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A VARIATIONAL PRINCIPLE FOR GRADIENT FLOWS OF NONCONVEX ENERGIES

GORO AKAGI AND ULISSE STEFANELLI

Abstract. We present a variational approach to gradient flows of energies of the form $E = \phi_1 - \phi_2$ where ϕ_1, ϕ_2 are convex functionals on a Hilbert space. A global parameter-dependent functional over trajectories is proved to admit minimizers. These minimizers converge up to subsequences to gradient-flow trajectories as the parameter tends to zero. These results apply in particular to the case of non λ-convex energies E. The application of the abstract theory to classes of nonlinear parabolic equations with nonmonotone nonlinearities is presented.

1. Introduction

Gradient flows arise ubiquitously in connection with the mathematical description of non-equilibrium systems. A variety of dissipative evolution problems can be variationally formulated in abstract spaces by letting the time-dependent state of the system $t \mapsto u(t) \in H$ belong to a real Hilbert space H, the nonconvex energy $E : (0,T) \times H \to (-\infty, +\infty]$ be defined, and imposing

$$u'(t) = -\mathrm{D}E(t, u(t)), \quad t > 0,$$

where $u' = \mathrm{d}u/\mathrm{d}t$ stands for the time derivative and $\mathrm{D}E$ represent some suitably defined *gradient* of the functional $E(t, \cdot)$. In what follows the symbol $(\cdot, \cdot)$ denotes the scalar product in H and $\|\cdot\|$ is the corresponding norm, whereas the norm in the normed space B is denoted by $\|\cdot\|_B$.

The focus of this paper is to present a variational approach to gradient flows in Hilbert spaces in the nonconvex case of

$$E(t,u) := \varphi^1(u) - \varphi^2(u) - (f(t), u) \quad \text{for } u \in D(E(t,\cdot)) \subset H, \ t \in (0,T), \tag{1.1}$$

where $D(E(t,\cdot)) := D(\varphi^1) \cap D(\varphi^2)$. Here $\varphi^1, \varphi^2 : H \to [0, +\infty]$ are convex, proper, and lower semicontinuous functionals and we assume $f \in L^2(0,T;H)$. In particular, we shall be considering strong solutions $u \in W^{1,2}(0,T;H)$ to the differential problem

$$u'(t) + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) \ni f(t) \ \text{ in } H, \ \text{for a.e } t \in (0,T), \tag{1.2}$$

$$u(0) = u_0 \tag{1.3}$$

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for u_0 belonging to a proper subset of $D(E(0,\cdot))$. Note that the operators $\partial\varphi^i : H \to 2^H$ $(i = 1, 2)$ are the *subdifferentials* of φ^i in the sense of convex analysis, namely, for all $u \in D(\varphi^i) := \{u \in H \colon \varphi^i(u) < \infty\}$, we define

$$\partial\varphi^i(u) := \big\{\xi \in H \colon \varphi^i(v) - \varphi^i(u) \geq (\xi, v - u) \quad \text{for all} \quad v \in H\big\}$$

and $D(\partial\varphi^i) := \{u \in D(\varphi^i) \colon \partial\varphi^i(u) \neq \emptyset\}$.

The existence of solutions to problem (1.2), (1.3) has been considered in [16, 17, 30, 31] and these abstract results have been indeed applied to nonlinear PDEs such as degenerate parabolic equations with blow-up terms and Allen-Cahn equations (see also [1] for a Banach space setting). The case where φ^2 is quadratic, for instance $\varphi^2(u) = \lambda\|u\|^2/2$ for some $\lambda \in \mathbb{R}$, is even more classical and usually referred to as λ-*convex* case (see [5] as a survey).

Our variational approach is based on the minimization

$$\min\{\mathcal{W}_\varepsilon(u) \mid u \in \mathcal{H} := L^2(0,T;H),\ u(0) = u_0\} \tag{1.4}$$

of the *Weighted Energy-Dissipation* (WED for short) functional $\mathcal{W}_\varepsilon : \mathcal{H} \to (-\infty, \infty]$ given by

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \Big(\frac{\varepsilon}{2}\|u'(t)\|^2 + E(t,u(t))\Big)\,\mathrm{d}t \\ \qquad\qquad \text{if} \ \ u \in W^{1,2}(0,T;H),\ E(\cdot,u(\cdot)) \in L^1(0,T),\ u(0) = u_0, \\ +\infty \qquad \text{else.} \end{cases}$$

Here $E(\cdot,u(\cdot)) \in L^1(0,T)$ means that $u(t) \in D(E(t,\cdot)) = D(\varphi^1) \cap D(\varphi^2)$ for a.e. $t \in (0,T)$ and the function $t \mapsto E(t,u(t))$ belongs to $L^1(0,T)$.

The relation between the minimization (1.4) of the parameter-dependent WED functional $\mathcal{W}_\varepsilon$ and the gradient flow (1.2)-(1.3) becomes apparent by (formally) computing the Euler-Lagrange system corresponding to (1.4), which reads

$$-\varepsilon u''(t) + u'(t) + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) \ni f(t) \ \text{ in } H, \quad 0 < t < T, \tag{1.5}$$

$$u(0) = u_0, \quad \varepsilon u'(T) = 0. \tag{1.6}$$

Indeed, the constrained minimization (1.4) of the WED functional corresponds to an elliptic-in-time regularization of the gradient-flow system (1.2)-(1.3). By letting u_ε denote a minimizer of $\mathcal{W}_\varepsilon$, the variational resolution of the gradient flow would require to prove that

$$u_\varepsilon \to u \quad \text{for some subsequence, and the limit } u \text{ solves (1.2)-(1.3).} \tag{1.7}$$

The main result of the paper consists in proving that the variational approach consisting in (1.4)+(1.7) can be rigorously ascertained in the nonconvex frame of assumption (1.1). Indeed, this result corresponds to an extension of the former [26], where the feasibility of the WED approach for λ-convex energies E is investigated. We shall here tackle the more general nonconvex case of (1.1) under some compactness (see (A2) below) and coercivity assumptions ((A3)-(A4), respectively). In particular, we assume that $\varphi^1 - \varphi^2$ is bounded from below and the monotone part $\partial\varphi^1$ dominates the anti-monotone part $-\partial\varphi^2$. These assumptions are general enough to include a variety of different nonlinear PDE problems. The reader is referred to Ôtani [30, 32, 31], Rossi & Savaré [33] and to Section 6 for an account on applications.

This extension of the WED approach (1.4)+(1.7) from the λ-convex case of [26] to the nonconvex case of (1.1) requires the treatment of a number of delicate issues. To begin with, the WED functional $\mathcal{W}_\varepsilon$ is a priori not lower semicontinuous, posing already the existence of a minimizer u_ε to (1.4) into question. Note that in the λ-convex case of [26] the functional $\mathcal{W}_\varepsilon$ was even uniformly convex for small ε, entailing indeed the existence of unique global minimizers. Secondly, under the choice (1.1) the limiting gradient flow (1.2)-(1.3) may show genuine nonuniqueness of solutions whereas solutions in the λ-convex case are unique. This is reflected also at the level of the WED functional $\mathcal{W}_\varepsilon$ which may indeed present many critical points. Finally, the limiting procedure (1.7) requires here the identification of *two* nonlinearities, of which one is antimonotone. In this respect, our compactness assumption, not needed in the λ-convex case, will turn out to be crucial.

Our proof of the existence of minimizers in (1.4) follows by a careful study of the Euler-Lagrange system (1.5)-(1.6). By suitably arguing by approximation, we shall prove that (1.5)-(1.6) admits at least one strong solution which minimizes $\mathcal{W}_\varepsilon$ globally. At the same time, we can check that all local minimizers of $\mathcal{W}_\varepsilon$ also solve (1.4).

As for the convergence result (1.7) we shall classically proceed by estimation and passage to the limit. Here, the crucial tool is a *maximal regularity* estimate valid indeed for all solutions of the Euler-Lagrange system (1.5)-(1.6). Combined with classical lower semicontinuity arguments, this estimate will entail the necessary compactness in order to identify the limit as $\varepsilon \to 0$.

1.1. **Related literature.** The elliptic-regularization approach to evolution problems is classical and has to be traced back at least to Lions and Oleinik [20, 29], see also [21]. The WED variational principle is mentioned even in the classical textbook by Evans [13, Problem 3, p. 487].

