\ &\lesssim \left(\sup_{i=1,\dots,N(y)} \Delta_i^2 \right) \int_a^{a'} |D(t,y)|^2 \, dt, \end{aligned}$$

where we denote by D the function defined by $D|_{\Omega_k} = \partial_x u|_{\Omega_k}$ for all k. Since $u|_{\Omega_k} \in C^\infty$, $D(\cdot,y)$ is in $L^2(]a,a'[)$ and $D(x,y)$ is in $L^2(\Omega^*)$. Since $\Delta_i \le (a'-a)$ we get

$$\|G_1\|_{0,\Omega^*}^2 \lesssim \|D\|_{0,\Omega^*}^2 = \sum_k \|\partial_x u\|_{0,\Omega_k}^2 \lesssim \sum_k |u^k|_{1,\Omega_k}^2. \tag{21}$$

Let us now consider the contribution of the jumps on the interface. For each $y \in]b,b'[$ and $n \in 1,\dots,N(y)$ the point $(a_n(y),y)$ belongs to $\Gamma_k \cap \Gamma_\ell$ for some k,ℓ depending on y and n. For such k,ℓ we set

$$\delta_n(y) = \frac{H_k + H_\ell}{2},$$

and we observe that, thanks to assumptions (A1–2), it holds

$$\sum_{n=1}^{N(y)} \delta_n(y) \lesssim 1.$$

By using a Cauchy-Schwartz inequality we can then bound

$$g_2(y) \lesssim \left(\sum_{n=1}^{N(y)} (\delta_n(y))^{-1} \, |[u](a_n(y),y)|^2 \right)^{1/2}.$$

Then

$$\begin{aligned}\|G_2\|^2_{0,\Omega^*} &= (a'-a)\int_b^{b'} |B_2(y)|^2\,dy \\ &\lesssim \int_b^{b'} dy \sum_{n=1}^{N(y)} (\delta_n(y))^{-1}\,|[u](a_n(y),y)]|^2 .\end{aligned}$$

We can now apply the following lemma which we will prove later on.

Lemma 3.2 *Let $\zeta \in L^1(\Sigma)$, $\zeta \geq 0$, then it holds*

$$\int_b^{b'} \sum_{n=0}^{N(y)} \zeta(a_n(y),y)\,dy \leq \int_\Sigma \zeta\,ds \tag{22}$$

Applying (22) for $\zeta(s) = (\delta(s))^{-1}|[u](s)|^2$, where (where for $s \in \Sigma$, $s \in \Gamma_k \cap \Gamma_\ell$, $\delta(s) = (H_k + H_\ell)/2 = \delta_{k,\ell}$) we obtain

$$\|G_2\|^2_{0,\Omega} \lesssim \int_\Sigma (\delta(s))^{-1}|[u](s)|^2\,ds \lesssim \sum_{k,\ell}\int_{\Gamma_k\cap\Gamma_\ell} \delta_{k,\ell}^{-1}\,|[u](s)|^2\,ds.$$

By recalling that, by assumption (A1b), $\delta_{k,\ell} \simeq H_k \simeq H_\ell$, adding and subtracting $\varphi(s)$ we get

$$\begin{aligned}\|G_2\|^2_{0,\Omega} &\lesssim \sum_{k,\ell}\int_{\Gamma_k\cap\Gamma_\ell} \Big((H_k)^{-1}|(u^k-\varphi)(s)|^2 \\ &\qquad\qquad + (H_\ell)^{-1}|(u^\ell-\varphi)(s)|^2\Big)\,ds \\ &\lesssim \sum_k (H_k)^{-1}\int_{\Gamma_k} |(u^k-\varphi)(s)|^2\,ds. \end{aligned}\tag{23}$$

Now we observe that, since constant functions belong (by assumption (A3a)) to Λ_h^k, condition (17) implies that $\int_{\Gamma_k}(u^k-\varphi)\,ds = 0$, which implies for all constants c,

$$\int_{\Gamma_k} |u^k-\varphi-c|^2\,ds = \int_{\Gamma_k}|u^k-\varphi|^2\,ds + \int_{\Gamma_k} c^2\,ds \geq \int_{\Gamma_k}|u^k-\varphi|^2\,ds.$$

Setting $c = \breve{u}^k + \breve{\varphi}^k$, $\breve{u}^k$ and $\breve{\varphi}^k$ being respectively the mean values of u^k and φ on Γ_k, we have

$$H_k^{-1}\int_{\Gamma_k}|u^k-\varphi|^2\,ds \lesssim H_k^{-1}\int_{\Gamma_k}|u^k-\breve{u}^k|^2\,ds + H_k^{-1}\int_{\Gamma_k}|\varphi-\breve{\varphi}^k|^2\,ds. \tag{24}$$

Now, using (14) and (9), we can bound

$$H_k^{-1}\textstyle\int_{\Gamma_k}|u^k-\breve{u}^k|^2\,ds \lesssim |u^k|^2_{1,\Omega_k}, \tag{25}$$

$$H_k^{-1}\textstyle\int_{\Gamma_k}|\varphi-\breve{\varphi}^k|^2\,ds \lesssim |\varphi|^2_{1/2,\Gamma_k}. \tag{26}$$

By substituting (25) and (26) in (24), thanks to (23), (19), (20) and (21) we get the thesis for all $(u, \varphi) \in \mathcal{Z}$ satisfying (17) with $u^k \in C^\infty(\bar{\Omega}_k)$. By density of $C^\infty(\bar{\Omega}_k)$ into $H^1(\Omega_k)$ we get the thesis. $\square$

Let us now come to the proof of (22).

