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VEM approximations of the Vector Potential Formulation of Magnetostatic problems

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Abstract. We consider, as a simple model problem, the application of Virtual Element Methods (VEM) to the linear Magnetostatic three-dimensional problem in the classical Vector Potential formulation. The Vector Potential is treated as a triple of $0-forms$ and is approximated by 3 nodal VEM spaces. However this is not done using three classical H^1-conforming nodal Virtual Elements, and instead we use the *Stokes Elements* of [18], introduced originally for the treatment of incompressible fluids.

Keywords. Virtual Element Methods, Serendipity, Magnetostatic problems, Vector Potential.

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1. Introduction

In recent times, for the discretization of PDEs, there has been a considerable interest in the use of decompositions of the computational domain in polytopes. See for instance [5, 8, 23, 24, 36, 37, 40, 41, 42, 45, 47, 53, 54] and the refereences therein.

Virtual Elements were introduced a few years ago [9, 13] for the discretization of H^1-conforming spaces to be used in the numerical approximation of PDEs on very general decompositions into polygons or polyhedra, and had a wide diffusion in the last years. On one hand they were extended to the discretization of more general Spaces, as H^1-nonconforming (e.g. [7]), $H(\mathrm{div})$-conforming, and $H(\mathbf{curl})$-conforming (e.g, [14, 16]). On the other hand they had other theoretical extensions through the Serendipity approach (see for instance [15]), and moreover their use has been extended to a wide variety of problems (see e.g. [1, 19, 34, 56] and the references therein). Also, the study of the possible usable decomposition and of the related interpolation errors made significant progresses in the last couple of years (see e.g. [17, 25, 27, 32, 49]).

The list of VEM contributions in the literature is nowadays quite large; we mention, e.g., [2, 6, 18, 20, 21, 28, 29, 30, 31, 38, 43, 44, 50, 51, 52, 55, 56, 57, 58] and the references therein.

Here we deal, as a simple model problem, with the classical magnetostatic problem in a smooth-enough bounded domain Ω in $\mathbb{R}^3$, simply connected with a connected boundary: given $\boldsymbol{j} \in H(\mathrm{div};\Omega)$

with $\mathrm{div}\boldsymbol{j} = 0$ in Ω, and given $\mu = \mu(\boldsymbol{x})$ with $0 < M_0 \le \mu \le M_1$,

$$\begin{cases} \text{find } \boldsymbol{H} \in H(\mathbf{curl};\Omega) \text{ and } \boldsymbol{B} \in H(\mathrm{div};\Omega) \text{ such that:} \\ \mathbf{curl}\boldsymbol{H} = \boldsymbol{j} \text{ and } \mathrm{div}\boldsymbol{B} = 0, \text{ with } \boldsymbol{B} = \mu\boldsymbol{H} \text{ in } \Omega, \\ \text{with the boundary conditions } \boldsymbol{B}\cdot\boldsymbol{n} = 0 \,(\text{or } \boldsymbol{H}\wedge\boldsymbol{n} = 0) \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

In some previous papers [10, 11] we dealt with two-dimensional and three-dimensional approximations of the above magneto-static problems using the variational formulation of Kikuchi [46]. Here, instead, we tackle the discretization of the problem in the (more classical) *Vector Potential* formulation (see e.g. [22] and the references therein). Other important contributions to the numerical approximation of Magnetostatic problems can be found, for instance, in [4, 39, 48] and the references therein.

As far as we know, the vector potential formulation has not yet been tackled with Virtual Elements, and the possible benefits due to the great freedom in the element shapes have not yet been investigated in practice. Here, in particular, we also take advantage from the use of the Virtual Element spaces introduced in [18] for dealing with Stokes problems (that however are used here in a slightly different way). This choice allows the use (for test and trial functions) of vector-valued fields that have a *constant divergence* in each element. We think that, together with the generality in the element geometry this could represent a nice feature (in particular for higher order approximations) when compared to more classical Finite Element formulations. We also point out that here the computed vector potential will have a divergence that is exactly zero.