The WED approach has being brought to new attention by Ilmanen [15], who used the method to tackle existence and partial regularity of the so-called Brakke mean-curvature flow of varifolds. The WED functional appears also in Hirano [14] in connection with the existence of periodic solutions of the gradient flow.

After a ten year lull, the WED formalism has been reconsidered by Mielke & Ortiz [24] in the context of rate-independent processes, see also the subsequent [25]. An application of the WED perspective is in Larsen, Ortiz, & Richardson [23], where a model for crack-front propagation in brittle materials is advanced.

As for the gradient flow-situation, a preliminary discussion on a linear case is recalled in [24] together with a first example of relaxation. Two additional examples of relaxation related with micro-structure evolution have been provided by Conti & Ortiz [11]. In the latter papers, the problem of ascertaining the limit $\varepsilon \to 0$ was left open and was indeed solved in [26]. In the paper [40] the relaxation of a WED functional related to mean curvature evolution of Cartesian surfaces and, more generally, linear growth functionals are considered. The reader is referred also to the recent paper by Bögelein, Duzaar, & Marcellini [7], where the WED approach is exploited in order to prove the existence of suitable variational solutions to the equation

$$u_t - \nabla \cdot f(x, u, \nabla u) + \partial_u f(x, u, \nabla u) = 0$$

where $u : \Omega \times (0, T) \to \mathbb{R}^d$ and the field f is convex in $(u, \nabla u)$. The theory for the gradient flow case has been also extended to the case of curves of maximal slope in metric spaces in [34, 35]. The doubly-nonlinear case of rate-dependent equations, corresponding indeed to a dissipation term of homogeneity $1 < p \neq 2$, has been tackled in the series of contributions [2, 3, 4].

Moving from Ilmanen's paper, De Giorgi conjectured in [12] that the WED functional procedure could be implemented in the hyperbolic setting of semilinear waves as well. Results in this direction are in [42] (for the finite-time case) and in Serra & Tilli [37] (for the infinite-time case). Extensions to mixed hyperbolic-parabolic semilinear equations and to some different classes of nonlinear energies are also available [18, 19, 38].

A functional close to WED (with ε fixed though) has been considered by Lucia, Muratov, & Novaga in connection with traveling waves in reaction-diffusion-advection problems [22, 27, 28].

2. Assumptions

We shall enlist here the assumptions which are assumed throughout the analysis.

(A1): The functionals φ^1 and φ^2 are lower semicontinuous and convex in H with nonempty effective domains $D(\varphi^i) := \{u \in H : \varphi^i(u) < \infty\} \neq \emptyset$ $(i = 1, 2)$. Moreover, $u_0 \in D(\partial\varphi^1)$ and $f \in L^2(0, T; H)$.

(A2): There exists a Banach space $(X, \|\cdot\|_X)$ compactly embedded in H such that

$$\|u\|_X \leq \ell_1(\|u\|)\left(\varphi^1(u) + 1\right) \quad \text{for all } u \in D(\varphi^1)$$

with a non-decreasing function ℓ_1 on $[0, \infty)$.

(A3): There exist constants $k_1 \in [0, 1)$, $C_1 \geq 0$ such that

$$\varphi^2(u) \leq k_1\varphi^1(u) + C_1 \quad \text{for all } u \in D(\varphi^1).$$

In particular, $D(\varphi^1) \subset D(\varphi^2)$.

(A4): It holds that $D(\partial\varphi^1) \subset D(\partial\varphi^2)$. Moreover, there exist a constant $k_2 \in [0,1)$ and a non-decreasing function ℓ_2 on $[0,\infty)$ such that

$$\|\xi\|^2 \le k_2 \|(\partial\varphi^1)^\circ(u)\|^2 + \ell_2(\|u\|)\left(\varphi^1(u)+1\right) \qquad (2.1)$$
$$\text{for all } \ u \in D(\partial\varphi^2),\ \xi \in \partial\varphi^2(u),$$

where $(\partial\varphi^1)^\circ(u)$ stands for the minimal section of $\partial\varphi^1(u)$ (see, e.g., [9]).

Remark 2.1. In [30], the existence result for (1.2)-(1.3) is established under similar assumptions to the above. However, (A4) is more restrictive compared to [30], where $\varphi^1(u)$ in the right-hand side of (2.1) is indeed replaced by $\ell(\varphi^1(u))$ for some non-decreasing function $\ell(\cdot)$ in $\mathbb{R}$. Such a restriction stems from additional difficulty in deriving energy estimates for the Euler-Lagrange system (1.5)-(1.6) (see §3.2 below for more details).

Note that assumption (A3) entails that the energy E is affinely bounded from below as

$$E(t,u) \ge (1-k_1)\varphi_1(u) - (f(t),u) - C_1 \quad \text{for all } \ u \in D(\varphi^1). \qquad (2.2)$$

In particular, owing to the classical Poincaré estimate in time, the WED functional $\mathcal{W}_\varepsilon$ turns out to be bounded from below on $\mathcal{H}$.

More precisely, the lower bound (2.2) in combination with the compactness assumption (A2) and the fact that $f \in L^2(0,T;H)$ entails that the sublevels of $\mathcal{W}_\varepsilon$ are bounded in $W^{1,2}(0,T;H) \cap L^2(0,T;X)$, hence compact in $C([0,T];H)$ due to the classical Aubin-Lions-Simon Lemma [39].

We warn the reader that in the following we will use the symbol C to indicate a positive constant, possibly depending on data, and specifically on k_1, C_1, $\varphi^1(u_0)$, $\|f\|_{\mathcal{H}}$, and T but independent of ε and the further approximation parameter λ. Note that C can change from line to line.

3. Solvability of Euler-Lagrange equations

We proceed to the study of the Euler-Lagrange system (1.5)-(1.6) and prove a solvability result. Let us begin by specifying the notion of solution we are interested in.

Definition 3.1 (Strong solutions of Euler-Langrange). *A function* $u \in W^{2,2}(0,T;H)$ *is called a* strong solution *of* (1.5)-(1.6) *if*

(i) $u(t) \in D(\partial\varphi^1)$ *for a.e.* $t \in (0,T)$.
(ii) *There exist* $\xi,\eta \in L^2(0,T;H)$ *such that, for a.e.* $t \in (0,T)$,

$$-\varepsilon u''(t) + u'(t) + \xi(t) - \eta(t) = f(t), \quad \xi(t) \in \partial\varphi^1(u(t)), \quad \eta(t) \in \partial\varphi^2(u(t)). \quad (3.1)$$

(iii) $u(0) = u_0,\ \ \varepsilon u'(T) = 0.$

The main result of this section reads as follows.

Theorem 3.2 (Existence of strong solutions for Euler-Lagrange). *There exists a strong solution u of* (1.5)-(1.6).

The rest of this section is devoted to a proof of this result. The argument consists of an approximation: we establish existence in the case of smooth functionals φ^2 and pass to the limit in order to handle the general case of (A1).

3.1. **Approximation.** We shall be considering the following family of approximating WED functionals $\mathcal{W}_{\varepsilon,\lambda} : \mathcal{H} \to (-\infty, \infty]$

$$\mathcal{W}_{\varepsilon,\lambda}(u) := \begin{cases} \displaystyle\int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \left(\frac{\varepsilon}{2}\|u'(t)\|^2 + E_\lambda(t, u(t)) \right) \mathrm{d}t \\ \qquad\qquad \text{if } \ u \in W^{1,2}(0,T;H), \ \varphi^1(u(\cdot)) \in L^1(0,T), \ u(0) = u_0, \\ +\infty \qquad \text{else,} \end{cases}$$

where the approximate energy is given by

$$E_\lambda(t,u) := \varphi^1(u) - \varphi^2_\lambda(u) - (f(t), u) \quad \text{ for } \ u \in H.$$

Here, $\varphi^2_\lambda \in C^{1,1}(H)$ stands for the Moreau-Yosida approximation of φ^2 in H, that is,

$$\varphi^2_\lambda(u) := \inf_{v \in H} \left\{ \frac{1}{2\lambda}\|u - v\|^2 + \varphi^2(v) \right\} = \frac{1}{2\lambda}\|u - J_\lambda u\|^2 + \varphi^2(J_\lambda u),$$

and J_λ is the resolvent for $\partial\varphi^2$ (see, e.g., [9]). In particular, it holds that

$$\varphi^2(J_\lambda u) \le \varphi^2_\lambda(u) \le \varphi^2(u) \quad \text{ and } \quad \varphi^2(J_\lambda u) \to \varphi^2(u) \quad \text{ for all } \ u \in H \tag{3.2}$$

as $\lambda \to 0$. Moreover, for all $u \in L^2(0,T;H)$ we observe that $\varphi^2_\lambda(u(\cdot))$ belongs to $L^1(0,T)$. Let us check that the constrained minimization problem (1.4) admits a solution when $\mathcal{W}_\varepsilon$ is replaced by the regularized $\mathcal{W}_{\varepsilon,\lambda}$.

