Recent techniques for PDE discretizations on polyhedral meshes

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This brief paper is an introduction to the papers published in a special issue devoted to review papers on recent techniques for discretizing Partial Differential Equations on general polygonal and polyhedral meshes. The number of different techniques to deal with discretizations on polygonal and polyhedral meshes is quite huge, and their history is quite long. Here we concentrate on the most recent techniques, including Mimetic Finite Differences, Virtual Element Methods, and the recent developments, in this directions, of Finite Volumes and Discontinuous Galerkin Methods.

Keywords: Polygonal meshes; polyhedral meshes; Mimetic Finite Differences; Virtual Element Methods; Finite Volumes; discontinuous Galerkin methods.

1. Preface

Several numerical methods for the approximation of Partial Differential Equations using polygonal and polyhedral meshes have been introduced and used since almost half a century. We just recall here the books,22,19 and the review papers,7,2,20,14,15,17

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In recent times the matter received an increasing attention, due to the combination of several factors that include the convenience in mesh generation, mesh deformations, fracture problems, composite materials, topology optimizations, mesh refinements and coarsening, and the like. Recent developments include the evolution of Mimetic Finite Differences in the direction of nodal unknowns or edge unknowns, the connections with Finite Volumes methods (see e.g. Ref. 13), and, possibly, the recent appearance of the Virtual Element Methods paradigm that allows a simpler conception, a much easier analysis, and a more straightforward generalization of several previous approaches. The present special issue is an attempt to present a variety of different directions that characterize these recent developments. Some of the papers included here are mostly review papers. These include the review on Finite Volumes for diffusion equations, the review on discontinuous Galerkin methods on very general geometries (including disconnected elements), and the guide on the actual implementation of Virtual Element Methods.

Reference 18 presents an overview and some new perspectives on the use of generalized barycentric coordinates in 2D and 3D diffusion problems and Reference 21 addresses the issue related to numerical integration on general polygonal meshes. Both papers discuss (among several other things) the idea of combining classical numerical integration with the exact integration made available (on some suitable part of the stiffness matrix) by the VEM approach for the satisfaction of the patch-test for the resulting numerical methods. The importance of an exact satisfaction of the patch-test is underlined in Ref 21. Extensions to nonlinear problems (as quasilinear equations or variational inequalities) are discussed in Ref. 1, while possible additional uses of the free parameters allowed by Mimetic Finite Differences (and Virtual Element Methods) are the object of Ref. 16.

In suggesting and collecting papers for this special issue we faced several difficult choices (of topics and authors). We had to neglect relevant applications as graphics, as well as very interesting theoretical works, as Ref. 10 on hp discontinuous Galerkin methods of polygonal meshes. Finally, the number of references that would have been appropriate for this Preface was indeed much bigger than that present here. We refer instead to the (numerous) references contained in the papers of this special issue.

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