Proof of Lemma 3.2: We first remark that, since by assumption $\zeta \geq 0$, it holds

$$\int_\Sigma \zeta(s)\,ds = \sum_{k,\ell} \int_{\Gamma_k \cap \Gamma_\ell} \zeta(s)\,ds \geq \sum_{\substack{k,\ell : \Gamma_k \cap \Gamma_\ell \text{ is not} \\ \text{parallel to the } x \text{ axis}}} \int_{\Gamma_k \cap \Gamma_\ell} \zeta(s)\,ds.$$

Now, if $\Gamma_k \cap \Gamma_\ell$ is not parallel to the x axis, it can be parameterized by the coordinate y: $\Gamma_k \cap \Gamma_\ell = \{(a_{k,\ell}(y), y), \quad y \in (b(k,\ell), b'(k,\ell))\}$. It is not difficult to see that

$$\int_{\Gamma_k \cap \Gamma_\ell} \zeta(s)\,ds = \int_{b(k,\ell)}^{b'(k,\ell)} \zeta((a_{k,\ell}(y), y))\,ds(y) \gtrsim \int_{b(k,\ell)}^{b'(k,\ell)} \zeta((a_{k,\ell}(y), y))\,dy.$$

Then

$$\int_\Sigma \zeta(s)\,ds \gtrsim \sum_{\substack{k,\ell:\ \Gamma_k \cap \Gamma_\ell \text{ is not} \\ \text{parallel to the } x \text{ axis}}} \int_{b(k,\ell)}^{b'(k,\ell)} \zeta((a_{k,\ell}(y), y))\,dy = \int_b^{b'} \sum_{n=0}^{N(y)} \zeta(a_n(y), y)\,dy$$

where the last equality descends by observing that for almost all $y \in (b, b')$ the two sets $\{a_n(y), n = 1, N(y)\}$ and $\{a_{k,\ell},\quad k, \ell \ s.t.\ \ b(k,\ell) < y < b'(k,\ell)\}$ coincide. $\square$

Remark 3.1 *By using the approach of [13], under slightly stronger assumptions on the multiplier space Λ_h than the extremely mild (A3a), a stronger discrete Poincaré type inequality of the form*

$$\|u\|^2_{0,\Omega_k} \lesssim \sum_k |u|^2_{1,\Omega_k}$$

can be proven to hold for all $(u, \varphi) \in \ker(\mathfrak{B})$. However some interesting case may not verify such assumptions, and therefore we prefer the weaker inequality (18), since this is sufficient for the analysis that we will carry out in the following.

We are now able to prove the coerciveness of the bilinear form $\mathfrak{A}_\gamma$ on the kernel of $\mathfrak{B}$.

Corollary 3.1 *There exists γ_0 such that for all γ, $0 < \gamma < \gamma_0$ it holds for a positive constant $\tilde{\alpha}(\gamma)$*

$$\mathfrak{A}_\gamma(Z_h, Z_h) \geq \tilde{\alpha}(\gamma) \|Z_h\|_{\mathcal{Z}}.$$

for all $Z_h \in \mathcal{Z}_h$ verifying $\mathfrak{B}(\mu_h, Z_h) = 0$ for all $\mu_h \in \Lambda_h$.

Proof: Let $Z_h = (u_h, \varphi_h) \in Z_h$ such that for all $\mu_h \in \Lambda_h$, $\mathfrak{B}(\mu_h, Z_h) = 0$. Then

$$\begin{aligned}
\mathfrak{A}_\gamma(Z_h, Z_h) &= \sum_k a^k(u_h^k, u_h^k) + \gamma \sum_k [u_h^k - \varphi_h, u_h^k - \varphi_h]_{\frac{1}{2},k} \\
&\geq \alpha \sum_k |u_h^k|^2_{1,\Omega_k} + \gamma \sum_k [u_h^k, u_h^k]_{\frac{1}{2},k} + \gamma \sum_k [\varphi_h, \varphi_h]_{\frac{1}{2},k} \\
&\quad -2 \left(\gamma \sum_k [u_h^k, u_h^k]_{\frac{1}{2},k}\right)^{1/2} \left(\gamma \sum_k [\varphi_h, \varphi_h]_{\frac{1}{2},k}\right)^{1/2} \\
&\geq \alpha \sum_k |u_h^k|^2_{1,\Omega_k} + \gamma(1-2) \sum_k [u_h^k, u_h^k]_{\frac{1}{2},k} \\
&\quad +\gamma b(1 - \frac{1}{2}) \sum_k [\varphi_h, \varphi_h]_{\frac{1}{2},k} \\
&\geq (\alpha - c_1^2 \gamma B) \sum_k |u_h^k|^2_{1,\Omega_k} + \gamma b \frac{1}{2} \sum_k |\varphi_h|^2_{1/2,\Gamma_k}
\end{aligned}$$

where b, B are the constants appearing in the bound (6) and c_1 is the constant stemming from the trace inequality $|u^k|_{1/2,\Gamma_k} \leq c_1 |u|_{1,\Omega_k}$ (see Proposition 3.1). Thus we have

$$\mathfrak{A}_\gamma(Z_h, Z_h) \geq \vartheta(\gamma) \left(\sum_k |u_h^k|^2_{1,\Omega_k} + \sum_k |\varphi_h|^2_{1/2,\Gamma_k} \right) \gtrsim \vartheta(\gamma) \left(\|u_h\|_V^2 + \|\varphi_h\|_\Phi^2 \right),$$

with

$$\vartheta(\gamma) = \min \left\{ (\alpha - c_1^2 B \gamma), \frac{b\gamma}{2} \right\}.$$

If $0 < \gamma < \alpha/(Bc_1^2)$, then $\vartheta(\gamma) > 0$ and then we have the thesis. □

Thanks to Proposition 3.3 and to Corollary 3.1 it is not difficult to prove Theorem 2.1 by a standard argument ([11]).