Lemma 3.3 (Minimization of $\mathcal{W}_{\varepsilon,\lambda}$). *For each $\varepsilon, \lambda > 0$, the functional $\mathcal{W}_{\varepsilon,\lambda}$ admits a global minimizer u_λ in $\{u \in \mathcal{H} \ : \ u(0) = u_0\}$.*

Proof. We first decompose $\mathcal{W}_{\varepsilon,\lambda}$ into the difference of two convex functionals:

$$\mathcal{W}_{\varepsilon,\lambda} = \mathcal{C}^1_\varepsilon - \mathcal{C}^2_{\varepsilon,\lambda},$$

where $\mathcal{C}^1_\varepsilon, \mathcal{C}^2_{\varepsilon,\lambda} : \mathcal{H} \to (-\infty, \infty]$ are given by

$$\mathcal{C}^1_\varepsilon(u) := \begin{cases} \displaystyle\int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \left(\frac{\varepsilon}{2}\|u'(t)\|^2 + \varphi^1(u(t)) - (f(t), u(t)) \right) \mathrm{d}t \\ \qquad\qquad \text{if } \ u \in W^{1,2}(0,T;H), \ \varphi^1(u(\cdot)) \in L^1(0,T), \ u(0) = u_0, \\ +\infty \qquad \text{else} \end{cases}$$

and

$$\mathcal{C}^2_{\varepsilon,\lambda}(u) := \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \varphi^2_\lambda(u(t))\,\mathrm{d}t.$$

One can easily check that $\mathcal{C}^1_\varepsilon$ is convex, proper, and lower semicontinuous in $\mathcal{H}$. On the other hand, $\mathcal{C}^2_{\varepsilon,\lambda}$ is convex and Fréchet differentiable in $\mathcal{H}$ because of the

Fréchet differentiability of φ^2_λ in H (see, e.g., [6, 9]). Hence, the whole $\mathcal{W}_{\varepsilon,\lambda}$ is lower semicontinuous on $\mathcal{H}$.

Owing to (A3), we now check that $\mathcal{W}_{\varepsilon,\lambda}$ is bounded from below. Indeed, we find that

$$E_\lambda(t,u) \overset{(3.2)}{\geq} E(t,u) \overset{(2.2)}{\geq} (1-k_1)\varphi^1(u) - (f(t),u) - C_1 \quad \text{for all} \ \ u \in D(\varphi^1).$$

On the other hand, from the elementary fact

$$\|u(t)\|^2 \leq 2t \int_0^t \|u'(s)\|^2 \,\mathrm{d}s + 2\|u_0\|^2,$$

we readily integrate by parts and find that

$$\int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \|u(t)\|^2 \,\mathrm{d}t \leq 2T \int_0^T e^{-t/\varepsilon} \|u'(t)\|^2 \,\mathrm{d}t + 2\|u_0\|^2.$$

Hence, we readily check that for any $\eta > 0$ there is a constant $C_\eta > 0$ such that

$$\begin{aligned}\int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} (f(t),u(t)) \,\mathrm{d}t &\leq \eta \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \|u(t)\|^2 \,\mathrm{d}t + \frac{C_\eta}{\varepsilon}\|f\|^2_{\mathcal{H}} \\ &\leq 2\eta T \int_0^T e^{-t/\varepsilon}\|u'(t)\|^2 \,\mathrm{d}t + 2\eta\|u_0\|^2 + \frac{C_\eta}{\varepsilon}\|f\|^2_{\mathcal{H}}.\end{aligned}$$

Therefore, setting $\eta = 1/(8T)$, one can deduce that

$$\begin{aligned}\mathcal{W}_{\varepsilon,\lambda}(u) \geq &\int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \left(\frac{\varepsilon}{4}\|u'(t)\|^2 + (1-k_1)\varphi^1(u(t))\right) \mathrm{d}t \\ &- C_1 - \frac{1}{4T}\|u_0\|^2 - \frac{C}{\varepsilon}\|f\|^2_{\mathcal{H}} \quad \text{for all} \ \ u \in \mathcal{H}. \qquad (3.3)\end{aligned}$$

Hence, the infimum of $\mathcal{W}_{\varepsilon,\lambda}$ over $\mathcal{H}$ is finite. Moreover, from relation (3.3) it follows that any minimizing sequence (u_n) for $\mathcal{W}_{\varepsilon,\lambda}$ fulfills the bound

$$\int_0^T \left(\|u_n'(t)\|^2 + \varphi^1(u_n(t))\right) \mathrm{d}t + \sup_{t\in[0,T]} \|u_n(t)\| \leq C_\varepsilon$$

for some constant $C_\varepsilon \geq 0$ (depending on ε). Assumption (A2) and the Aubin-Lions-Simon compactness lemma (see [39, Thm. 3]) entail that (u_n) is precompact in $C([0,T];H) \hookrightarrow \mathcal{H}$. Eventually, the lower semicontinuity of $\mathcal{W}_{\varepsilon,\lambda}$ in $\mathcal{H}$ entails that a nonrelabeled subsequence of (u_n) converges strongly to a minimizer of $\mathcal{W}_{\varepsilon,\lambda}$. □

Our next step consists in proving that global minimizers $\mathcal{W}_{\varepsilon,\lambda}$ are solutions of the corresponding Euler-Lagrange system.

Lemma 3.4 (Global minimizers solve Euler-Lagrange). *For each $\varepsilon, \lambda > 0$, every global minimizer u_λ of $\mathcal{W}_{\varepsilon,\lambda}$ solves*

$$-\varepsilon u_\lambda''(t) + u_\lambda'(t) + \partial\varphi^1(u_\lambda(t)) - \mathrm{D}\varphi^2_\lambda(u_\lambda(t)) \ni f(t) \ \ \textit{in } H, \quad 0 < t < T, \qquad (3.4)$$

$$u_\lambda(0) = u_0, \quad u_\lambda'(T) = 0, \qquad (3.5)$$

where $\mathrm{D}\varphi^2_\lambda$ is the Fréchet differential of the $C^{1,1}$ functional φ^2_λ.

Proof. Let u_λ be a global minimizer of $\mathcal{W}_{\varepsilon,\lambda}$ over $\mathcal{H}$. We claim that

$$\partial \mathcal{C}^1_\varepsilon(u_\lambda) - \mathrm{D}\mathcal{C}^2_{\varepsilon,\lambda}(u_\lambda) \ni 0 \tag{3.6}$$

where $\mathrm{D}\mathcal{C}^2_{\varepsilon,\lambda}$ is the Fréchet differential of $\mathcal{C}^2_{\varepsilon,\lambda}$ in $\mathcal{H}$. Indeed, since u_λ minimizes $\mathcal{W}_{\varepsilon,\lambda}$ over $\mathcal{H}$, it follows that

$$\mathcal{W}_{\varepsilon,\lambda}(u_\lambda) \le \mathcal{W}_{\varepsilon,\lambda}(v) \quad \text{for all } \ v \in \mathcal{H},$$

which particularly implies

$$\mathcal{C}^1_\varepsilon(u_\lambda) - \mathcal{C}^1_\varepsilon(v) \le \mathcal{C}^2_{\varepsilon,\lambda}(u_\lambda) - \mathcal{C}^2_{\varepsilon,\lambda}(v) \quad \text{for all } \ v \in D(\mathcal{C}^1_\varepsilon).$$

