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VIRTUAL ELEMENT METHOD FOR PLATE BENDING PROBLEMS

FRANCO BREZZI¹,²,³ AND L. DONATELLA MARINI⁴,²

Abstract. We discuss the application of Virtual Elements to linear plate bending problems, in the Kirchhoff-Love formulation. As we shall see, in the Virtual Element environment the treatment of the $C^1$-continuity condition is much easier than for traditional Finite Elements. The main difference consists in the fact that traditional Finite Elements, for every element $K$ and for every given set of degrees of freedom, require the use of a space of polynomials (or piecewise polynomials for composite elements) for which the given set of degrees of freedom is unisolvent. For Virtual Elements instead we only need unisolvence for a space of smooth functions that contains a subset made of polynomials (whose degree determines the accuracy). As we shall see the non-polynomial part of our local spaces does not need to be known in detail, and therefore the construction of the local stiffness matrix is simple, and can be done for much more general geometries.

Keywords: High-order MFD; Plate bending problems; Virtual Elements

1. Introduction

The Finite Element literature of the last fifty years contains a big variety of $H^1$-conforming (in practice, $C^0$) elements of various degrees, with different features. On the other hand, the list of available $H^2$-conforming (in practice, $C^1$) elements is much more slim. Among the most commonly used are the composite Hsieh-Clough-Tocher element (with cubic accuracy) and its reduced version (with quadratic accuracy), the Argyris element (with quintic accuracy) and its reduced Bell version (with quartic accuracy). But a much bigger effort has been devoted to alternative formulations that could avoid the use of $C^1$ finite elements, including mixed formulations (essentially, based on the Hellinger-Reissner principle) or the use of the Reissner-Mindlin model for thin plates and shells (instead of moderately thick). Dual hybrid elements of Pian and co-authors (see e.g. [32], [33]) could be seen as intermediate $C^1$ formulations, as the displacements are indeed $C^1$ but they are defined only at the interelement boundaries (see also [11] and in particular [17]). And several among the most common and useful applications of nonconforming elements (as for instance the Morley element) can indeed be seen as an effort to bypass $C^1$ continuity.

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We refer for instance to the books [1], [24], [6], [39] (or, for more mathematically oriented ones, to [18], [10], [9]) and to the references therein.

In this paper we want to show that the newly born Virtual Element Method [4] is able to tackle the construction of $C^1$-approximations in a much swifter way, and on very general polygonal elements, and we propose a few examples of elements that could surely be interesting in the approximation of the solution of plate bending problems in the Kirchhoff-Love formulation.

Virtual Element Methods could be seen as an evolution of Mimetic Finite Differences and related methods (see e.g. [23], [25], [28], [29], or the more recent [8], [19] [20], [37]). In particular they are strictly related to the more theoretical versions of MFD ([26], [14], [16] [15] [12]), and specially to their last versions concerning higher order methods ([31], [22] [3], [2]).

As we shall see, the basic idea consists in choosing first some degrees of freedom at the interelement boundaries that could identify in a unique way the traces of globally $C^1$ functions that are polynomials of degree $\leq r$ on each edge, with normal derivative of degree $\leq s$ on each edge. Then we add a suitable amount of internal degrees of freedom, and we define the discrete subspace inside the elements by means of a local plate bending problem. Needless to say, the actual solution of these local problems will not be required, not even in an approximate way.

Actually, we will show that, having constructed the discrete subspace $V_h$ as above, if the boundary and internal degrees of freedom are chosen properly, then

a) in each element $K$, the restriction of the discrete space to $K$ contains the space $\mathbb{P}_k$ of polynomials of degree $\leq k$, where $k$ depends on $r$ and $s$,

b) for every polynomial $p_k \in \mathbb{P}_k(K)$, and for every non-polynomial element $v_h$ of $V_h$, the contribution of $K$ to the energy bilinear form $a(p_k, v_h)$ can be computed exactly, using only the degrees of freedom of $v_h$ (and not the fact that $v_h$ solves a local plate problem),

c) we can then easily complete the computation of the energy bilinear form $a(v_h, w_h)$ (when neither $v_h$ nor $w_h$ is a polynomial in $\mathbb{P}_k$) in an almost arbitrary way, in order to ensure stability.

Note that, if we have a), b), and c) we can ensure a sort of $k-th$ order patch test, meaning that, on any patch of elements: if the true solution is a global polynomial of degree $\leq k$, then our discrete solution will coincide exactly with the true solution.

For other attempts to deal with elements of a general shape we refer to [5], [7], [21], [30], [34], [35], [36], [38] and the references therein. We point out, however, that here we do not use numerical integration.

An outline of the paper is as follows. In Section 2 we recall the continuous problem and fix some notation. In Section 3 we present in an abstract framework the Virtual Element approach, state the basic assumptions and prove a convergence theorem. In Section 4 we explicitly show how to construct a $C^1$ VEM-approximation of the plate
bending problem with the optimal convergence rate, namely, order $k$, with $k \geq 2$ whenever the discrete space $V_h$ contains locally the polynomials of degree $k$.

Throughout the paper we shall use the common notation for the Sobolev spaces $H^m(\mathcal{D})$ for $m$ a nonnegative integer and $\mathcal{D}$ an open bounded domain. In particular (see e.g. [27], [18]) the $L^2(\mathcal{D})$ scalar product and norm will be indicated by $(\cdot, \cdot)_{0,\mathcal{D}}$ or $(\cdot, \cdot)_\mathcal{D}$ and $\| \cdot \|_{0,\mathcal{D}}$ or $\| \cdot \|_\mathcal{D}$, respectively. Moreover, for $m$ a nonnegative integer, the $m$th seminorm of the function $\varphi$ will be defined by

$$\|\varphi\|_{m,\mathcal{D}} := \sum_{|\alpha|=m} \left| \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right|^2_{0,\mathcal{D}}$$

where for the nonnegative multi-index $\alpha = (\alpha_1, \alpha_2)$ we denoted as usual $|\alpha| = \alpha_1 + \alpha_2$.