Remark 3.2 *Though for simplicity we assumed that (6) holds for all η, ξ in $H^{1/2}(\Gamma_k)$, it is not difficult to see that in order for Theorem 2.1 to hold we only need that the bound*

$$[\eta, \eta]_{1/2,\Gamma_k} \gtrsim |\eta|_{1/2,\Gamma_k}$$

holds just for the elements of $\Phi_h|_{\Gamma_k}$, rather than for all the elements of Φ.

4 Analysis of the interface problem

The linear system stemming from Problem (7) takes the form

$$\begin{pmatrix} \mathbb{A} + \gamma\mathbb{E} & \mathbb{B}^T & -\gamma\mathbb{F}^T \\ \mathbb{B} & 0 & \mathbb{C}^T \\ -\gamma\mathbb{F} & \mathbb{C} & \gamma\mathbb{D} \end{pmatrix} \begin{pmatrix} \underline{u}_h \\ \underline{\lambda}_h \\ \underline{\varphi}_h \end{pmatrix} = \begin{pmatrix} \underline{f}_h \\ 0 \\ 0 \end{pmatrix},$$

where $\underline{u}_h$, $\underline{\lambda}_h$, $\underline{\varphi}_h$ denote the unknown coefficient vectors of u_h, λ_h and φ_h in the bases chosen for V_h, Λ_h and Φ_h, while the matrices $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}$ and $\mathbb{F}$ correspond to the different bilinear forms appearing at the left hand side of (7). The solution of such a linear system can be reduced to solving a linear system of the form

$$\left[\gamma\mathbb{D} - \begin{pmatrix} \gamma\mathbb{F} & -\mathbb{C} \end{pmatrix} \begin{pmatrix} \mathbb{A}+\gamma\mathbb{E} & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma\mathbb{F}^T \\ -\mathbb{C}^T \end{pmatrix} \right] \underline{\varphi}_h = \underline{b}_h \tag{27}$$

on the only unknown vector $\underline{\varphi}_h$, for a suitable right hand side $\underline{b}_h$. The solution of (27) by any iterative method implies solving, at each iteration, a linear system of the form

$$\begin{pmatrix} \mathbb{A}+\gamma\mathbb{E} & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix} \mathbf{x} = \mathbf{b}$$

which, as already recalled in the introduction, can be done in parallel, thanks to the block diagonal structure of the matrix at the left hand side, which reflects the uncoupling of the local subdomain problems.

The performance of several preconditioners for the Schur complement matrix $\mathbb{S}$

$$\mathbb{S} = \gamma\mathbb{D} - \begin{pmatrix} \gamma\mathbb{F} & -\mathbb{C} \end{pmatrix} \begin{pmatrix} \mathbb{A}+\gamma\mathbb{E} & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma\mathbb{F}^T \\ -\mathbb{C}^T \end{pmatrix}$$

appearing in such reduced linear system relies on the spectral properties of the bilinear form $s : \Phi_h \times \Phi_h \to \mathbb{R}$ corresponding to $\mathbb{S}$ (see [9, 5]). It is not difficult to see that such a bilinear form is defined by

$$s(\varphi_h, \psi_h) = \gamma \sum_k [\varphi_h - u_h^k(\varphi_h), \psi_h]_{\frac{1}{2},\Gamma_k} + \sum_k \int_{\Gamma_k} \lambda_h^k(\varphi_h)\psi_h \, ds \tag{28}$$

where $(u_h^k(\varphi_h), \lambda_h^k(\varphi_h)) \in V_h^k \times \Lambda_h^k$ denotes the solution of the problem

$$\left\{ \begin{array}{lcl} \text{for all } v_h^k \in V_h^k, \;\; \mu_h^k \in \Lambda_h^k : & & \\ a^k(u_h^k(\varphi), v_h^k) + \gamma[u_h^k(\varphi_h), v_h^k]_{\frac{1}{2},\Gamma_k} - \int_{\Gamma_k} \lambda_h^k(\varphi_h) v_h^k \, ds & = & \gamma[\varphi_h, v_h^k]_{\frac{1}{2},\Gamma_k}, \\ \int_{\Gamma_k} u_h^k(\varphi_h)\mu_h^k \, ds & = & \int_{\Gamma_k} \varphi_h \mu_h^k \, ds. \end{array} \right. \tag{29}$$

We can now prove the following Theorem.

Theorem 4.1 *For all $\gamma < \gamma_0$ (γ_0 given by Theorem 2.1), for all $\varphi_h, \psi_h \in \Phi_h$ it holds*

$$s(\varphi_h, \psi_h) \lesssim \|\varphi_h\|_\Phi \|\psi_h\|_\Phi \tag{30}$$

and

$$s(\varphi_h, \varphi_h) \gtrsim \|\varphi_h\|_\Phi^2, \tag{31}$$

(the implicit constant in (31) depending on γ).