Let now $\theta \in (0,1)$ and $w \in D(\mathcal{C}^1_\varepsilon)$ be arbitrarily fixed. Set $v = u_\lambda + \theta(w - u_\lambda) \in D(\mathcal{C}^1_\varepsilon)$. Then, from the convexity of $\mathcal{C}^1_\varepsilon$, we obtain

$$\theta\left(\mathcal{C}^1_\varepsilon(u_\lambda) - \mathcal{C}^1_\varepsilon(w)\right) \le \mathcal{C}^2_{\varepsilon,\lambda}(u_\lambda) - \mathcal{C}^2_{\varepsilon,\lambda}(u_\lambda + \theta(w-u_\lambda)) \quad \text{for all } \ w \in D(\mathcal{C}^1_\varepsilon).$$

Dividing both sides by $\theta > 0$ and taking a limit as $\theta \to 0$, from the Fréchet differentiability of $\mathcal{C}^2_{\varepsilon,\lambda}$, we derive

$$\mathcal{C}^1_\varepsilon(u_\lambda) - \mathcal{C}^1_\varepsilon(w) \le \left(\mathrm{D}\mathcal{C}^2_{\varepsilon,\lambda}(u_\lambda), u_\lambda - w\right)_{\mathcal{H}} \quad \text{for all } \ w \in D(\mathcal{C}^1_\varepsilon).$$

Thus we conclude that $\partial\mathcal{C}^1_\varepsilon(u_\lambda) \ni \mathrm{D}\mathcal{C}^2_{\varepsilon,\lambda}(u_\lambda)$.

Let us recall from [26] the representation of the relation $[u,g] \in \partial\mathcal{C}^1_\varepsilon$. Indeed, u minimizes the convex functional

$$v \mapsto \mathcal{C}^1_\varepsilon(v) - \int_0^T (g(t), v(t))\,\mathrm{d}t = \mathcal{C}^1_\varepsilon(v) - \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon}\left(\varepsilon e^{t/\varepsilon} g(t), v(t)\right)\,\mathrm{d}t$$

defined on $\mathcal{H}$. Hence, by applying Theorem 3.1 of [26], one finds that u solves

$$\begin{aligned} -\varepsilon u'' + u' + \partial\varphi^1(u) \ni f + \varepsilon e^{t/\varepsilon} g \ \ \text{in } H, \quad 0 < t < T,\\ u(0) = u_0, \quad u'(T) = 0, \quad u \in W^{2,2}(0,T;H). \end{aligned}$$

In addition u satisfies the following *maximal regularity estimate*:

$$\|\varepsilon u''\|^2_{\mathcal{H}} + \|u'\|^2_{\mathcal{H}} + \|\xi\|^2_{\mathcal{H}} \le \|f + \varepsilon e^{t/\varepsilon} g\|^2_{\mathcal{H}} + \varepsilon\|(\partial\varphi^1)^\circ(u_0)\|^2 + 2\varphi^1(u_0), \tag{3.7}$$

where $\xi := f + \varepsilon e^{\varepsilon/t} g + \varepsilon u'' - u'$ is a section of $\partial\varphi^1(u)$ and $(\partial\varphi^1)^\circ(u_0)$ stands for the minimal section of $\partial\varphi^1(u_0)$. The Fréchet differential of the functional $\mathcal{C}^2_{\varepsilon,\lambda}$ reads

$$\mathrm{D}\mathcal{C}^2_{\varepsilon,\lambda}(u) = \frac{e^{-t/\varepsilon}}{\varepsilon}\mathrm{D}\varphi^2_\lambda(u(\cdot)) \quad \text{for all } \ u \in \mathcal{H}.$$

By exploiting these representations of $\partial\mathcal{C}^1_\varepsilon$ and $\mathrm{D}\mathcal{C}^2_{\varepsilon,\lambda}$ and performing an easy computation, (3.6) is rewritten in the form of (3.4)-(3.5). $\square$

3.2. **Uniform estimates and convergences.** The next step toward a proof of Theorem 3.2 consists in deriving uniform estimates independently of λ (as well as of ε) and pass to the limit as $\lambda \to 0$. We start from the former.

Lemma 3.5 (Uniform estimates I). *There exists a constant $M_1 \geq 0$ depending on k_1, C_1, $\|u_0\|$, $\varphi^1(u_0)$, $\|f\|_{\mathcal{H}}$, and T but independent of ε, λ such that*

$$\varepsilon\|u_\lambda'(0)\|^2 + \int_0^T \|u_\lambda'(t)\|^2 \,\mathrm{d}t + \varphi^1(u_\lambda(T)) \leq M_1, \tag{3.8}$$

$$\sup_{t\in[0,T]} \|u_\lambda(t)\| \leq M_1, \tag{3.9}$$

$$\int_0^T \varphi^1(u_\lambda(t)) \,\mathrm{d}t \leq M_1. \tag{3.10}$$

Proof. Let us start by testing (3.4) on $u_\lambda'(t)$ in order to get

$$-\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\lambda'(t)\|^2 + \|u_\lambda'(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\varphi^1(u_\lambda(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\varphi^2_\lambda(u_\lambda(t)) = (f(t), u_\lambda'(t)). \tag{3.11}$$

By integrating over $(0,T)$. Then using the final condition $u_\lambda'(T) = 0$ we obtain

$$\begin{aligned}\frac{\varepsilon}{2}\|u_\lambda'(0)\|^2 + \frac{1}{2}\int_0^T \|u_\lambda'(t)\|^2 \,\mathrm{d}t + \varphi^1(u_\lambda(T)) - \varphi^2_\lambda(u_\lambda(T))\\ \leq \varphi^1(u_0) - \varphi^2_\lambda(u_0) + \frac{1}{2}\int_0^T \|f(t)\|^2 \,\mathrm{d}t.\end{aligned}$$

Since $u_0 \in D(\varphi^1) \subset D(\varphi^2)$ by (A3), the right-hand side above is bounded. Indeed, we have

$$\varphi^1(u_0) - \varphi^2_\lambda(u_0) \overset{(3.2)}{\leq} \varphi^1(u_0) - \varphi^2(J_\lambda u_0) \leq \varphi^1(u_0).$$

Moreover, again by (A3), we can compute

$$\varphi^1(u) - \varphi^2_\lambda(u) \overset{(3.2)}{\geq} \varphi^1(u) - \varphi^2(u) \overset{(A3)}{\geq} (1-k_1)\varphi^1(u) - C_1 \quad \text{for all} \;\; u \in D(\varphi^1).$$

Hence, it follows that

$$\varepsilon\|u_\lambda'(0)\|^2 + \int_0^T \|u_\lambda'(t)\|^2 \,\mathrm{d}t + \varphi^1(u_\lambda(T)) \leq C,$$

which, together with $u_\lambda(0) = u_0$, implies

$$\sup_{t\in[0,T]} \|u_\lambda(t)\| \leq C.$$

By integrating (3.11) over $(0,t)$ and using (A3) we obtain

$$\begin{aligned}&\frac{\varepsilon}{2}\|u_\lambda'(0)\|^2 + \frac{1}{2}\int_0^t \|u_\lambda'(\tau)\|^2 \,\mathrm{d}\tau + (1-k_1)\varphi^1(u_\lambda(t)) - C_1\\ &\leq \varphi^1(u_0) - \varphi^2_\lambda(u_0) + \frac{\varepsilon}{2}\|u_\lambda'(t)\|^2 + \frac{1}{2}\int_0^T \|f(t)\|^2 \,\mathrm{d}t \quad \text{for all} \;\; t \in (0,T).\end{aligned}$$

Integrating both sides over $(0,T)$ again, we obtain

$$\begin{aligned}&(1-k_1)\int_0^T \varphi^1(u_\lambda(t))\,\mathrm{d}t\\&\leq T\left(C_1+\varphi^1(u_0)+\frac{1}{2}\int_0^T \|f(t)\|^2\,\mathrm{d}t\right)+\frac{\varepsilon}{2}\int_0^T \|u_\lambda'(t)\|^2\,\mathrm{d}t,\end{aligned}$$

which together with (3.8) implies

$$\int_0^T \varphi^1(u_\lambda(t))\,\mathrm{d}t \leq C$$

and estimate (3.10) follows. □

LEMMA 3.6 (Uniform estimates II). *There exists a constant $M_2 \geq 0$ depending on M_1, k_2, T, $\ell_2(M_1)$, $\|f\|_{\mathcal{H}}$, $\varphi^1(u_0)$ and $\varepsilon\|(\partial\varphi^1)^\circ(u_0)\|^2$ but independent of ε, λ such that*

$$\int_0^T \|\varepsilon u_\lambda''(t)\|^2\,\mathrm{d}t+\int_0^T \|\xi_\lambda(t)\|^2\,\mathrm{d}t \leq M_2, \tag{3.12}$$

$$\int_0^T \|\mathrm{D}\varphi_\lambda^2(u_\lambda(t))\|^2\,\mathrm{d}t \leq M_2. \tag{3.13}$$