A layout of the paper is as follows: in the next Section 2 we will introduce some basic notation, and recall some well known properties of polynomial spaces. Nothing is new there. In Section 3 we will first recall, in Subsection 3.1, the Vector Potential approach to (1.1) and its variational formulation. Then, in Subsection 3.2 we present the *local* two-dimensional Virtual Element spaces (of nodal type) to be used on the inter-element boundaries. Here we use a simpler (although less powerful) version of the Serendipity spaces of [15], corresponding, roughly, to the approach that is called *lazy choice* there. Note that, instead, we do not use three-dimensional Serendipity elements (always with the aim of keeping the presentation as simple as possible) to reduce the number of degrees of freedom inside the polyhedrons. Actually, as is well known, in a three-dimensional problem it is more important to reduce the number of degrees of freedom *on faces* (where static condensation is quite cumbersome to perform), than to reduce the number of degrees of freedom internal to polyhedrons (that can be tackled by static condensation, which is practically done in an almost automatic way by several recent direct solvers).

In Subsection 3.4 we then discuss the Virtual Element spaces to be used *inside* each polyhedron. As we said, on each face of the boundary we use a simplified version of the Serendipity elements of [15] , and inside the polyhedron we use spaces inspired by[18], and we avoid 3D Serendipity versions. Note that however we still have some gain in the number of internal degrees of freedom from the use of a constant divergence. Then in Subsection 3.5 we discuss which quantities (in our discrete spaces) are actually *computable*, out of the degrees of freedom.

In Section 4 introduce the *global* Virtual Element spaces. We discuss their most important properties, and then we use them to define the discretized problem and to show existence and uniqueness of its solution.

In Section 5 we prove the a priori error bounds. First we bound the error between exact and approximate solutions in terms of the approximation errors (of the exact solution within the Virtual Element Spaces). Then we recall some (already classic) assumptions on the decompositions that allow to estimate the approximation errors, and we use them to derive the error estimates.

2. Notation and well known properties of Polynomial spaces

In two dimensions, we will denote by $\boldsymbol{x}$ the indipendent variable, using $\boldsymbol{x} = (x, y)$ or (more often) $\boldsymbol{x} = (x_1, x_2)$ following the circumstances. We will also use $\boldsymbol{x}^{\perp} := (-x_2, x_1)$, and in general, for a vector $\boldsymbol{v} \equiv (v_1, v_2)$: $\boldsymbol{v}^{\perp} := (-v_2, v_1)$. Moreover, for a vector $\boldsymbol{v}$ and a scalar q we will write

$$\mathrm{rot}\boldsymbol{v} := \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}, \qquad \mathbf{rot}\, q := (\frac{\partial q}{\partial y}, -\frac{\partial q}{\partial x})^T. \tag{2.1}$$

We recall some commonly used functional spaces. On a domain $\mathcal{O} \subseteq \mathbb{R}^3$ we have

$$\begin{aligned}
&H(\mathrm{div}; \mathcal{O}) = \{\boldsymbol{v} \in [L^2(\mathcal{O})]^3 \text{ with } \mathrm{div}\boldsymbol{v} \in L^2(\mathcal{O})\},\\
&H_0(\mathrm{div}; \mathcal{O}) = \{\boldsymbol{\varphi} \in H(\mathrm{div}; \mathcal{O}) \text{ with } \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \text{ on } \partial\mathcal{O}\},\\
&H(\mathbf{curl}; \mathcal{O}) = \{\boldsymbol{v} \in [L^2(\mathcal{O})]^3 \text{ with } \mathbf{curl}\boldsymbol{v} \in [L^2(\mathcal{O})]^3\},\\
&H_0(\mathbf{curl}; \mathcal{O}) = \{\boldsymbol{v} \in H(\mathbf{curl}; \mathcal{O}) \text{ with } \boldsymbol{v} \wedge \boldsymbol{n} = 0 \text{ on } \partial\mathcal{O}\},\\
&H^1(\mathcal{O}) = \{q \in L^2(\mathcal{O}) \text{ with } \mathbf{grad}\, q \in (L^2(\mathcal{O}))^3\},\\
&H^1_0(\mathcal{O}) = \{q \in H^1(\mathcal{O}) \text{ with } q = 0 \text{ on } \partial\mathcal{O}\}.
\end{aligned}$$