When $\mathcal{D} \equiv \Omega$ the subscript $\mathcal{D}$ will often be omitted.

2. The Continuous Problem

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain occupied by the plate, let $\Gamma$ be its boundary, and let $f \in L^2(\Omega)$ be a transversal load acting on the plate. The Kirchoff-Love model for thin plates (see e.g. [18]) corresponds to look for the transversal displacement $w$ solution of

$$D \Delta^2 w = f \quad \text{in } \Omega,$$

where $D = Et^3/12(1 - \nu^2)$ is the bending rigidity, $t$ the thickness, $E$ the Young modulus, and $\nu$ the Poisson’s ratio.

Assuming for instance the plate to be clamped all over the boundary, equation (2.1) is supplemented with the boundary conditions

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma.$$

The variational formulation of (2.1)-(2.2) is:

(2.3) \[ \left\{ \begin{array}{l} \text{Find } w \in V := H^2_0(\Omega) \text{ solution of } \\
 a(w,v) = (f,v) \quad \forall v \in H^2_0(\Omega), \end{array} \right. \]

where, as we said, $(\cdot, \cdot)$ is the usual scalar product in $L^2(\Omega)$, and the energy bilinear form $a(\cdot, \cdot)$ is given by

$$a(w,v) = D \left[ (1 - \nu) \int_\Omega \frac{\partial^2 w}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} \, dx + \nu \int_\Omega \Delta w \Delta v \, dx \right].$$

In (2.4) $v_{ij} = \partial v/\partial x_i$, $i = 1,2$, and we used the summation convention of repeated indices. Thus, for instance:

$$w_{ij} v_{ij} = (w_{11} v_{11} + 2w_{12} v_{12} + w_{22} v_{22}).$$

Setting $\|v\|_V := \|v\|_{2,\Omega}$, it is easy to see that, thanks to the boundary conditions in $V$ and to the Poincaré inequality, this is indeed a norm on $V$. Moreover

$$\exists M > 0 \quad \text{such that } \quad a(u,v) \leq M \|u\|_V \|v\|_V \quad u, v \in V,$$
\[ \exists \alpha > 0 \text{ such that } a(v, v) \geq \alpha \|v\|_V^2, \quad v \in V. \]

Hence, (2.3) has a unique solution, and (see, e.g. [27])

\[ \|w\|_V \leq C\|f\|_{L^2(\Omega)}. \]

Before going to the discretization of (2.3) let us fix some notation that will be used later on. On a domain \( D \subset \mathbb{R}^2 \), with boundary \( \partial D \), we denote by \( \mathbf{n} = (n_1, n_2) \) the outward unit normal vector to \( \partial D \), and by \( \mathbf{t} = (t_1, t_2) \) the unit tangent vector in the counterclockwise ordering of the boundary. For \( v \in H^2(\Omega) \), let \( M = M_{ij}(v) \ i, j = 1, 2 \) be the moment tensor given by the stress-strain relation

\[
\begin{bmatrix}
M_{11} \\
M_{22} \\
M_{12}
\end{bmatrix} = D
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1 - \nu
\end{bmatrix}
\begin{bmatrix}
v_{/11} \\
v_{/22} \\
v_{/12}
\end{bmatrix},
\]

and let \( M_n = M_{ij}n_j \) be its normal vector. We then denote by \( M_{nn}(v) := M_{ij}n_in_j \) the normal bending moment, by \( M_{nt}(v) := M_{ij}n_it_j \) the twisting moment, and by \( Q_n(v) := M_{ij}/n_j \) the normal shear force.

Always for a generic domain \( D \) we also set, as in (2.4)

\[ a^D(w, v) = D \left[ (1 - \nu) \int_D w_{/ij}v_{/ij} \, dx + \nu \int_D \Delta w \Delta v \, dx \right], \]

keeping the simpler notation \( a(w, v) \) only when \( D = \Omega \).

After integrating by parts twice we have

\[ a^D(w, v) = \int_D D \Delta^2 w v \, dx + \int_{\partial D} M_{nn}(w) \frac{\partial v}{\partial n} \, dt - \int_{\partial D} \left( Q_n(w) + \frac{\partial M_{nt}(w)}{\partial t} \right) v \, dt. \]

When \( D = \Omega \) and \( v \in V \) the last two terms in (2.11) disappear, due to the boundary conditions (2.2), but (2.11) will be useful later on when dealing with the discrete problem (and, in any case, whenever boundary conditions of a different type are considered on some part of the boundary). In the sequel we shall often use \( v_{/n} \) and \( v_{/t} \) for \( \partial v/\partial n \) and \( \partial v/\partial t \), respectively.

3. The discrete problem. Abstract framework

Let \( \{T_h\}_h \) be a sequence of decompositions of \( \Omega \) into elements \( K \), and let \( E_h \) be the set of edges \( e \) of \( T_h \). We want to construct a finite dimensional space \( V_h \subset H^2_0(\Omega) \), a bilinear form \( a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R} \), and an element \( f_h \in V_h' \) such that the discrete problem:

\[ \begin{align*}
\text{Find } w_h \in V_h \text{ such that } \\
a_h(w_h, v_h) = \langle f_h, v_h \rangle & \quad \forall v_h \in V_h,
\end{align*} \]

has a unique solution \( w_h \), and good approximation properties hold. Namely, if \( k \geq 2 \) is the target degree of accuracy, and the solution \( w \) of (2.3) is smooth enough, we want to have

\[ \|w - w_h\|_V \leq C h^{k-1} |w|_{k+1, \Omega}, \]
where $C$, here and in the following formulae, is, as usual, a positive constant independent of $h$. Note that (3.2), as written, implies our $k-th$ order Patch-Test, since when $w \in \mathbb{P}_k$ then $|w|_{k+1} = 0$.