Proof. The maximal regularity estimate (3.7) entails that

$$\begin{aligned}&\|\varepsilon u_\lambda''\|_{\mathcal{H}}^2+\|u_\lambda'\|_{\mathcal{H}}^2+\|\xi_\lambda\|_{\mathcal{H}}^2\\&\leq \|\mathrm{D}\varphi_\lambda^2(u_\lambda(\cdot))+f\|_{\mathcal{H}}^2+\varepsilon\|(\partial\varphi^1)^\circ(u_0)\|^2+2\varphi^1(u_0)\\&\overset{\text{(A4)}}{\leq} k_2\|\xi_\lambda\|_{\mathcal{H}}^2+\ell_2\left(\sup_{t\in[0,T]}\|u_\lambda(t)\|\right)\left(\int_0^T \varphi^1(u_\lambda(t))\,\mathrm{d}t+T\right)\\&\quad+\int_0^T \left(2\|\mathrm{D}\varphi_\lambda^2(u_\lambda(t))\|\|f(t)\|+\|f(t)\|^2\right)\,\mathrm{d}t+\varepsilon\|(\partial\varphi^1)^\circ(u_0)\|^2+2\varphi^1(u_0).\end{aligned}$$

Hence, by using (A4) once more and using Young's inequality for the third term in the right-hand side above, we derive from (3.8)-(3.10) with (A1), in particular $u_0 \in D(\partial\varphi^1)$, that

$$\|\varepsilon u_\lambda''\|_{\mathcal{H}}^2+\|u_\lambda'\|_{\mathcal{H}}^2+\|\xi_\lambda\|_{\mathcal{H}}^2 \leq C,$$

which together with (A4) implies

$$\int_0^T \|\mathrm{D}\varphi_\lambda^2(u_\lambda(t))\|^2\,\mathrm{d}t \leq C.$$

Note that C depends only on $M_1, k_2, T, \ell_2(M_1), \|f\|_{\mathcal{H}}, \varphi^1(u_0)$ and $\varepsilon\|(\partial\varphi^1)^\circ(u_0)\|^2$. □

REMARK 3.7. The assumption $u_0 \in D(\partial\varphi^1)$ of (A1) is only used to derive the uniform estimates of Lemma 3.6, where the boundedness of $\varepsilon\|(\partial\varphi^1)^\circ(u_0)\|^2$ is crucial.

In fact, in the same spirit of [26] this assumption can be relaxed as

$$u_0 \in \overline{D(\partial\varphi^1)} \text{ and there exist } u_{0,\varepsilon} \in D(\partial\varphi^1) \text{ satisfying } u_{0,\varepsilon} \to u_0 \text{ in } H \\ \text{and } \varphi^1(u_{0,\varepsilon}) + \varepsilon\|\partial\varphi^1(u_{0,\varepsilon})\|^2 \le C$$

by replacing u_0 of the minimization (1.4) with $u_{0,\varepsilon}$. Such data u_0 correspond to the nonlinear interpolation class $\mathscr{B}_{1/2,\infty}(\partial\varphi^1)$ between $D(\partial\varphi^1)$ and its closure in H introduced by D. Brézis [8] (see also [10] and [32]).

Let us now move on to the limit passage as $\lambda \to 0$. Owing to the obtained uniform estimates, up to some not relabeled subsequence, we have

$$\begin{aligned} u_\lambda \to u &\quad \text{weakly in } W^{2,2}(0,T;H), \\ \xi_\lambda \to \xi &\quad \text{weakly in } L^2(0,T;H), \\ \mathrm{D}\varphi^2_\lambda(u_\lambda(\cdot)) \to \eta &\quad \text{weakly in } L^2(0,T;H). \end{aligned}$$

Using (A2) together with (3.9) and (3.10) and by virtue of the classical Aubin-Lions-Simon Lemma [39, Thm. 3], we have

$$u_\lambda \to u \quad \text{strongly in } C([0,T];H).$$

From the demiclosedness of maximal monotone operators, we also obtain the identification $[u(t),\xi(t)] \in \partial\varphi^1$. The inclusion $[u(t),\eta(t)] \in \partial\varphi^2$, for a.e. $t \in (0,T)$ follows then by a standard monotonicity argument (see, e.g., §1.2 of [6, Chap.II]). Since X is compactly embedded in H (equivalently, $H \equiv H^* \hookrightarrow X^*$ compactly), possibly extracting a further subsequence we derive from (3.8) and (3.12) that

$$u'_\lambda \to u' \quad \text{strongly in } C([0,T];X^*).$$

In particular, $u'(T) = 0$. We hence conclude that u solves (3.4)-(3.5) in the strong sense.

4. Minimization of WED functionals

Let us eventually turn our attention to the minimization of WED functionals $\mathcal{W}_\varepsilon$ over $\mathcal{H}$. Assumption (A2) and the lower bound (2.2) entail coercivity for $\mathcal{W}_\varepsilon$ in $W^{1,2}(0,T;H) \cap L^1(0,T;X)$ (see (3.3)). Still, the functional $\mathcal{W}_\varepsilon$ might not be lower semicontinuous, so that the Direct Method of the calculus of variations cannot be directly used. We resort instead to proving that some solution of the Euler-Lagrange system is indeed a global minimizer of $\mathcal{W}_\varepsilon$.

Theorem 4.1 (Minimization of WED functionals). *For each $\varepsilon > 0$ fixed, let u_ε be the strong solution of* (1.5)-(1.6) *obtained in Theorem 3.2. Then, u_ε is a global minimizer of $\mathcal{W}_\varepsilon$ on $\mathcal{H}$.*

Proof. Let u_λ be a global minimizer of $\mathcal{W}_{\varepsilon,\lambda}$ on $\mathcal{H}$, namely

$$\mathcal{W}_{\varepsilon,\lambda}(u_\lambda) \le \mathcal{W}_{\varepsilon,\lambda}(v) \quad \text{for all } v \in \mathcal{H}. \tag{4.1}$$

We note that, for any $v \in D(\mathcal{W}_\varepsilon) \subset D(\mathcal{C}^1_\varepsilon)$,

$$\mathcal{W}_{\varepsilon,\lambda}(v) = \mathcal{C}^1_\varepsilon(v) - \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \varphi^2_\lambda(v(t))\, \mathrm{d}t \to \mathcal{W}_\varepsilon(v) \quad \text{as} \quad \lambda \to 0,$$

since $\varphi^2_\lambda(v(t)) \to \varphi^2(v(t))$ for a.e. $t \in (0,T)$ and $\varphi^2_\lambda(v(\cdot)) \le \varphi^2(v(\cdot)) \in L^1(0,T)$. As for the left-hand-side of (4.1), by the convergences obtained in §3.2 and (3.2), we infer that

$$\liminf_{\lambda \to 0} \mathcal{W}_{\varepsilon,\lambda}(u_\lambda) \ge \liminf_{\lambda\to 0} \mathcal{C}^1_\varepsilon(u_\lambda) - \limsup_{\lambda \to 0} \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \varphi^2(u_\lambda(t))\, \mathrm{d}t.$$

By the definition of subdifferential, we note that

$$\varphi^2(u_\lambda(t)) - \varphi^2(u(t)) \le \left(\mathrm{D}\varphi^2_\lambda(u_\lambda(t)), u_\lambda(t) - u(t)\right),$$

which implies

$$\limsup_{\lambda \to 0} \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \varphi^2(u_\lambda(t))\, \mathrm{d}t \le \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \varphi^2(u(t))\, \mathrm{d}t.$$

Moreover, by the lower semicontinuity of $\mathcal{C}^1_\varepsilon$ on $\mathcal{H}$,

$$\liminf_{\lambda \to 0} \mathcal{C}^1_\varepsilon(u_\lambda) \ge \mathcal{C}^1_\varepsilon(u).$$

We conclude that

$$\liminf_{\lambda\to 0} \mathcal{W}_{\varepsilon,\lambda}(u_\lambda) \ge \mathcal{W}_\varepsilon(u).$$

Consequently, we obtain $\mathcal{W}_\varepsilon(u) \le \mathcal{W}_\varepsilon(v)$ for all $v \in D(\mathcal{W}_\varepsilon)$. □

We next check that every (possibly local) minimizer of $\mathcal{W}_\varepsilon$ is a strong solution of the Euler-Lagrange equation.