Proof: We first remark that, by applying a standard argument, it is not difficult to prove that

$$|u_h^k(\varphi_h)|_{1,\Omega_k} + \|\lambda_h^k(\varphi_h)\|_{\Lambda^k} \lesssim |\varphi_h|_{1/2,\Gamma_k},$$

which easily implies (30). As far as coercivity (31) is concerned, we observe that, by definition of $u_h(\varphi_h)$ and $\lambda_h(\varphi_h)$ it holds

$$\begin{aligned}\int_{\Gamma_k} \varphi_h \lambda_h(\varphi_h)\,ds &= \int_{\Gamma_k} \lambda_h(\varphi_h) u_h(\varphi_h)\,ds \\ &= a^k(u_h^k(\varphi_h), u_h^k(\varphi_h)) + \gamma[u_h^k(\varphi_h), u_h^k(\varphi_h)]_{\frac{1}{2},\Gamma_k} - \gamma[\varphi_h, u_h^k(\varphi_h)]_{\frac{1}{2},\Gamma_k}\end{aligned}$$

By substituting this into the definition of s we obtain

$$\begin{aligned}s(\varphi_h,\varphi_h) &= \sum_k \Big(a^k(u_h^k(\varphi_h), u_h^k(\varphi_h)) + \gamma[u_h^k(\varphi_h), u_h^k(\varphi_h)]_{\frac{1}{2},\Gamma_k} \\ &\qquad + \gamma[\varphi_h,\varphi_h]_{\frac{1}{2},\Gamma_k} - 2\gamma[\varphi_h, u_h^k(\varphi_h)]_{\frac{1}{2},\Gamma_k}\Big) \\ &\geq \sum_k \left((\alpha - \gamma B c_1^2)|u_h^k(\varphi_h)|_{V^k}^2 + \frac{\gamma b}{2}|\varphi_h|_{1/2,\Gamma_k}^2 \right),\end{aligned}$$

where, again, b, B are the constants appearing in the bound (6) and c_1 is the constant stemming from the trace inequality $\sum_k |u^k|^2_{H^{1/2}(\Gamma_k)} \leq c_1 \sum_k |u|_{H^1(\Omega_k)^2}$. We then conclude as in the proof of Corollary 3.1. □

5 Uniform inf-sup condition

Assumption (A3) requires that an inf-sup condition holds with respect to the norms $\|\cdot\|_{1,\Omega_k}$ and $\|\cdot\|_{\Lambda^k}$, uniformly with respect to the diameter H_k of the subdomain Ω_k. The aim of this section is to give some sufficient condition for (A3) to hold in terms of standard assumptions on the discretization spaces V_h^k and Λ_h^k. More precisely let V_h^k and Λ_h^k satisfy the following assumptions:

(A4) for some (sub)space W_h^k of the trace of V_h^k on Γ_k, ($W_h^k \subseteq V_h^k|_{\Gamma_k}$), the following direct and inverse estimate hold (h_k acts here as the "mesh size" of W_h^k):

(a) for all $\eta \in H^1(\Omega_k)$ with $\int_{\Gamma_k} \eta\,ds = 0$

$$\inf_{\eta_h \in W_h^k} |\eta - \eta_h|_{1/2,\Gamma_k} \lesssim h_k^{1/2} |\eta|_{1,\Gamma_k}; \tag{32}$$

(b) for all $\eta_h \in W_h^k$ with $\int_{\Gamma_k} \eta_h\,ds = 0$

$$|\eta_h|_{1/2,\Gamma_k} \lesssim h_k^{-1/2} |\eta_h|_{0,\Gamma_k}; \tag{33}$$

(A5) a bounded projector $\pi_h^k : L^2(\Gamma_k) \to W_h^k$ exists, verifying for all $\zeta \in L^2(\Gamma_k)$

$$\int_{\Gamma_k} (\zeta - \pi_h^k \zeta)\lambda_h \, ds = 0, \qquad \forall \lambda_h \in \Lambda_h^k, \tag{34}$$

and, for all $\zeta \in H^{1/2}(\Gamma_k)$ with $\int_{\Gamma_k} \zeta \, ds = 0$,

$$\|\zeta - \pi_h^k \zeta\|_{0,\Gamma_k} \lesssim h_k^{1/2} |\zeta|_{1/2,\Gamma_k}. \tag{35}$$

Remark that given V_h^k satisfying assumption (A4), the choice $\Lambda_h^k = W_h^k$ satisfies assumption (A5), with π_h^k equals to the L^2 orthogonal projection.

The idea for proving (A3) is to use Fortin's argument with a projector built with the aid of π_h^k. In order to prove the H^1 boundedness of such a projector we start to prove the following Lemma.

Lemma 5.1 *If (A3a), (A4) and (A5) are satisfied then for all $\zeta \in H^{1/2}(\Gamma_k)$ it holds*

$$|\pi_h^k \zeta|_{1/2,\Gamma_k} \lesssim |\zeta|_{1/2,\Gamma_k}.$$

Proof: First of all we remark that (34), (35) and (A3a) imply respectively

$$\text{a)} \quad \int_{\Gamma_k} \pi_h^k \zeta \, ds = \int_{\Gamma_k} \zeta \, ds \qquad \text{and} \qquad \text{b)} \quad \pi_h^k c = c, \quad \text{for all constants } c. \tag{36}$$