For an integer $s \geq -1$ we will denote by $\mathbb{P}_s$ the space of polynomials of degree $\leq s$. Following a common convention, $\mathbb{P}_{-1} \equiv \{0\}$ and $\mathbb{P}_0 \equiv \mathbb{R}$. Moreover, for $s \geq 0$

$$\mathbb{P}^h_s := \{\text{homog pol. of degree } s\}, \text{ and } \mathbb{P}^0_s(\mathcal{O}) := \{q \in \mathbb{P}_s \text{ s. t. } \int_{\mathcal{O}} q \, \mathrm{d}\mathcal{O} = 0\}. \tag{2.2}$$

For $d = 1, 2, 3$ we denote the dimension of the space $\mathbb{P}_s$ in d space dimensions by $\pi_{d,s}$:

$$\pi_{1,s} = s + 1, \quad \pi_{2,s} = \frac{(s+1)(s+2)}{2}, \quad \pi_{3,s} = \frac{(s+1)(s+2)(s+3)}{6} \tag{2.3}$$

Obviously, in d space dimensions, the (common) value of the dimension of $\mathbb{P}^0_s$ and of the space $\nabla(\mathbb{P}_s)$ will be equal to $\pi_{d,s} - 1$. The following decompositions of polynomial vector spaces are well known and will be useful in what follows.

$$(\mathbb{P}_s)^3 = \mathbf{curl}((\mathbb{P}_{s+1})^3) \oplus \boldsymbol{x}\mathbb{P}_{s-1}, \quad \text{and} \quad (\mathbb{P}_s)^3 = \mathbf{grad}(\mathbb{P}_{s+1}) \oplus \boldsymbol{x} \wedge (\mathbb{P}_{s-1})^3. \tag{2.4}$$

Taking the **curl** of the second of (2.4) we also get :

$$\mathbf{curl}(\mathbb{P}_s)^3 = \mathbf{curl}(\boldsymbol{x} \wedge (\mathbb{P}_{s-1})^3) \tag{2.5}$$

which used in the first of (2.4) gives:

$$(\mathbb{P}_s)^3 = \mathbf{curl}(\boldsymbol{x} \wedge (\mathbb{P}_s)^3) \oplus \boldsymbol{x}\mathbb{P}_{s-1}. \tag{2.6}$$

In what follows, when dealing with the *faces* of a polyhedron (or of a polyhedral decomposition) we shall use two-dimensional differential operators that act on the restrictions to faces of scalar functions that are defined on a three-dimensional domain. Similarly, for vector valued functions we will use two-dimensional differential operators that act on the restrictions to faces of the tangential components. In many cases, no confusion will be likely to occur; however, to stay on the safe side, we will often use a superscript τ to denote the tangential components of a three-dimensional vector, and a subscript f to indicate the two-dimensional differential operator. Hence, to fix ideas, if a face has equation $x_3 = 0$ then $\boldsymbol{x}^\tau := (x_1, x_2)$ and, say, $\mathrm{div}_f \boldsymbol{v}^\tau := \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$.

3. The problem and the local spaces

3.1. The Vector Potential formulation

We recall the classical Vector Potential Formulation. The idea is to present the *magnetic induction field* $\boldsymbol{B}$ $(= \mu\boldsymbol{H})$ as the **curl** of a vector potential $\boldsymbol{A}$:

$$\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A}. \tag{3.1}$$

Then the solenoidal property $\mathrm{div}\boldsymbol{B} = 0$ will be automatically satisfied, and the Ampère law becomes

$$\mathbf{curl}\,\boldsymbol{H} \equiv \mathbf{curl}(\mu^{-1}\mathbf{curl}\,\boldsymbol{A}) = \boldsymbol{j}. \tag{3.2}$$

In turn the boundary condition $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ will be satisfied if we require that $\boldsymbol{A}\wedge\boldsymbol{n} = 0$ on $\partial\Omega$. Hence we define the space

$$\mathcal{A} := H_0(\mathbf{curl};\Omega)\cap H(\mathrm{div};\Omega) \tag{3.3}$$