Let us first recall the basic assumptions that we need (see e.g. [12]).

**H0** - There exists an integer $N$ and a positive real number $\gamma$ such that for every $h$ and for every $K \in T_h$:

- the number of edges of $K$ is $\leq N$,
- the ratio between the shortest edge and the diameter $h_K$ of $K$ is bigger than $\gamma$, and
- $K$ is star-shaped with respect to every point of a ball of radius $\gamma h_K$.

**H1** - We assume that we are given, for each $h$:

- a space $V_h \subset V$; for each element $K$ we will denote

$$V^K_h = \text{restriction of } V_h \text{ to } K$$

- a symmetric bilinear form $a_h$ from $V_h \times V_h$ to $\mathbb{R}$ that can be split as

$$a_h(u_h, v_h) = \sum_K a^K_h(u_h, v_h) \quad \forall u_h, v_h \in V_h,$$

where $a^K_h$ is a bilinear symmetric form on $V^K_h \times V^K_h$,

- an element $f_h \in V'_h$.

Similarly, we split the bilinear form $a(\cdot, \cdot)$ and the norm $\| \cdot \|_V$ as

$$a(u, v) = \sum_K a^K(u, v) \quad \forall u, v \in V, \quad \|v\|_V = \left( \sum_K |v|^2_{V,K} \right)^{1/2} \quad \forall v \in V.$$  

Since in what follows we shall also deal with functions in $H^2(T_h) := \prod_K H^2(K)$, we also need to define a broken $H^2$ semi-norm:

$$|v|_{h,V} := \left( \sum_K |v|^2_{V,K} \right)^{1/2} \quad \forall v \in \prod_K H^2(K).$$

3.1. **An abstract convergence theorem.** Together with **H0** and **H1** we further assume the following properties.

**H2** - There exists an integer $k \geq 2$ (that will determine our order of accuracy) such that for all $h$, and for all $K$ in $T_h$,

- $k$-Consistency: $\forall p \in \mathbb{P}_k, \forall v_h \in V_h$

$$a^K_h(p, v_h) = a^K(p, v_h).$$

- Stability: $\exists$ two positive constants $\alpha_s$ and $\alpha^*$, independent of $h$ and of $K$, such that

$$\forall v_h \in V_h \quad \alpha_s a^K(v_h, v_h) \leq a^K_h(v_h, v_h) \leq \alpha^* a^K(v_h, v_h).$$
We note that the symmetry of $a_h$, (3.8) and the continuity of $a^K$ easily imply the continuity of $a_h$ with

\[
(3.9) \quad a_h^K(u, v) \leq \left( a_h^K(u, u) \right)^{1/2} \left( a_h^K(v, v) \right)^{1/2} \leq \alpha^* \left( a^K(u, u) \right)^{1/2} \left( a^K(v, v) \right)^{1/2} \leq \alpha^* M \|u\|_{V,K} \|v\|_{V,K} \quad \text{for all } u \text{ and } v \text{ in } V_h.
\]

In turn, (3.8) and (3.9) easily imply

\[
(3.10) \quad \forall v \in V_h \quad \alpha_* a(v, v) \leq a_h(v, v) \leq \alpha^* a(v, v).
\]

and

\[
(3.11) \quad a_h(u, v) \leq \alpha^* M \|u\|_V \|v\|_V \quad \text{for all } u \text{ and } v \in V_h.
\]

We have the following convergence theorem.

**Theorem 3.1.** Under the Assumptions H1-H2 the discrete problem: Find $w_h \in V_h$ such that

\[
(3.12) \quad a_h(w_h, v_h) = < f_h, v_h > \quad \forall v_h \in V_h,
\]

has a unique solution $w_h$. Moreover, for every approximation $w_I$ of $w$ in $V_h$ and for every approximation $w_\pi$ of $w$ that is piecewise in $P_k$, we have

\[
\|w - w_h\|_V \leq C \left( \|w - w_I\|_V + \|w - w_\pi\|_{h,V} + \|f - f_h\|_{V_h'} \right)
\]

where $C$ is a constant depending only on $\alpha$, $\alpha_*$, $\alpha^*$, $M$ and, with the usual notation, the norm in $V_h'$ is defined as

\[
(3.13) \quad \|f - f_h\|_{V_h'} := \sup_{v_h \in V_h} \frac{< f - f_h, v_h >}{\|v_h\|_V}.
\]

**Proof.** Existence and uniqueness of the solution of (3.12) is a consequence of (3.8) and (2.7). Next, setting

\[
(3.14) \quad \delta_h := w_h - w_I
\]

we have

\[
\alpha_* \alpha \|\delta_h\|_V^2 \leq ( \text{ use (2.7) and (3.10)} \leq \alpha_* a(\delta_h, \delta_h) \leq \alpha_* a(\delta_h, \delta_h) \text{ use (3.14)} = a_h(w_h, \delta_h) - a_h(w_I, \delta_h) \text{ (use (3.12) and (3.4))} = < f_h, \delta_h > - \sum_K a_h^K(w_I - w_\pi, \delta_h) \text{ (use } \pm w_\pi) = < f_h, \delta_h > - \sum_K a_h^K(w_I - w_\pi, \delta_h) + a^K(w_\pi, \delta_h) \text{ (use } \pm w \text{ and (3.5))} = < f_h, \delta_h > - \sum_K a_h^K(w_I - w_\pi, \delta_h) + a^K(w_\pi - w, \delta_h) - a(w, \delta_h)
\]
Now we can use (2.3) in (3.15) to get

\[ \alpha \alpha \| \delta_h \|^2_V \leq <f_h, \delta_h> - \sum_K \left( a^K_h(w_I - w, \delta_h) + a^K(w, w, \delta_h) \right) - (f, \delta_h) \]