Let $\check{\pi}_h^k : H^{1/2}(\Gamma_k)/\mathbb{R} \to W_h^k/\mathbb{R}$ be the projection induced by the $H^{1/2}(\Gamma_k)$ seminorm, ($H^{1/2}(\Gamma_k)/\mathbb{R}$ and $W_h^k/\mathbb{R}$ denote here the zero mean value subspaces of $H^{1/2}(\Gamma_k)$ and W_h^k respectively): for any $\check{\zeta} \in H^{1/2}(\Gamma_k) \sim \{\zeta \in H^{1/2}(\Gamma_k), \quad \int_{\Gamma_k} \zeta \, ds = 0\}$:

$$|\check{\pi}_h^k \check{\zeta} - \check{\zeta}|_{1/2,\Gamma_k} = \inf_{\substack{\zeta_h \in W_h^k \\ \int_{\Gamma_k} \zeta_h \, ds = 0}} |\zeta_h - \check{\zeta}|_{1/2,\Gamma_k}.$$

For $\zeta \in H^{1/2}(\Gamma_k)$ let

$$\check{\zeta} = \zeta - \frac{1}{|\Gamma_k|} \int_{\Gamma_k} \zeta \, ds.$$

Since trivially $|\check{\pi}_h^k \check{\zeta}|_{1/2,\Gamma_k} \leq |\check{\zeta}|_{1/2,\Gamma_k} = |\zeta|_{1/2,\Gamma_k}$, by applying a triangular inequality and (33) (by (36a) $\pi_h^k \check{\zeta} - \check{\pi}_h^k \check{\zeta}$ has zero mean value) we get

$$|\pi_h^k \zeta|_{1/2,\Gamma_k} = |\pi_h^k \check{\zeta}|_{1/2,\Gamma_k} \lesssim |\check{\zeta}|_{1/2,\Gamma_k} + h_k^{-1/2} \|\pi_h^k \check{\zeta} - \check{\pi}_h^k \check{\zeta}\|_{0,\Gamma_k}. \tag{37}$$

Now we have

$$h_k^{-1/2} \|\pi_h^k \check{\zeta} - \check{\pi}_h^k \check{\zeta}\|_{0,\Gamma_k} \lesssim h_k^{-1/2} \left(\|\pi_h^k \check{\zeta} - \check{\zeta}\|_{0,\Gamma_k} + \|\check{\zeta} - \check{\pi}_h^k \check{\zeta}\|_{0,\Gamma_k} \right). \tag{38}$$

We first observe that thanks to (35) it holds

$$\|\pi_h^k\check{\zeta} - \check{\zeta}\|_{0,\Gamma_k} \lesssim h_k^{1/2}|\check{\zeta}|_{1/2,\Gamma_k} = h_k^{1/2}|\zeta|_{1/2,\Gamma_k}. \tag{39}$$

In order to bound $\|\check{\zeta} - \check{\pi}_h^k\check{\zeta}\|_{0,\Gamma_k}$ we use the classical Nitsche's type duality argument which we have to show to holds with constants independent of the size of Ω_k. At first we make the following remark: denoting by $(\cdot,\cdot)_{1/2,\Gamma_k}$ the $H^{1/2}(\Gamma_k)/\mathbb{R}$ scalar product, let $\tau = \tau(f) \in H^{1/2}(\Gamma_k)/\mathbb{R}$ be the unique solution of the problem

$$(\tau,\xi)_{1/2,\Gamma_k} = \int_{\Gamma_k} f\xi\, ds, \qquad \text{for all } \xi \in H^{1/2}(\Gamma_k)/\mathbb{R}.$$

We have: $f \in L^2(\Gamma_k)$ implies that $\tau(f) \in H^1(\Gamma_k)$ and

$$|\tau(f)|_{1,\Gamma_k} \le \|f\|_{0,\Gamma_k}. \tag{40}$$

This descends from a scaling argument. In fact, if we re-scale the above equation to a reference curve $\hat{\Gamma}$ of length 1, the rescaled solution $\hat{\tau}(\hat{f})$ ($\hat{f}$ being the rescaled data) satisfies:

$$(\hat{\eta}(\hat{f}),\hat{\xi})_{1/2,\Gamma_k} = \tilde{H}_k \int_{\Gamma_k} \hat{f}\hat{\xi}\, d\hat{s}, \qquad \tilde{H}_k = |\Gamma_k| \simeq H_k.$$

It is not difficult to see (since $\hat{\Gamma}$ is a closed curve we can for instance use a Fourier transform argument) that

$$|\hat{\tau}(\hat{f})|_{1,\hat{\Gamma}} \lesssim \tilde{H}_k\|\hat{f}\|_{0,\hat{G}}$$

and, by rescaling back to Γ_k this yields precisely

$$\tilde{H}_k^{1/2}|\tau|_{1,\Gamma_k} \lesssim \tilde{H}_k\tilde{H}_k^{-1/2}\|f\|_{0,\Gamma_k} \qquad \Longrightarrow \qquad (40).$$