It is easy to check that

$$\|\boldsymbol{v}\|^2_{\mathcal{A}} := \|\mu^{-1/2}\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\boldsymbol{v}\|^2_{0,\Omega} \tag{3.4}$$

is a (Hilbert) norm on $\mathcal{A}$ (remember that we assumed Ω to be simply connected with connected boundary). Moreover, under mild regularity assumptions on Ω (see e.g. [33] and, mostly, the references therein) we also have

$$c_1\|\boldsymbol{v}\|_{1,\Omega} \le \|\boldsymbol{v}\|_{\mathcal{A}} \le c_2\|\boldsymbol{v}\|_{1,\Omega} \quad \forall\boldsymbol{v}\in\mathcal{A} \tag{3.5}$$

with c_1 and c_2 depending on Ω and μ_0, μ_1.

For the sake of simplicity, we will just assume, from now on, that Ω is convex and that μ is constant. Needless to say, the results hold in more general situations. For the treatment of Virtual Element discretizations of problems with variable coefficients we refer, for instance, to [12] and the references therein.

Here we will use one of the most classical variational formulations of the vector-potential equations: following for instance [22] we consider the problem

$$\begin{cases} \text{find } \boldsymbol{A}\in\mathcal{A} \text{ such that:} \\ a(\boldsymbol{A},\boldsymbol{v}) := \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\Omega + \int_\Omega \mathrm{div}\boldsymbol{A}\,\mathrm{div}\boldsymbol{v}\,\mathrm{d}\Omega = \int_\Omega \boldsymbol{j}\cdot\boldsymbol{v}\,\mathrm{d}\Omega \quad \forall\boldsymbol{v}\in\mathcal{A}. \end{cases} \tag{3.6}$$

It is clear that

$$a(\boldsymbol{v},\boldsymbol{v}) = \|\boldsymbol{v}\|^2_{\mathcal{A}} \tag{3.7}$$

so that (3.6) has a unique solution in $\mathcal{A}$. Then we check that the solution of (3.6) verifies $\mathrm{div}\boldsymbol{A} = 0$. For this we take $\varphi\in H^1_0(\Omega)$ such that $\Delta\varphi = \mathrm{div}\boldsymbol{A}$, and then we take $\boldsymbol{v} = \mathbf{grad}\varphi$ (that clearly belongs to $\mathcal{A}$). Then $\mathbf{curl}\boldsymbol{v} = 0$ and $\int_\Omega \boldsymbol{j}\cdot\boldsymbol{v}\,\mathrm{d}\Omega = 0$ as well (since $\mathrm{div}\boldsymbol{j} = 0$). Hence from (3.6) we have $\mathrm{div}\boldsymbol{A} = 0$. It also follows immediately that for $\boldsymbol{B} := \mathbf{curl}\boldsymbol{A}$ one gets $\mathrm{div}\boldsymbol{B} = 0$. Moreover from (3.6), using $\mathrm{div}\boldsymbol{A} = 0$ and integrating by parts, we have now that $\mathbf{curl}(\mu^{-1}\mathbf{curl}\boldsymbol{A}) = \boldsymbol{j}$. Hence setting $\boldsymbol{H} := \mu^{-1}\boldsymbol{B}$ we have $\mathbf{curl}\boldsymbol{H} = \mathbf{curl}(\mu^{-1}\boldsymbol{B}) = \mathbf{curl}(\mu^{-1}\mathbf{curl}\boldsymbol{A}) = \boldsymbol{j}$. Finally, on the boundary $\partial\Omega$ we have $\boldsymbol{B}\cdot\boldsymbol{n} = \mathrm{rot}(\boldsymbol{A}\wedge\boldsymbol{n}) = 0$.

Remark 3.1. The extension of the formulation to the case where $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ only on a subset Γ of the boundary (and $\boldsymbol{H}\wedge\boldsymbol{n} = 0$ on the remaining part) is immediate by substituting the space (3.3) with

$$\mathcal{A} := \{\boldsymbol{v}\in H(\mathbf{curl};\Omega)\cap H(\mathrm{div};\Omega) \text{ with } \boldsymbol{v}\wedge\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

In the following, in order to keep the notation simpler, we stick to (3.3), the extension to the more general case being trivial.