(3.16)

\[ =<f_h, \delta_h>- (f, \delta_h) - \sum_K \left( a^K_h(w_I - w, \delta_h) + a^K(w, w, \delta_h) \right) \]

Finally we use (3.13), (3.9), and the continuity of each \( a^K \) in (3.16) to obtain

\[ \| \delta_h \|^2_V \leq C \left( \| f - f_h \|_{V_h} + \| w_I - w \|_{h_V} + \| w - w \|_{h,V} \right) \| \delta_h \|_V \]

for some constant \( C \) depending only on \( \alpha, \alpha_*, \alpha^*, \) and \( M \). Then the result follows easily by the triangle inequality. \( \square \)

4. Construction of \( V_h, a_h, \) and \( f_h \)

We shall construct now a space \( V_h \), a bilinear form \( a_h \), and a right-hand side \( f_h \) satisfying assumptions \( H1 - H2 \). Our family of elements will depend on three integer indices \( (r, s, m) \), related to the degree of accuracy \( k \geq 2 \) by:

(4.1)

\[ r = \max\{3, k\}, \quad s = k - 1, \quad m = k - 4. \]

**Remark 4.1.** As we shall see in a while, the indices \( r \) and \( s \) will be related, respectively, to the polynomial degree of the functions in \( V_h \), and to the polynomial degree of their normal derivative, on each edge of the decomposition. On the other hand, the index \( m \) will be related (in a way to be made precise) to degrees of freedom internal to each element.

(4.2)

\[ r = 5, \quad s = 3, \quad m = 0 \quad \text{and} \quad k = 4 \]

(that might produce a Bell-like type of element). What we really need is actually

\[ r \geq \max\{3, k\}, \quad s \geq k - 1, \quad m \geq k - 4. \]

**Remark 4.2.** As it will be clear when constructing the spaces, condition (4.1) will not allow, for instance, to take the simple choice

(4.3)

On the other hand, the present choice allows to consider a family of discretizations that depends only on one parameter (here, \( k \)). Choosing (4.3), instead, we should actually deal with a family of spaces \( V_h \) depending on four parameters \( r, s, m, \) and \( k \), with a considerable complication in the notation and little conceptual gain. However, in developing our theory we should keep an eye on more general cases, like (4.3) or other possible approaches for which our theory would apply almost unchanged (but whose comprehensive treatment would imply a much more complicated notation).

**Remark 4.3.** We can easily attach to each vertex \( \xi \) a characteristic length \( h_\xi \), taken, for instance, as the average of the diameters of the elements having \( \xi \) as a vertex. To an edge \( e \) we can attach, as characteristic length \( h_e \) the length \( |e| \) of the edge itself, while to an element \( K \) we can attach as characteristic length \( h_K \) its diameter.
4.1. Local construction of $V_h$. Let $K$ be a generic polygon in $\mathcal{T}_h$, let $k \geq 2$, and let $r, s, m$ be given by (4.1). We define $V_h^K$ (that will be the local version of our discretized space $V_h$) as

$$\begin{align*}
(4.4) \quad V_h^K := \{ v \in H^2(K) \text{ s.t. } \Delta^2 v \in \mathbb{P}_m(K), v_{|e} \in \mathbb{P}_r(e), (v_{/n})_{|e} \in \mathcal{P}_s(e), \forall e \in \partial K \}.
\end{align*}$$

We shall make use of the following notation: for $t$ a nonnegative integer, and $e$ an edge with midpoint $x_e$, we denote by $\mathcal{M}_t^e$ the set of $t + 1$ normalized monomials

$$\begin{align*}
(4.5) \quad \mathcal{M}_t^e := \left\{ 1, \frac{x - x_e}{h_e}, \left( \frac{x - x_e}{h_e} \right)^2, \ldots, \left( \frac{x - x_e}{h_e} \right)^t \right\},
\end{align*}$$

Similarly, for a two-dimensional domain $K$ with diameter $h_K$ and barycenter $\mathbf{x}_K$ we denote by $\mathcal{M}_t^K$ the set of $(t + 1)(t + 2)/2$ normalized monomials

$$\begin{align*}
(4.6) \quad \mathcal{M}_t^K := \left\{ \left( \frac{x - \mathbf{x}_K}{h_K} \right)^{\beta}, \quad |\beta| \leq t \right\},
\end{align*}$$

where, as usual, for a nonnegative multiindex $\beta = (\beta_1, \beta_2)$ we set $|\beta| = \beta_1 + \beta_2$ and $x^\beta = x_1^{\beta_1} x_2^{\beta_2}$. In $K$ we define the following degrees of freedom:

$$\begin{align*}
(4.7) & \quad \text{The value of } v(\xi) \quad \forall \text{ vertex } \xi \\
(4.8) & \quad \text{The values of } h_\xi v_{1/2}(\xi) \text{ and } h_\xi v_{/2}(\xi) \quad \forall \text{ vertex } \xi \\
(4.9) & \quad \text{For } r > 3, \text{ the moments } \frac{1}{h_e} \int_e q(\xi)v(\xi)\,d\xi \quad \forall q \in \mathcal{M}_{r-4}^e, \quad \forall \text{ edge } e \\
(4.10) & \quad \text{For } s > 1, \text{ the moments } \int_e q(\xi)v_{/n}(\xi)\,d\xi \quad \forall q \in \mathcal{M}_{s-2}^e, \quad \forall \text{ edge } e \\
(4.11) & \quad \text{For } m \geq 0, \text{ the moments } \frac{1}{h_K^2} \int_K q(x)v(x)\,dx \quad \forall q \in \mathcal{M}_m^K.
\end{align*}$$

Note that all the quantities (4.7)-(4.11) have the same dimension, and hence in a homothetic blow up of $K$ would scale in the same manner.