Now, setting $\check{\eta} = \check{\zeta} - \check{\pi}_h^k\check{\zeta} \in H^{1/2}(\Gamma_k)$ we remark that by the definition of $\check{\pi}_h^k$ it holds $(\check{\eta},\xi_h)_{1/2,\Gamma_k} = (\check{\zeta} - \check{\pi}_h^k\check{\zeta},\xi_h)_{1/2,\Gamma_k} = 0$ for all $\xi_h \in W_h^k/\mathbb{R} \sim \{\xi_h \in W_h^k : \int_{\Gamma_k}\xi_h\, ds = 0\}$. Then, thanks to (32) and (40) we can write

$$\begin{aligned}
\|\check{\eta}\|_{0,\Gamma_k} &= \frac{\int_{\Gamma_k}|\check{\eta}|^2\, ds}{\|\check{\eta}\|_{0,\Gamma_k}} = \frac{(\tau(\check{\eta}),\check{\eta})_{1/2,\Gamma_k}}{\|\check{\eta}\|_{0,\Gamma_k}} \\
&= \frac{(\tau(\check{\eta}) - \pi_h^k\tau(\check{\eta}),\check{\eta})_{1/2,\Gamma_k}}{\|\check{\eta}\|_{0,\Gamma_k}} \\
&\lesssim \frac{|\tau(\check{\eta}) - \pi_h^k\tau(\check{\eta})|_{1/2,\Gamma_k}|\check{\eta}|_{1/2,\Gamma_k}}{\|\check{\eta}\|_{0,\Gamma_k}} \\
&\lesssim \frac{h_k^{1/2}|\tau(\check{\eta})|_{1,\Gamma_k}}{\|\check{\eta}\|_{0,\Gamma_k}}|\check{\eta}|_{1/2,\Gamma_k} \lesssim h_k^{1/2}|\check{\eta}|_{1/2,\Gamma_k}.
\end{aligned}$$

Thus

$$|\check{\zeta}-\check{\pi}_h^k\check{\zeta}|_{0,\Gamma_k}\lesssim h_k^{1/2}|\check{\zeta}-\check{\pi}_h^k\check{\zeta}|_{1/2,\Gamma_k}\lesssim h_k^{1/2}|\check{\zeta}|_{1/2,\Gamma_k}=h_k^{1/2}|\zeta|_{1/2,\Gamma_k}. \tag{41}$$

By combining (39), (41) and (38) and substituting the resulting bound in (37) we get the thesis. □

We can now prove the following Lemma:

Lemma 5.2 *If (A3a), (A4–5) are satisfied then it holds uniformly in* H_k

$$\inf_{\lambda_h^k\in\Lambda_h^k}\sup_{u_h^k\in V_h^k}\frac{\int_{\Gamma_k}u_h^k\lambda_h^k\,ds}{\|u_h^k\|_{1,\Omega_k}\|\lambda_h^k\|_{\Lambda^k}}\gtrsim 1. \tag{42}$$

Proof: Thanks to the definition of $\|\cdot\|_{\Lambda^k}$ it is easily seen that the continuous inf-sup condition

$$\inf_{\lambda\in H^{-1/2}(\Gamma_k)}\sup_{u\in H^1(\Omega_k)}\frac{\int_{\Gamma_k}u\lambda\,ds}{\|u\|_{V^k}\|\lambda\|_{\Lambda^k}}\geq 1$$

holds. Then, by Fortin's argument, (42) is verified provided that a bounded projector $\Pi_h^k : H^1(\Omega_k)\to V_h^k$ exists, such that, for all $v\in H^1(\Omega_k)$, for all $\lambda_h\in\Lambda_h^k$ it holds $\int_{\Gamma_k}(v-\Pi_h^k v)\lambda_h\,ds=0$. Let Π_h^k be defined by

$$\int_{\Omega_k}\nabla\Pi_h^k u\nabla v_h^k\,d\boldsymbol{x}=0,\qquad \forall v_h^k\in V_h^k\cap H_0^1(\Omega_k),\qquad \Pi_h^k u|_{\Gamma_k}=\pi_h^k(u|_{\Gamma_k}).$$

By (34) $\Pi_h^k u$ trivially satisfies for all $v\in H^1(\Omega_k)$

$$\int_{\Gamma_k}(v-\Pi_h^k v)\lambda_h^k\,ds=0,\qquad \forall\lambda_h^k\in\Lambda_h^k.$$

Then we just need to show that such projector is bounded (uniformly in H_k) with respect to the H^1 norm. It is not difficult to check (see for instance [9]) that

$$|\Pi_h^k u|_{1,\Omega_k}\lesssim|\pi_h^k u|_{1/2,\Gamma_k}\lesssim|u|_{1,\Omega_k}, \tag{43}$$

and that

$$\|\Pi_h^k u\|_{0,\Omega_k}\lesssim H_k^{1/2}\|\pi_h^k u\|_{0,\Gamma_k}+H_k|\pi_h^k u|_{1/2,\Gamma_k}. \tag{44}$$

Now, thanks to (8) we have

$$H_k^{1/2}\|u\|_{0,\Gamma_k}\lesssim\|u\|_{0,\Omega_k}+H_k|u|_{1,\Omega_k}. \tag{45}$$

By combining (44) and (45) we obtain

$$\begin{aligned}\|\Pi_h^k u\|_{0,\Omega_k} &\lesssim H_k^{1/2}\|\pi_h^k u\|_{0,\Gamma_k}+H_k|\pi_h^k u|_{1/2,\Gamma_k}\\ &\lesssim H_k^{1/2}\|u\|_{0,\Gamma_k}+H_k|u|_{1/2,\Gamma_k}\\ &\lesssim \|u\|_{0,\Omega_k}+H_k|u|_{1,\Omega_k}\end{aligned}$$

which added to (43) implies $\|\Pi_h^k u\|_{1,\Omega_k}\lesssim\|u\|_{1,\Omega_k}$. By Fortin's argument we can then conclude that (42) holds. □