THEOREM 4.2 (Minimizers solve Euler-Lagrange). *Let $\varepsilon > 0$ be fixed and let u_ε be a global or local minimizer of $\mathcal{W}_\varepsilon$ over $\mathcal{H}$. Then, u_ε is a strong solution of* (1.5)-(1.6) *on* $[0,T]$.

Proof. In case $\mathcal{W}_\varepsilon$ admits a unique minimizer, Theorem 4.1 ensures that it coincides with the strong solution of (1.5)-(1.6) whose existence has been proved in Theorem 3.2.

Let us define, for every $\delta > 0$,

$$\mathcal{W}^\delta_\varepsilon(u) := \mathcal{W}_\varepsilon(u) + \frac{1}{2\delta} \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \|u(t) - u_\varepsilon(t)\|^2 \, \mathrm{d}t \quad \text{for } u \in \mathcal{H}.$$

In case u_ε is a (possibly non-unique) global minimizer of $\mathcal{W}_\varepsilon$ over $\mathcal{H}$, we have

$$\mathcal{W}^\delta_\varepsilon(u_\varepsilon) = \mathcal{W}_\varepsilon(u_\varepsilon) < \mathcal{W}_\varepsilon(u) + \frac{1}{2\delta}\int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \|u(t) - u_\varepsilon(t)\|^2 \, \mathrm{d}t = \mathcal{W}^\delta_\varepsilon(u)$$

for all $u \in \mathcal{H} \setminus \{u_\varepsilon\}$ and any $\delta > 0$. Hence u_ε is the unique global minimizer of $\mathcal{W}^\delta_\varepsilon$.

In case u_ε is a local minimizer but not a global one, there exists $r > 0$ such that

$$\mathcal{W}_\varepsilon(u_\varepsilon) \le \mathcal{W}_\varepsilon(v) \quad \text{for all} \;\; v \in B(u_\varepsilon; r)$$

where $B(u_\varepsilon; r)$ indicates the neighborhood of u_ε given by

$$B(u_\varepsilon; r) := \left\{ v \in \mathcal{H} \colon \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \|u(t) - u_\varepsilon(t)\|^2 \,\mathrm{d}t < r^2 \right\}.$$

In particular, $B(u_\varepsilon; r)$ contains no global minimizer. Let us take $\delta > 0$ such that

$$m + \frac{r^2}{2\delta} > \mathcal{W}_\varepsilon(u_\varepsilon),$$

where m is the minimum value of $\mathcal{W}_\varepsilon$ over $\mathcal{H}$. Then, we observe that

$$\mathcal{W}_\varepsilon^\delta(u_\varepsilon) = \mathcal{W}_\varepsilon(u_\varepsilon) < m + \frac{r^2}{2\delta} \le \mathcal{W}_\varepsilon(u) + \frac{1}{2\delta} \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \|u(t) - u_\varepsilon(t)\|^2 \,\mathrm{d}t = \mathcal{W}_\varepsilon^\delta(u)$$

for all $u \notin B(u_\varepsilon; r)$. As for $u \in B(u_\varepsilon; r) \setminus \{u_\varepsilon\}$, one can readily check $\mathcal{W}_\varepsilon^\delta(u_\varepsilon) < \mathcal{W}_\varepsilon^\delta(u)$ as in the last case. Therefore, we conclude that

$$\mathcal{W}_\varepsilon^\delta(u_\varepsilon) < \mathcal{W}_\varepsilon^\delta(u) \quad \text{for all } u \in \mathcal{H} \setminus \{u_\varepsilon\}.$$

In particular, u_ε is the unique (global) minimizer of $\mathcal{W}_\varepsilon^\delta$ over $\mathcal{H}$. As we readily observe that

$$\begin{aligned} \mathcal{W}_\varepsilon^\delta(u) = \mathcal{W}_\varepsilon(u) &+ \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \left\{ \frac{1}{2\delta} \|u(t)\|^2 - \left(u(t), \frac{u_\varepsilon(t)}{\delta} \right) \right\} \mathrm{d}t \\ &+ \frac{1}{2\delta} \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \left\| u_\varepsilon(t) \right\|^2 \,\mathrm{d}t, \end{aligned}$$

the minimization of $\mathcal{W}_\varepsilon^\delta$ over $\mathcal{H}$ is equivalently rewritten as that of

$$\hat{\mathcal{W}}_\varepsilon^\delta(u) := \mathcal{W}_\varepsilon(u) + \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \left\{ \frac{1}{2\delta} \|u(t)\|^2 - \left(u(t), \frac{u_\varepsilon(t)}{\delta} \right) \right\} \mathrm{d}t,$$

which also satisfies assumptions (A1)-(A4) with $\varphi^1(u)$ and $f(t)$ replaced by $\widehat{\varphi}^1(u) := \varphi^1(u) + \|u\|^2/(2\delta)$ and $\widehat{f}(t) := f(t) + u_\varepsilon(t)/\delta$, respectively. Moreover, $\partial\widehat{\varphi}^1(u) = \partial\varphi^1(u) + u/\delta$. Therefore the unique global minimizer u_ε of $\widehat{\mathcal{W}}_\varepsilon^\delta$ is a strong solution of the Euler-Lagrange system for $\widehat{\mathcal{W}}_\varepsilon^\delta$, i.e.,

$$-\varepsilon u''(t) + u'(t) + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) + \frac{1}{\delta}(u(t) - u_\varepsilon(t)) \ni f(t).$$

By substituting $u = u_\varepsilon$, one deduces that u_ε solves (1.5)-(1.6) as well. □

5. Convergence of minimizers as $\varepsilon \to 0$

The focus of this section is to check that minimizers of $\mathcal{W}_\varepsilon$ converge to a solution of the gradient flow (1.2)-(1.3) as $\varepsilon \to 0$. Let us first introduce the following:

Definition 5.1 (Strong solution of the gradient flow). *A function $u \in W^{1,2}(0,T;H)$ is called a* strong solution *of* (1.2), (1.3) *if*

(i) *$u(t) \in D(\partial\varphi^1)$ for a.e. $t \in (0,T)$.*

(ii) *There exists functions* $\xi, \eta \in L^2(0,T;H)$ *such that*

$$u'(t) + \xi(t) - \eta(t) = f(t), \quad \xi(t) \in \partial\varphi^1(u(t)), \quad \eta(t) \in \partial\varphi^2(u(t)) \quad \textit{in } H$$

for a.e. $t \in (0,T)$.

(iii) $u(0) = u_0$.

Our convergence result reads as follows:

THEOREM 5.2 (Convergence of minimizers). *Let* $\varepsilon_n \to 0$ *and let* u_{ε_n} *be global or local minimizers of* $\mathcal{W}_{\varepsilon_n}$ *over* $\mathcal{H}$. *Then, there exists a not relabeled subsequence such that*

$$u_{\varepsilon_n} \to u \quad \textit{weakly in } W^{1,2}(0,T;H) \textit{ and strongly in } C([0,T];H)$$

where u *is a strong solution of the gradient flow* (1.2)-(1.3).