Figure 1 shows the first two elements of the family determined by the d.o.f. (4.7)-(4.11): for both elements we have that $v \in \mathbb{P}_3(e)$ on each edge, but $\frac{\partial v}{\partial n_{|e}} \in \mathbb{P}_1(e)$ for $k = 2$, and $\frac{\partial v}{\partial n_{|e}} \in \mathbb{P}_2(e)$ for $k = 3$. These elements are the extension to polygonal domains of the Hsieh-Clough-Tocher triangle ($k = 3$) and of its reduced version ($k = 2$).

We note that he degrees of freedom (4.7) and (4.8) are always needed to ensure $C^1$-continuity at the vertices. Moreover, on each edge, they provide $v$ and $v_{/t}$ at the endpoints of the edge: enough information to identify uniquely a polynomial of degree $\leq 3$. This, together with (4.9), explains the requirement $r \geq 3$. Indeed, to identify a polynomial of degree $r$, together with (4.7)-(4.8) we need $r - 3$ additional information (as for instance (4.9)). At the same time, conditions (4.8) provide, for every edge, $v_{/n}$ at the endpoints of the edge, thus allowing to identify uniquely a polynomial of degree $\leq 3$. For $s > 1$, to identify a polynomial of degree $\leq s$, we would need the
additional $s - 1$ conditions (4.10). On the other hand, the d.o.f. (4.11) are equivalent to prescribe $P_m^K v$ in $K$, where, for $m$ a nonnegative integer,

$$P_m^K v$$

is the $L^2(K)$–projector operator onto the space $P_m(K)$.

We summarize the above discussion in the following proposition

**Proposition 4.1.** In each element $K$ the d.o.f. (4.7), (4.8), and (4.9) uniquely determine a polynomial of degree $\leq r$ on each edge of $K$, the degrees of freedom (4.8) and (4.10) uniquely determine a polynomial of degree $\leq s$ on each edge of $K$, and the d.o.f. (4.11) are equivalent to prescribe $P_m^K v$ in $K$.

Let $\ell$ be the number of edges (and hence of vertices) of $K$, and let $G^K$ be the set of degrees of freedom corresponding to $K$, whose total number amounts to

$$T := T_\partial + T_0; \quad \text{with } T_\partial = \{3\ell + \ell(r-3) + \ell(s-1)\} \equiv \ell(r+s-1),$$

and

$$T_0 = \{(m+1)(m+2)\over 2\} \equiv m^2 + 3m + 2 \over 2.$$  

Now we have to show how to associate to the above degrees of freedom a function in $K$. It is important to point out from the very beginning that we do not need to be able to compute such a function, but just to show that it exists and it has certain properties.

**Proposition 4.2.** The degrees of freedom (4.7)–(4.11) are unisolvent in $V^K_h$ (as defined in (4.4)).

**Proof.** According to Proposition 4.1, to prove the proposition it is enough to show that, for each $K \in \mathcal{T}_h$, a function $u$ in $V^K_h$ such that

$$u = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial K$$

(4.14)

and

$$P_m^K u = 0 \quad \text{in } K$$

(4.15)
is actually identically zero in \( K \). To this end, we show that \( \Delta^2 u = 0 \) in \( K \), which joined with (4.14) gives \( u \equiv 0 \). We first solve, for every \( q \in \mathbb{P}_m(K) \), the following auxiliary problem: \( \text{Find } \psi = \psi(q) \in H_0^2(K) \text{ such that:} \\
\tag{4.16} a^K(\psi, v) = (q, v)_0,K \quad \forall v \in H_0^2(K), \)
that could also be written as
\[
\tag{4.17} D\Delta^2 \psi = q \quad \text{in } K, \quad \psi|_{\partial K} = \frac{\partial \psi}{\partial n}|_{\partial K} = 0, \quad \text{or else } \psi = (D\Delta_{0,K}^2)^{-1}(q).
\]
Next, we consider the map \( R \), from \( \mathbb{P}_m(K) \) into itself, defined by
\[
\tag{4.18} R(q) := P^K_m((D\Delta_{0,K}^2)^{-1}(q)) \equiv P^K_m(\psi(q)) \quad \forall q \in \mathbb{P}_m(K).
\]
We claim that the operator \( R \) defined by (4.18) is an isomorphism from \( \mathbb{P}_m(K) \) into itself. Indeed, from (4.18) and the definition of \( P^K_m \) we have, for every \( q \in \mathbb{P}_m(K) \):
\[
(R(q), q)_{0,K} = (P^K_m(D\Delta_{0,K}^2)^{-1}(q), q)_{0,K} = (P^K_m(\psi(q)), q)_{0,K}
\]
\[
= (\psi(q), q)_{0,K} = a^K(\psi(q), \psi(q)).
\]
Since \( \psi \in H_0^2(K) \) we have then that
\[
\tag{4.19} \{ R(q) = 0 \} \Rightarrow \{ a^K(\psi(q), \psi(q)) = 0 \} \Rightarrow \{ \psi(q) = 0 \} \Rightarrow \{ q = 0 \}.
\]
We note that, if \( u \in V^K_h \) is such that \( u = \partial u / \partial n = 0 \) on \( \partial K \), then
\[
\tag{4.20} P^K_m u = P^K_m((D\Delta_{0,K}^2)^{-1}(D\Delta^2 u)) = R(D\Delta^2 u).
\]
Hence, if \( P^K_m u = 0 \) as in (4.15), we can say that
\[
P^K_m u = 0 \Rightarrow R(D\Delta^2 v) = 0 \Rightarrow D\Delta^2 v = 0,
\]
and the proof is concluded. \( \square \)

**Remark 4.4.** We point out once more that we will not try to compute the solutions of the local bi-harmonic problems involved in the definition of (4.4). On the other hand we will always assume that we can compute polynomials on \( \partial K \), and hence also integral of polynomials over \( K \). Indeed, for instance,
\[
\tag{4.21} \int_K x_1^2 = \frac{1}{3} \int_{\partial K} x_1^3 n_1
\]
and so on.