6 An Example

For all k let $\mathcal{T}_u^k$ be a triangulation of Ω_k, with mesh size h_k^u. Let $\mathcal{T}_\lambda^k$ be a quasi-uniform decomposition of Γ_k into intervals I with mesh-size h_k^λ, such that all nodes of $\mathcal{T}_\lambda^k$ are also nodes of $\mathcal{T}_u^k$. Moreover let $\mathcal{T}_\varphi$ be a decomposition of Σ in intervals J such that for all k all vertex of Ω_k are nodes of the decomposition. Let h_k^φ be the mesh-size of $\mathcal{T}_\varphi$ restricted to Γ_k. Remark that, in our framework, independently of $\mathcal{T}_u^k$ and $\mathcal{T}_\lambda^k$, it is allowed to take $\mathcal{T}_\varphi$ such that the decomposition is uniform on each $\Gamma_k \cap \Gamma_\ell$, which can facilitate the construction of good preconditioners for the discrete Steklov-Poincaré operator. We can discretize the three fields V, Λ and Φ by three P1 finite element spaces. More precisely, let

$$\begin{aligned} V_h^k &= \{v \in C^0(\Omega_k):\ v|_T \in P_1(T)\ \forall T \in \mathcal{T}_u^k\},\\ \Lambda_h^k &= \{\mu \in C^0(\Gamma_k):\ \mu|_I \in P_1(I)\ \forall I \in \mathcal{T}_\lambda^k\},\\ \Phi_h &= \{\psi \in C^0(\Sigma):\ \psi|_J \in P_1(J)\ \forall J \in \mathcal{T}_\varphi\}, \end{aligned}$$

where as usual P_1 denotes the space of polynomials of degree lower or equal than one. We observe that such a choice falls in the framework here considered. In fact, by the assumptions on the nodes of $\mathcal{T}_\lambda^k$, the inclusion $\Lambda_h^k \subseteq V_h^k|_{\Gamma_k}$ holds. Then, setting $W_h^k = \Lambda_h^k$ (A4) and (A5) hold for $h_k = h_k^\lambda$, with $\pi_h^k : L^2(\Gamma_k) \to W_h^k$ taken as the L^2 orthogonal projection. The bounds (32), (33) and (35) are classical (see [15]). It is not difficult to check (by using a scaling argument) that the constants in the bounds are indeed independent of H_k. By applying Lemma 5.2 we can then conclude that (A3) holds. The error is then estimated by Theorem 2.1. To this end we need to bound the three terms

$$\inf_{v_h \in V_h} \|u - v_h\|_V, \qquad \inf_{\mu_h \in \Lambda_h} \left\| \frac{\partial u}{\partial \nu_k^a} - \mu_h \right\|_\Lambda, \quad \text{and} \quad \inf_{\psi_h \in \Psi} \|u - \psi_h\|_\Psi.$$

As usual let us introduce the two Lagrange interpolation operators $I_h^{u,k} : H^2(\Omega_k) \to V_h^k$, $I_h^\varphi : H^1(\Sigma) \to \Phi_h$ corresponding respectively to the grids $\mathcal{T}_u^k$, $\mathcal{T}_\varphi$, and the L^2 projection $P_h^{\lambda,k} : L^2(\Gamma_k) \to \Lambda_h^k$. The bounds

$$\|u - I_h^{u,k} u\|_{1,\Omega_k} \lesssim h_k^u |u|_{2,\Omega_k} \qquad \text{and} \qquad |u - I_h^\varphi \varphi|_{1/2,\Gamma_k} \lesssim h_k^\varphi |u|_{3/2,\Gamma_k}$$

are classical. Let us then concentrate on the Λ^k component of the error. It is well known that for all $\lambda \in H^{1/2}(\Gamma_k)$ it holds

$$\|\lambda - P_h^{\lambda,k}\lambda\|_{0,\Gamma_k} \lesssim (h_k^\lambda)^{1/2} |\lambda|_{1/2,\Gamma_k}.$$

Let us bound $\|\lambda - P_h^{\lambda,k}\lambda\|_{\Lambda^k}$. We have, recalling (9)

$$\begin{aligned} \|\lambda - P_h^{\lambda,k}\lambda\|_{\Lambda^k} &= \sup_{v \in H^1(\Omega_k)} \frac{\int_{\Gamma_k} (\lambda - P_h^{\lambda,k}\lambda) v \, ds}{\|v\|_{1,\Omega_k}} = \frac{\int_{\Gamma_k} (\lambda - P_h^{\lambda,k}\lambda)(v - P_h^{\lambda,k} v)\, ds}{\|v\|_{1,\Omega_k}} \\ &\lesssim h_k^\lambda |\lambda|_{1/2,\Gamma_k} \frac{|v|_{1/2,\Gamma_k}}{\|v\|_{1,\Omega_k}} \lesssim h_k^\lambda |\lambda|_{1/2,\Gamma_k}. \end{aligned}$$

Then, thanks to the definition of the V and Λ norms, and to the characterization (6) of the Φ norm, using the trace bounds (10) we can conclude that

$$\|u - u_h\|_V + \left\| \frac{\partial u}{\partial \nu^a} - \lambda_h \right\|_\Lambda + \|u - u_h\|_\Phi \lesssim \left(\sum_k (h_k^u + h_k^\lambda + h_k^\varphi)^2 |u|^2_{2,\Omega_k} \right)^{1/2}.$$

7 Conclusion

We proved that for the stabilized three fields formulation proposed in [8], under quite general assumptions (some of which actually weaker of the ones in [8]) on the geometrical decomposition and on the discretization spaces for the unknowns u and λ, and *under no assumption on the discretization space relative to the interface unknown* φ stability and convergence bounds hold with constant independent of the number of subdomains or of their size. The same holds for the discrete Steklov-Poincaré operator, obtained by reducing the stabilized three fields formulation to a problem on the interface unknown. The corresponding bilinear form is proven to be continuous and coercive *uniformly* with respect to the size and number of subdomains. Such method is then well suited to massively parallel implementation and, if coupled with optimal preconditioner the resulting algorithm will be potentially scalable.