Proof. For simplicity, we write ε instead of ε_n. Let u_ε be global or local minimizers of $\mathcal{W}_\varepsilon$ over $\mathcal{H}$. Then by Theorem 4.2, they are strong solutions of the Euler-Lagrange system (1.5)-(1.6). Recalling Lemmas 3.5 and 3.6 along with the limiting procedure in §3, one can derive

$$\varepsilon\|u_\varepsilon'(0)\|^2 + \int_0^T \|u_\varepsilon'(t)\|^2 \, \mathrm{d}t + \int_0^T \varphi^1(u_\varepsilon(t)) \, \mathrm{d}t \le C, \tag{5.1}$$

$$\sup_{t \in [0,T]} \|u_\varepsilon(t)\| \le C, \tag{5.2}$$

$$\int_0^T \|\varepsilon u_\varepsilon''(t)\|^2 \, \mathrm{d}t + \int_0^T \|\xi_\varepsilon(t)\|^2 \, \mathrm{d}t \le C, \tag{5.3}$$

$$\int_0^T \|\eta_\varepsilon(t)\|^2 \, \mathrm{d}t \le C. \tag{5.4}$$

Here, $\xi_\varepsilon(t)$ and $\eta_\varepsilon(t)$ are sections of $\partial\varphi^1(u_\varepsilon(t))$ and $\partial\varphi^2(u_\varepsilon(t))$, respectively, as in (3.1). By taking not relabeled subsequences one has

$$\begin{aligned} u_\varepsilon &\to u \quad \text{weakly in } W^{1,2}(0,T;H) \text{ and strongly in } C([0,T];H), \\ \xi_\varepsilon &\to \xi \quad \text{weakly in } L^2(0,T;H), \\ \eta_\varepsilon &\to \eta \quad \text{weakly in } L^2(0,T;H). \end{aligned}$$

Moreover, by the demiclosedness of maximal monotone operators, we also conclude that $[u(t), \xi(t)] \in \partial\varphi^1$ and $[u(t), \eta(t)] \in \partial\varphi^2$ for a.e. $t \in (0,T)$. This completes the proof. □

6. Applications to nonlinear PDEs

We shall now present some applications of the abstract theory to classes of nonlinear parabolic equations with nonmonotone terms. The crucial point is of course to fulfill the control conditions (A3)-(A4). We illustrate here three examples, showing indeed such control. All examples consists of parabolic equations of the form

$$u_t - \operatorname{div}\gamma(x,\nabla u) + g(u) = 0$$

for possibly nonlinear functions γ and g (precise assumptions and statements are below).

In Subsection 6.1 we give the detail of a quasilinear example where the balancing of monotone and nonmonotone terms occurs within in the term $g(u)$ only. This corresponds to the case of Allen-Cahn equations. Subsection 6.2 is then devoted to the case when the whole term $g(u)$ is antimonotone and gets balanced by the term $\operatorname{div}\gamma(\nabla u)$, corresponding indeed to the case of semilinear heat equations. Eventually, Subsection 6.3 deals with quasilinear equations where γ is itself nonmonotone. In particular, conditions (A3)-(A4) are there realized by splitting the term $\operatorname{div}\gamma(\nabla u)$ into a linear monotone and a smooth nonmonotone part. It should be clear that these examples are meant to illustrate the applicability of the theory and can indeed be extended and combined in different ways.

Throughout this section, let Ω be a nonempty, open, connected, and bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

6.1. **Quasilinear Allen-Cahn equations.** We consider the problem

$$\begin{aligned} \partial_t u - \Delta_p u + W'(u) &= f \quad \text{in } \ \Omega\times(0,T), && (6.1)\\ u = 0 \quad \text{or} \quad \partial_\nu u &= 0 \quad \text{on } \ \partial\Omega\times(0,T), && (6.2)\\ u(\cdot,0) &= u_0 \quad \text{in } \ \Omega, && (6.3) \end{aligned}$$

where $\partial_t = \partial/\partial t$, $f = f(x,t)$ is given, $\partial_\nu u$ denotes the outer normal derivative of u on $\partial\Omega$, and W is the double-well potential given by

$$W(u) := \frac{1}{m}|u|^m - \frac{1}{q}|u|^q \quad \text{for } \ u\in\mathbb{R}$$

with exponents $1 < q < m < \infty$. Then $W'(u) = |u|^{m-2}u - |u|^{p-2}u$. Here Δ_p is the so-called *p-Laplacian* given by

$$\Delta_p u := \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right), \quad 1<p<\infty.$$

In order to reduce the Allen-Cahn system (6.1)–(6.3) to our abstract setting, let $H = L^2(\Omega)$ and

$$X = \begin{cases} W^{1,p}_0(\Omega)\cap L^m(\Omega) & \text{for the Dirichlet case,}\\ W^{1,p}(\Omega)\cap L^m(\Omega) & \text{for the Neumann case} \end{cases}$$

endowed with the norm

$$\|u\|_X := \left(\|u\|^2_{L^m(\Omega)} + \|\nabla u\|^2_{L^p(\Omega)}\right)^{1/2}.$$

Define the convex functionals φ^1, φ^2 on H by

$$\varphi^1(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x + \dfrac{1}{m}\int_\Omega |u|^m \,\mathrm{d}x & \text{if } \ u \in X, \\ +\infty & \text{else} \end{cases}$$

and

$$\varphi^2(u) = \begin{cases} \dfrac{1}{q}\displaystyle\int_\Omega |u|^q \,\mathrm{d}x & \text{if } \ u \in L^q(\Omega), \\ +\infty & \text{else.} \end{cases}$$

Then, φ^1 and φ^2 are lower semicontinuous on H with $D(\varphi^1) = X$ and $D(\varphi^2) = L^q(\Omega)$. Moreover, $\partial\varphi^1(u)$ and $\partial\varphi^2(u)$ coincide with $-\Delta_p u + |u|^{m-2}u$ and $|u|^{q-2}u$, respectively, in the sense of distribution.

The Euler-Lagrange equation (1.5)-(1.6) can be written in the following form

$$-\varepsilon\partial_t^2 u_\varepsilon + \partial_t u_\varepsilon - \Delta_p u_\varepsilon + W'(u_\varepsilon) = f \quad \text{in } \ \Omega \times (0,T), \tag{6.4}$$

$$u_\varepsilon = 0 \quad \text{or} \quad \partial_\nu u_\varepsilon = 0 \quad \text{on } \ \partial\Omega \times (0,T), \tag{6.5}$$

$$u_\varepsilon(\cdot,0) = u_0 \quad \text{in } \ \Omega. \tag{6.6}$$

We make the following assumption on data.

(H1): $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in X$ satisfies $-\Delta_p u_0 + |u_0|^{m-2}u_0 \in L^2(\Omega)$, and

$$q < m, \quad 2 < \max\{m, p^*\} \quad \text{with} \quad p^* := \begin{cases} \dfrac{Np}{N-p} & \text{if } \ p < N, \\ +\infty & \text{else.} \end{cases} \tag{6.7}$$

Assumption (A1) obviously holds. Then, (A2) is easily checked, because the boundedness of $\varphi^1(u_n)$ and (6.7) entail that (u_n) is precompact in $H = L^2(\Omega)$. One can derive (A3) from the fact that $q < m$. Finally, let us comment on the validity of (A4). For any $k_2 \in (0,1)$, using $q < m$, one can take $C_2 > 0$ such that

$$\big\||u|^{q-2}u\big\|^2 = \|u\|_{L^{2(q-1)}(\Omega)}^{2(q-1)} \le k_2 \|u\|_{L^{2(m-1)}(\Omega)}^{2(m-1)} + C_2 = k_2 \big\||u|^{m-2}u\big\|^2 + C_2$$

for any $u \in L^{2(m-1)}(\Omega)$. On the other hand

$$\big\||u|^{m-2}u\big\|^2 + \|\Delta_p u\|^2 \le \big\|-\Delta_p u + |u|^{m-2}u\big\|^2 \quad \text{for all } \ u \in D(\partial\varphi^1).$$

Hence (A4) holds true.

By virtue of Theorems 3.2, 4.1, 4.2 and 5.2, we can hence conclude the following:

THEOREM 6.1 (Quasilinear Allen-Cahn equations). *Assume* (H1). *Then, the following* (i)-(iii) *hold.*

(i) *For every $\varepsilon > 0$, the corresponding* WED *functional $\mathcal{W}_\varepsilon$ admits a global minimizer.*
(ii) *Let u_ε be a global or local minimizer of $\mathcal{W}_\varepsilon$. Then, u_ε strongly solves the Euler-Lagrange equation* (6.4)-(6.6) *in $\Omega \times (0,T)$.*

(iii) *For every sequence* $\varepsilon_n \to 0$ *there exists a not relabeled subsequence* u_{ε_n} *such that* $u_{\varepsilon_n} \to u$ *weakly in* $W^{1,2}(0,T;L^2(\Omega)) \cap L^p(0,T;Z) \cap L^m(0,T;L^m(\Omega))$ *and strongly in* $C([0,T];L^2(\Omega))$ *(where we set* $Z := W^{1,p}(\Omega)$ *or* $Z := W_0^{1,p}(\Omega)$ *for Neumann or Dirichlet boundary conditions, respectively). The limit* u *strongly solves the Allen-Cahn problem* (6.1)-(6.3) *in* $\Omega \times (0,T)$.