### 4.2. Global construction of \( V_h \).
For every decomposition \( \mathcal{T}_h \), and for every \( k \geq 2 \) we are now ready to define the space \( V_h \) as
\[
\tag{4.22} V_h = \{ v \in V : v|_e \in \mathbb{P}_r(e), \frac{\partial v}{\partial n}|_e \in \mathbb{P}_s(e) \ \forall e \in \mathcal{E}_h, \ \Delta^2 v|_K \in \mathbb{P}_m(K) \ \forall K \in \mathcal{T}_h \},
\]
where \( r, s, \) and \( m \) are always given by (4.1) (although, actually, they could be any triple that satisfies (4.3)). It follows from the above discussion that the global degrees of freedom in \( V_h \) could then be taken as
\[
\tag{4.23} \text{The value of } v(\xi) \quad \forall \text{ internal vertex } \xi
\]
\[
\tag{4.24} \text{The values of } h_{\xi} v_{1/2}(\xi) \text{ and } h_{\xi} v_{1/2}(\xi) \quad \forall \text{ internal vertex } \xi
\]
For $r > 3$, the moments $\frac{1}{h_e} \int_{e} q(\xi)v(\xi) d\xi \quad \forall q \in \mathcal{M}_{e}^{r-4}, \quad \forall$ internal edge $e$

(4.25) 

For $s > 1$, the moments $\int_{e} q(\xi)v_{/n}(\xi) d\xi \quad \forall q \in \mathcal{M}_{e}^{s-2}, \quad \forall$ internal edge $e$

(4.26) 

For $m \geq 0$, the moments $\frac{1}{h_K} \int_{K} q(x)v(x) dx \quad \forall q \in \mathcal{M}_{m}^{K} \quad \forall$ element $K$.

(4.27) 

The dimension of $V_h$, that is, the total number of degrees of freedom in $V_h$ is then given by

$$G = 3N_V + N_E(r - 3 + s - 1) + N_K T_0,$$

where $N_V$ is the number of internal vertices of $\mathcal{T}_h$, $N_E$ the number of internal edges, $N_K$ the number of elements, and $T_0$ is still given by (4.13). Needless to say, in the construction of $v_h \in V_h$ one should take into account the values of $v_h$ and $v_{h/n}$ at the boundary as well, setting them equal to zero.

**Remark 4.5.** At this point we can order (in a rather arbitrary way) the degrees of freedom of $V_h$

\begin{equation}
(4.29) \quad g_1, \ g_2, \ldots, g_G.
\end{equation}

Accordingly we will have a characteristic length $h_i$ attached to each degree of freedom $g_i$, depending on its nature, according to Remark 4.3. It is obvious that, in the computer code, we will have chosen, once and for all, an orientation for the normal unit vector, say $n_e$, to each internal edge $e$.

**Remark 4.6.** It follows easily from the above construction that for every smooth enough $w$ there exists a unique element $w_I \in V_h$ such that

\begin{equation}
(4.30) \quad g_i(w - w_I) = 0 \quad \forall i = 1, 2, \ldots, G.
\end{equation}

Moreover, by the usual Bramble-Hilbert/Deny-Lions technique (see e.g. [18]) and using a scaling argument to get around the variability in the geometry (see e.g. [13]) it is not difficult to see that, for any nonnegative indices, we can prove that

\begin{equation}
(4.31) \quad \|w - w_I\|_{\alpha, \Omega} \leq C h^{\beta - \alpha} |w|_{\beta, \Omega} \quad \alpha = 0, 1, 2, \quad \alpha \leq \beta \leq k + 1
\end{equation}

(with a constant $C$ independent of $h$) as in the usual Finite Element framework.

We point out that (as it happens also for the classical finite elements), different choices of degrees of freedom might be used for the same space. However a different choice of the local degrees of freedom can induce a different choice of the global degrees of freedom, that in turn could alter the continuity property of the resulting functions, giving rise to a different global space. Let us see a couple of examples that, in our opinion, are particularly meaningful. Let us consider the value $k = 5$ (and, according to (4.1), $(r, s, m) = (5, 4, 1)$), and the corresponding degrees of freedom in our construction (4.7)-(4.11). They are: the values of $w$, $w_x$, $w_y$ at the vertices, plus, on every edge, the moments up to the order two of $w$ and of $w_{/n}$, plus the moments up to the order one of $w$ inside the element. Taking instead $k = 4$ with $r = 5$, $s = 3$, and $m = 0$ (using this time (4.3)) we would have the same degrees of freedom minus the moments of order one of $w_{/n}$ on each edge. The two cases are presented in Figure 2, with obvious meaning of the symbols.
Figure 2. Local d.o.f. for the (5,4,1) element with $k = 5$ (left), and for the (5,3,0) element with $k = 4$ (right)

Note that, however, we could as well have used as local degrees of freedom the ones in Figure 3, where (with classical notation) the circles at the vertices represent the six degrees of freedom "values of $(w, w_x, w_y, w_{xx}, w_{xy}, w_{yy})$ at the vertices.

Figure 3. Alternative d.o.f. for the elements of Figure 2. We have an "Argyris-like" element on the left and a "Bell-like" element on the right.