3.2. **The local spaces on faces**

We assume that we are given a sequence of decompositions $\{\mathcal{T}_h\}_h$ of the computational domain Ω into polyhedrons P. For every polyhedron P we define

$$h_{\mathrm{P}} := \mathrm{diameter of P} \tag{3.8}$$

and for every decomposition $\mathcal{T}_h$ we set

$$|h| := \sup_{\mathrm{P}\in\mathcal{T}_h} h_{\mathrm{P}} \tag{3.9}$$

We also assume that each P is simply connected and convex, with all its faces also simply connected and convex. (For the treatment of non-convex faces we refer to [15]). More detailed assumptions will be given in Section 5. We also assume, for the sake of simplicity, that

$$\mu \text{ is piecewise constant on } \mathcal{T}_h. \tag{3.10}$$

For the treatment of Virtual Element discretizations of problems with variable coefficients we refer, for instance, to [12] and references therein.

We will now design the Virtual Element approximation of (3.6) of order $k \geq 1$ on $\mathcal{T}_h$. We begin with the definition of the local spaces, and in particular we start by defining suitable VEM spaces on the faces. We are going to use, essentially, a particular choice of Serendipity nodal Virtual Element spaces of [15]. For this, for every integer $k \geq 1$ and for every face f we consider the Virtual Element space

$$\widetilde{V}_k(f) := \{v \in C^0(\overline{f}) \text{ such that } v_{|e} \in \mathbb{P}_k(e)\ \forall \text{ edge } e, \text{ and } \Delta_f v \in \mathbb{P}_k(f)\}. \tag{3.11}$$

In $\widetilde{V}_k(f)$ we have the natural degrees of freedom

$$\bullet \text{ value of } v(\nu), \text{ for every vertex } \nu \text{ of } f, \tag{3.12}$$

$$\bullet \text{ (for } k \geq 2\text{) value of } \int_e v\, q_{k-2}\, \mathrm{d}e,\ \forall q_{k-2} \in \mathbb{P}_{k-2}(e), \text{ for every edge } e \text{ of } f, \tag{3.13}$$

$$\bullet \text{ value of } \int_f v\, q_k\, \mathrm{d}f,\ \forall q_k \in \mathbb{P}_k(f). \tag{3.14}$$

In $\widetilde{V}_k(f)$ we want to identify a subspace that contains all polynomials of degree $\leq k$ but uses less degrees of freedom. For this, we will use a simplified version of the Serendipity elements of [15]. We consider first the space of $\mathbb{P}_k$-bubbles on f

$$B_k(f) := \{q \in \mathbb{P}_k(f) \text{ such that } q_{|\partial f} \equiv 0\}. \tag{3.15}$$

Note that $B_k \equiv \{0\}$ for $k \leq 2$, regardless of the number of edges of f (that, obviously, will always be ≥ 3), so that the first non-trivial bubble appears on a triangular face for $k = 3$. In general, the dimension of $B_k(f)$ will always verify

$$\text{dimension of } B_k(f) =: \beta_k(f) \leq \pi_{2,k-3} \tag{3.16}$$

where $\pi_{2,k-3}$ is the number of $\mathbb{P}_k$ bubbles on a triangle. We recall that we assumed, for the sake of simplicity, that f is convex, and we define a projector $v \to \Pi_{k,f} v$ from $\widetilde{V}_k(f)$ to $\mathbb{P}_k$ as the *least squares solution* of the system:

$$\bullet \int_{\partial f} (v - \Pi_{k,f} v)\, q_k\, \mathrm{d}s = 0\ \forall q_k \in \mathbb{P}_k(f) \tag{3.17}$$

$$\bullet \text{ (for } k \geq 3\text{) } \int_f (v - \Pi_{k,f} v)\, q_{k-3}\, \mathrm{d}f = 0\ \forall q_{k-3} \in \mathbb{P}_{k-3}(f). \tag{3.18}$$