Remark 6.2. The convex decomposition of the double-well potential W can be applied to more general settings. Indeed, let $f := W'$ be of class C^1 in $\mathbb{R}$ and decompose its derivative f' as follows:

$$f' = (f')_+ - (-f')_+,$$

where $(\cdot)_+$ stands for the positive part. As we clearly observe that

$$f(u(x)) = f(0) + \int_0^{u(x)} (f'(s))_+ \,\mathrm{d}s - \int_0^{u(x)} (-f'(s))_+ \,\mathrm{d}s,$$

one can define two functionals φ^1 and φ^2 as $\partial\varphi^1(u)(x) = -\Delta_p u(x) + f(0) + \int_0^{u(x)} (f'(s))_+ \,\mathrm{d}s$ and $\partial\varphi^2(u)(x) = \int_0^{u(x)} (-f'(s))_+ \,\mathrm{d}s$.

6.2. **Sublinear heat equations.** Consider the following problem

$$\partial_t u - \Delta u - |u|^{q-2}u = f \quad \text{in } \Omega \times (0,T), \tag{6.8}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{6.9}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega \tag{6.10}$$

with $1 < q < 2$. Suppose that

(H2): $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $1 < q < 2$.

Let $H = L^2(\Omega)$ and $X = H_0^1(\Omega)$ and define φ^1 on H by

$$\varphi^1(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 \,\mathrm{d}x & \text{if } u \in X, \\ +\infty & \text{else,} \end{cases} \tag{6.11}$$

and φ^2 as in Subsection 6.1. Then assumptions (A1) and (A2) can be checked as in §6.1. By virtue of the continuous embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ and Young's inequality, for any $k_1 \in (0,1)$ there exists $C_1 \geq 0$ such that

$$\varphi^2(u) = \frac{1}{q}\|u\|^q_{L^q(\Omega)} \leq C\|\nabla u\|^q \leq k_1\varphi^1(u) + C_1 \quad \text{for all } u \in X.$$

Thus (A3) follows. As for (A4), for any $k_2 \in (0,1)$, one can take $C_2 \geq 0$ such that

$$\begin{aligned} \left\||u|^{q-2}u\right\|^2 &= \|u\|^{2(q-1)}_{L^{2(q-1)}(\Omega)} \\ &\leq C\|u\|^{2(q-1)}_{H^2(\Omega)} \\ &\leq C\left\|\Delta u\right\|^{2(q-1)} \leq k_2\left\|\Delta u\right\|^2 + C_2 \end{aligned}$$

as $q < 2$. We used here the continuous embedding $H^2(\Omega) \hookrightarrow L^{2(q-1)}(\Omega)$ as well as elliptic estimates.

The Euler-Lagrange equations for WED functionals $\mathcal{W}_\varepsilon$ in the current setting can be written in the form:

$$-\varepsilon\partial_t^2 u_\varepsilon + \partial_t u_\varepsilon - \Delta u_\varepsilon - |u_\varepsilon|^{q-2}u_\varepsilon = f \quad \text{in } \Omega\times(0,T), \tag{6.12}$$

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{6.13}$$

$$u_\varepsilon(\cdot,0) = u_0 \quad \text{in } \Omega. \tag{6.14}$$

THEOREM 6.3 (Sublinear heat equation). *Assume* (H2). *Then, the same conclusions of Theorem* 6.1 *hold. More precisely, for any* $\varepsilon > 0$ *there exists a global minimizer of the corresponding* WED *functional* $\mathcal{W}_\varepsilon$. *Moreover, any* (*local or global*) *minimizer* u_ε *solves* (6.12)-(6.14). *Furthermore, for any* $\varepsilon_n \to 0$, *one can take a nonrelabeled subsequence* u_{ε_n} *such that* $u_{\varepsilon_n} \to u$ *weakly in* $W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ *and strongly in* $C([0,T];L^2(\Omega))$ *and the limit* u *is a strong solution of the sublinear heat problem* (6.8)-(6.10).

6.3. **Quasilinear equations.** Let us consider the following

$$\partial_t u - \Delta u + k\mathrm{div}\,(b(x,\nabla u)) = f \quad \text{in } \Omega\times(0,T), \tag{6.15}$$

$$u = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{6.16}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega. \tag{6.17}$$

Here, we assume that

(H3): the field $b = (b_i) : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies

$$|\mathrm{div}_x b(x,\xi)| \le C(|\xi|+1), \quad |\nabla_\xi b(x,\xi)| \le C, \tag{6.18}$$

where $\mathrm{div}_x b(x,\xi) := \sum_{i=1}^N \partial b_i(x,\xi)/\partial x_i$ and $\nabla_\xi b(x,\xi) := (\partial b_i(x,\xi)/\partial\xi_j)$, for all $\xi\in\mathbb{R}^N$ and a.e. $x\in\Omega$, and has the form $b(x,\xi) = \nabla_\xi F(x,\xi)$ where $F : \Omega\times\mathbb{R}^n \to [0,+\infty)$ is such that $F \in C^{1,1}(\Omega\times\mathbb{R}^n)$, $F(x,\cdot)$ is convex and $F(x,0) = 0$ for all $x\in\Omega$, $|F(x,\xi)| \le C_0(1+|\xi|^2)$ for some $C_0 > 0$, and $u_0 \in H^2(\Omega)\cap H^1_0(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$.

Then, by letting $H = L^2(\Omega)$ and $X = H^1_0(\Omega)$ (other boundary conditions would also be amenable), defining φ^1 as in (6.11), and letting

$$\varphi^2(u) = \begin{cases} k\displaystyle\int_\Omega F(x,\nabla u(x))\,\mathrm{d}x & \text{if } u\in X,\\ +\infty & \text{else,}\end{cases}$$

we can reformulate (6.15)-(6.17) in the abstract form of (1.2)-(1.3). Our abstract result applies whenever the constant $k \ge 0$ is chosen to be small enough. Indeed, assumptions (A1) and (A2) follow as in Subsection 6.1 and (A3) is immediate as soon as $k < 1/(2C_0)$. As for (A4) we exploit (6.18) and compute almost everywhere

$$\begin{aligned}|\mathrm{div}\, b(x,\nabla u(x))| &= \left|\mathrm{div}_x\, b(x,\nabla u(x)) + \nabla_\xi b(x,\nabla u(x)) : \mathrm{D}^2 u(x)\right|\\ &\le C(1+|\nabla u(x)| + |\mathrm{D}^2 u(x)|),\end{aligned}$$

where $\mathrm{D}^2u(x)$ stands for the Hessian matrix of $u(x)$. It follows that

$$\|\mathrm{div}\, b(\cdot, \nabla u(\cdot))\|^2 \leq C_3 \left(1 + \|\nabla u\|^2 + \|\Delta u\|^2\right)$$

for some $C_3 > 0$. Hence, Assumption (A4) is fulfilled as soon as $k < 1/C_3$.

The Euler-Lagrange equations for WED functionals $\mathcal{W}_\varepsilon$ in the current setting can be written in the form:

$$-\varepsilon\partial_t^2 u_\varepsilon + \partial_t u_\varepsilon - \Delta u_\varepsilon - k\mathrm{div}\, b(x, \nabla u_\varepsilon) = f \quad \text{in } \Omega \times (0,T), \tag{6.19}$$

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{6.20}$$

$$u_\varepsilon(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{6.21}$$

Theorem 6.4 (Quasilinear equation). *Assume* (H3) *and take* $k < 1/\max\{2C_0, C_3\}$. *Then, the same conclusions of Theorem* 6.1 *hold. More precisely, for any* $\varepsilon > 0$ *there exists a global minimizer of the corresponding* WED *functional* $\mathcal{W}_\varepsilon$. *Moreover, any* (*local or global*) *minimizer* u_ε *solves* (6.19)-(6.21). *Furthermore, for any* $\varepsilon_n \to 0$, *one can take a nonrelabeled subsequence* u_{ε_n} *such that* $u_{\varepsilon_n} \to u$ *weakly in* $W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ *and strongly in* $C([0,T];L^2(\Omega))$ *and the limit* u *is a strong solution of the quasilinear problem* (6.15)-(6.17).

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