The theoretical treatment of the case of Figure 3 could be done in a way practically identical to the one followed here. As already pointed out in Remark 4.2 we chose to skip the comprehensive treatment of all possible cases allowed by our theory, in order to avoid a too cumbersome notation.

4.3. Construction of $a_h$. We are left to show how to construct a (computable!) symmetric bilinear form $a_h$ satisfying (3.7) and (3.8). We shall work on a generic element $K \in T_h$, and we recall that we denoted by $V_h^K$ the restriction of $V_h$ to $K$. First of all, we observe that the local degrees of freedom allow us to compute exactly $a^K(p, v)$ for any $p \in \mathbb{P}_k(K)$ and for any $v \in V_h^K$. Indeed, recalling (2.11), we have

\begin{equation}
(4.32) \quad a^K(p, v) = D \int_K \Delta^2 p \, v \, dx + \int_{\partial K} [M_{nn}(p) \frac{\partial v}{\partial n} + (Q_n(p) + \partial M_{nt}(p)/\partial t)v] \, dt.
\end{equation}

We note then that $\Delta^2 p$ belongs to $\mathbb{P}_{k-4}(K)$; for $m \geq k - 4$, the integral over $K$ that appears in (4.32) is then computable using only the values of the internal degrees of freedom of $v$, without actually knowing $v$. On the other hand, on each edge of $K$ both $M_{nn}(p)$ (which belongs to $\mathbb{P}_{k-2}(e)$) and $Q_n(p) + \partial M_{nt}(p)/\partial t$ (which belongs to $\mathbb{P}_{k-3}(e)$) are polynomials, as well as $v$ and $v/n$. From the degrees of freedom on the
boundary we can easily compute both \( v \) and \( v_{/n} \), and in conclusion all the terms in the right-hand side of (4.32) can be easily computed without knowing \( v \) inside.

Now for every \( \varphi \in C^0(\overline{K}) \) we define its quasi-average \( \bar{\varphi} \) as the constant function (over \( K \)) whose value

\[
\bar{\varphi} := \frac{1}{\ell} \sum_{i=1}^{\ell} \varphi(\mathbf{x}^i)
\]

is given by the average of the values that \( \varphi \) assumes at the \( \ell \) vertices \( \mathbf{x}^i \), \( i = 1, 2, \ldots, \ell \) of \( K \). Next, we introduce the operator \( \Pi^K : V^K_h \rightarrow \mathbb{P}_k(K) \subset V^K_h \) defined as the solution of

\[
\begin{align*}
\begin{cases}
 a^K(\Pi^K \psi, q) = a^K(\psi, q) & \forall \psi \in V^K_h, \forall q \in \mathbb{P}_k(K) \\
 \Pi^K \psi = \bar{\psi}, & \nabla \Pi^K \psi = \nabla \bar{\psi}.
\end{cases}
\end{align*}
\]

We note that for \( v \in \mathbb{P}_k(K) \) the first equation in (4.34) implies \( (\Pi^K v)_{/ij} = v_{/ij} \) for \( i, j = 1, 2 \), that joined with the second equation gives easily

\[
(4.35) \quad \Pi^K v = v \quad \forall v \in \mathbb{P}_k(K).
\]

Choosing \( a^K_h(u, v) = a^K(\Pi^K u, \Pi^K v) \) would now ensure property (3.7), but not, in general, property (3.8). Hence we need to add a suitable term capable to ensure (3.8). Let then \( S^K(u, v) \) be a symmetric positive definite bilinear form, to be chosen to verify

\[
(4.36) \quad c_0 a^K(v, v) \leq S^K(v, v) \leq c_1 a^K(v, v), \quad \forall v \in V^K_h \text{ with } \Pi^K v = 0,
\]

for some positive constants \( c_0, c_1 \) independent of \( K \) and \( h_K \). Then set

\[
(4.37) \quad a^K_h(u, v) := a^K(\Pi^K u, \Pi^K v) + S^K(u - \Pi^K u, v - \Pi^K v).
\]

**Proposition 4.3.** The bilinear form (4.37) constructed with the above procedure satisfies both the consistency property (3.7) and the stability property (3.8).

**Proof.** Property (3.7) follows immediately from (4.35) and (4.34): for \( p \in \mathbb{P}_k(K) \), (4.35) implies \( S^K(p - \Pi^K p, v - \Pi^K v) = 0 \, \forall v \), and hence

\[
(4.38) \quad a^K(p, v) = a^K(\Pi^K p, \Pi^K v) = a^K(\Pi^K p, v) = a^K(p, v).
\]

Moreover, from (4.34) we have \( a^K(\psi - \Pi^K \psi, q) = 0 \) for all \( q \in \mathbb{P}_k(K) \) and for all \( \psi \in V^K_h \), so that by Pythagoras theorem we have

\[
(4.39) \quad a^K(u - \Pi^K u, v - \Pi^K v) + a^K(\Pi^K v, \Pi^K v) = a^K(v, v) \quad \forall v \in V^K_h.
\]

Property (3.8) follows then easily from (4.36) and (4.39). Indeed setting \( \alpha := \max\{1, c_1\} \) we have

\[
(4.40) \quad a^K(v, v) \leq a^K(\Pi^K v, \Pi^K v) + c_1 a^K(v - \Pi^K v, v - \Pi^K v)
\]

\[
\leq \max\{1, c_1\} \left( a^K(\Pi^K v, \Pi^K v) + a^K(v - \Pi^K v, v - \Pi^K v) \right)
\]

\[
= \alpha a^K(v, v),
\]
and similarly, setting $\alpha_* := \min\{1, c_0\}$:

$$
\tag{4.41}
a^K_h(v, v) \geq \min\{1, c_0\} \left( a^K(\Pi^K v, \Pi^K v) + a^K(v - \Pi^K v, v - \Pi^K v) \right) = \alpha_* a^K(v, v).
$$

\[ \square \]