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Appendix

In this appendix we want to prove that, in the case of a geometrically conforming decomposition, (A2) is a direct consequence of (A1). We recall that a decomposition $\bar{\Omega} = \cup \bar{\Omega}_k$ is said to be *geometrically conforming* if any vertex A of any subdomain Ω_k is also a vertex of all the subdomains Ω_ℓ to which it belongs.
Let l_c be the line of equation $\boldsymbol{x}_2 = c$, and let $K_c = \{k : \text{ length}(l_c \cap \Omega_k) > 0\}$. For simplicity assume that $K_c = \{1, 2, \ldots, N_c\}$.
For $k \in K_c$ let $\varepsilon_k = \text{length}(l_c \cap \Omega_k)$. It trivially holds

$$\sum_k \varepsilon_k \leq diam(\Omega).$$

Now, let $k \in K_c$. Two situations may occur:

(i) l_c crosses Ω_k in two opposite edges;

(ii) l_c crosses Ω_k in two adjacent edges e_1 and e_2.

If (i) occurs, then $\varepsilon_k \simeq H_k$. If, on the other hand (ii) occurs, let A denote the common vertex to e_1 and e_2. For some $k' = k'(k)$, $k'' = k''(k)$, with $k' < k < k''$ the set $k' \leq \ell \leq k''$ will individuate those subdomains sharing A as a vertex and crossed by the line l_c. Since, letting ω_k denote the internal angle to Ω_k with vertex A, clearly it must hold $\sum_{\ell=k'+1}^{k''-1} \omega_\ell \leq \pi$, thanks to the angle condition in assumption (A1),

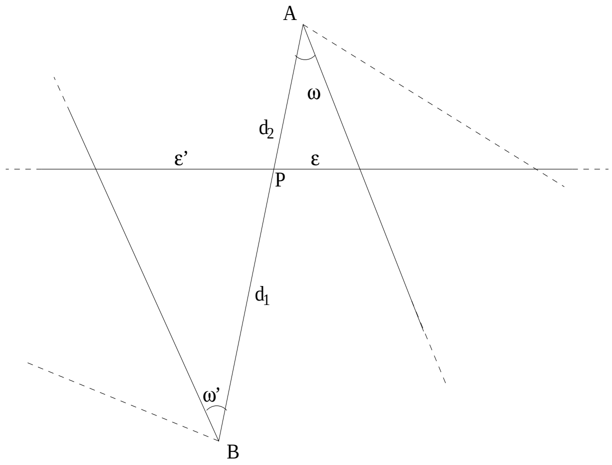
$$C_1(k'' - k' - 1) \leq \sum_{\ell=k'+1}^{k''-1} \omega_\ell \leq \pi \qquad \Longrightarrow \qquad (k'' - k') \leq C'.$$

Let us then consider the subdomain $\Omega_{k'}$. Again either (i) or (ii) occur. In the first case, we have $\varepsilon_{k'} \simeq H_{k'}$. In the second case, by the definition of k', denoting by B the common vertex to the two edges crossed by l_c we will have that $A \neq B$ (otherwise, the subdomain $\omega_{k'-1}$, also crossed by l_c would also share A as a vertex, contraddicting the definition of k') and we will be in the situation depicted in figure 7
In this case it is not difficult to see that $\varepsilon_{k'} + \varepsilon_{k'+1} \gtrsim H_{k'}$. In fact, letting ω and ω' be the two angles respectively internal to $\Omega_{k'+1}$ with vertex A and internal to $\Omega_{k'}$ with vertex B, and denoting by P the crossing point of l_c with $[A, B]$ (see figure 7) it holds, (thanks to the angle condition of assumption (A1) which both ω and ω' satisfy)

$$\begin{aligned} \varepsilon_{k'} &\geq \text{length}[B, P] \sin(\omega') &\gtrsim \text{length}[B, P] \\ \varepsilon_{k'+1} &\geq \text{length}[A, P] \sin(\omega) &\gtrsim \text{length}[A, P] \end{aligned}$$

whence

$$\varepsilon_{k'} + \varepsilon_{k'+1} \gtrsim \text{length}[A, P] + \text{length}[B, P] = \text{length}[A, B] \simeq H_{k'}.$$



Then, since thanks to the grading of the decomposition, for all ℓ, $k' \le \ell \le k''$ it holds $H_{k'} \simeq H_\ell \simeq H_{k''}$, we get

$$\sum_{\ell=k'}^{k''} \varepsilon_\ell \gtrsim H_{k'} + H_{k''} \gtrsim \sum_{\ell=k'}^{k''} H_\ell.$$

Now we can conclude by observing that

$$\sum_{k=1}^{N_c} H_k \le \sum_{k=1}^{N_c} \sum_{\ell=k'(k)}^{k''(k)} H_\ell \lesssim \sum_{k=1}^{N_c} \sum_{\ell=k'(k)}^{k''(k)} \varepsilon_\ell \lesssim \sum_{k=1}^{N_c} \varepsilon_k \lesssim 1,$$

where for Ω_k in the case (i) we set $k'(k) = k''(k) = k$.