### 4.4. Choice of $S^K$

In general, the choice of the bilinear form $S^K$ might depend on the problem and on the degrees of freedom. From (4.36) it is clear that $S^K$ must scale like $a^K(\cdot, \cdot)$ on the kernel of $\Pi^K$. Since in (4.23)-(4.27) we took care to have degrees of freedom of the same dimension, then all the elements of the canonical basis $\varphi_1, \ldots, \varphi_T$ defined by

$$
\tag{4.42}
g_i(\varphi_j) = \delta_{ij}, \quad i, j = 1, T,
$$

will scale in the same way (and in particular as the ones associated to “point value” degrees of freedom). Hence the choice

$$
S^K(v, w) = D \sum_{i=1}^{T} g_i(v)g_i(w)(h_i)^{-2},
$$

(with $h_i$ as in Remark 4.5) would clearly do the job.

**Remark 4.7.** Clearly, if we decide to use degrees of freedom involving second derivatives, as the ones in Figure 3, we should scale them by $h_2^2$ in order to apply the above argument.

### 4.5. Construction of the right-hand side.

In order to build the loading term $\langle f_h, v_h \rangle$ for $v_h \in V_h$ in a simple and easy way it is convenient to have internal degrees of freedom in $V_h$, and this means, according to (4.1) and (4.11), that we need $k \geq 4$. In this case we define $f_h$ on each element $K$ as the $L^2(K)$—projection of the load $f$ onto the space of piecewise polynomials of degree $m = k - 4$, that is,

$$
f_h = P^K_{k-4} f \quad \text{on each } K \in T_h.
$$

Then, always for $k \geq 4$, the associated loading term

$$
\langle f_h, v_h \rangle = \sum_{K \in T_h} \int_K f_h v_h \, dx \equiv \sum_{K \in T_h} \int_K (P^K_{k-4} f) v_h \, dx = \sum_{K \in T_h} \int_K f \, (P^K_{k-4} v_h) \, dx
$$

can be exactly computed using the degrees of freedom for $V_h$ that represent the internal moments, see (4.11).

For $k > 4$, that is, $m = k - 4 \geq 1$, standard $L^2$ orthogonality and approximation estimates on star-shaped domains yield

$$
\tag{4.43}
\langle f_h, v_h \rangle = \sum_{K \in T_h} \int_K (P^K_{k-4} f - f)(v_h - P^K_{1} v_h) \, dx \leq C h^{k-3} \left( \sum_{K \in T_h} |f|_{k-3,K}^2 \right)^{1/2} \|v_h\|_{V,h}.
$$
and thus, recalling (3.13),

\[(4.44) \quad \|f - f_h\|_{V_h^k} \leq Ch^{k-1}\left(\sum_{K \in T_h} |f|_{k-3,K}^2\right)^{1/2}.
\]

Hence, for \( k > 4 \) Theorem 3.1 ensures the optimal \( O(h^{k-1}) \) error bound. For \( k \leq 4 \) we need some additional tricks. In view of Remark 4.4 we assume that we are able to compute exactly the integral

\[(4.45) \quad \int_K p_1 q_1 \, dx
\]

for every pair \( p_1 \) and \( q_1 \) of polynomials of degree \( \leq 1 \) on \( K \). In each case \( k = 2, 3, 4 \) the term \( < f_h, v_h > \) will then be defined by

\[(4.46) \quad < f_h, v_h >_K = \int_K \tilde{P}_1^k f \, \overline{P}_1^k v_h
\]

where \( \tilde{P}_1^k \) and \( \overline{P}_1^k \) are projectors on the space of polynomials of degree \( \leq 1 \) that will change from one value of \( k \) to another.

For \( k = 2 \) and for \( k = 3 \) we take

\[(4.47) \quad f_{h|K} \equiv \tilde{P}_1^k f := P_{k-2}^k f, \quad \overline{P}_1^k v_h \equiv \tilde{v}_h := \hat{v}_h + (k-2)(x - x_K) \cdot \nabla v_h \quad \text{on each } K.
\]

Note that for both values \( k = 2 \) and \( k = 3 \) we have that \( \tilde{v}_h \in \mathbb{P}_{k-2}(K) \). Hence on each \( K \) we have

\[(4.48) \quad < f_h, v_h >_K - (f, v_h)_K = (P_{k-2}^k f - f, \tilde{v}_h)_K + (f, \tilde{v}_h - v_h)_K
\]

\[(4.49) \quad = (f, \tilde{v}_h - v_h)_K \leq Ch_K^{k-1}\|f\|_{0,K}\|v_h\|_{k-1,K}.
\]

For \( k = 4 \) we take instead

\[(4.50) \quad f_{h|K} \equiv \tilde{P}_1^k f := P_1^k f, \quad \overline{P}_1^k v_h \equiv \tilde{v}_h := P_0^k v_h + (x - x_K) \cdot \nabla v_h \quad \text{on each } K.
\]

We have on each \( K \)

\[(4.51) \quad < f_h, v_h >_K - (f, v_h)_K = (P_1^k f - f, \tilde{v}_h)_K + (f, \tilde{v}_h - v_h)_K
\]

\[(4.52) \quad = (f, \tilde{v}_h - v_h)_K \leq Ch^3_K\|f\|_{1,K}\|v_h\|_{2,K}
\]

where, going from second to third line we took advantage of the fact that \( \tilde{v}_h - v_h \) has zero mean value over \( K \). Note that, for \( k = 4 \), we have \( m = 0 \) in (4.1), and from (4.11) we have the right to use \( P_0^k v_h \) instead of \( \hat{v}_h \), since \( P_0^k v_h \) is now part of the degrees of freedom.

References

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