Proposition 3.2. *For a triangular face f the system* (3.17)-(3.18) *has a unique solution. In the other cases, the system* (3.17)-(3.18) *is over-determined (i.e. it has more equations than unknowns) but its least square solution is unique.*

Proof. We note that the number of nontrivial equations in (3.17) is equal to the dimension of $\mathbb{P}_k$ minus the dimension of $B_k(f)$. For a *triangular* f, the dimension of $B_k(f)$ is equal to the dimension of $\mathbb{P}_{k-3}$, so that (3.17)-(3.18) is a square system, and it is immediate to check that the associated matrix is non-singular. For a *non triangular* face f the dimension of $B_k(f)$, according to (3.16), is *smaller* than $\pi_{2,k-3}$, and the number of equations of the system (3.17)-(3.18) is equal to $[\pi_{2,k} - \beta_k(f)] + [\pi_{2,k-3}]$, that is *bigger* than $\pi_{2,k}$ (the number of unknowns). To see that the least-squares solution is uniquely determined we have to check that for $v = 0$ the only solution is given by $\Pi_{k,f} v = 0$. For this, observe that if p, in $\mathbb{P}_k(f)$, vanishes on ∂f then either $p \equiv 0$, or p must have the form

$$p = b_\eta q^*_{k-\eta} \tag{3.19}$$

where:

- η (> 3) is the minimum number of straight lines necessary to cover ∂f,
- b_η is a polynomial in $\mathbb{P}_\eta(f)$ that vanishes on ∂f and is positive inside (remember, f is convex)
- $q^*_{k-\eta} \in \mathbb{P}_{k-\eta}(f) \subset \mathbb{P}_{k-3}(f)$.

For $v = 0$ (3.18) would then imply (taking $q_{k-3} = q^*_{k-\eta}$) that

$$0 = \int_f p\, q^*_{k-\eta}\, \mathrm{d}f = \int_f b_\eta (q^*_{k-\eta})^2\, \mathrm{d}f, \tag{3.20}$$

and finally $p = 0$. ■

Remark 3.3. It is easy to see that: if the face f has more that 3 edges (and $\eta > 3$, meaning that the boundary of f cannot be covered using only three straight lines), then the projection operator $\Pi_{k,f}$ could be defined using in (3.18) only the polynomials of degree $k-4$. Similarly, for a bigger and bigger η we could use fewer and fewer polynomials in (3.18). However, we decided that the present choice (reminiscent of what is called *the lazy choice* in [15]) has the advantage of working in general, and to avoid the necessity to detect more delicate situations (as, for instance, the case of a quadrilateral with an internal angle very close to π radiants).

Once the projection operator $\Pi_{k,f}$ has been defined, we can introduce for every face f the Serendipity VEM space $V_{S,k}(f)$.

Definition 3.4. The Serendipity VEM space $V_{S,k}(f)$ is defined as the subspace of $\widetilde{V}_k(f)$ made of elements v such that

$$\int_f (v - \Pi_{k,f} v)\, q^h_s\, \mathrm{d}f = 0 \quad \forall\, q^h_s \in \mathbb{P}^h_s(f),\ \forall \text{ non-negative integer } s \in [k-2, k]. \tag{3.21}$$

It is easy to see that a uni-solvent set of degrees of freedom for $V_{S,k}(f)$ is given by

$$\bullet \text{ value of } v(\nu), \text{ for every vertex } \nu \text{ of } f, \tag{3.22}$$

$$\bullet \text{ (for } k \ge 2) \text{ value of } \int_e v\, q_{k-2}\, \mathrm{d}e,\ \forall q_{k-2} \in \mathbb{P}_{k-2}(e), \text{ for every edge } e \text{ of } f, \tag{3.23}$$

$$\bullet \text{ (for } k \ge 3) \text{ value of } \int_f v\, q_{k-3}\, \mathrm{d}f,\ \forall q_{k-3} \in \mathbb{P}_{k-3}(f), \tag{3.24}$$

and consequently its dimension is given by $(kN_\nu(f) + \pi_{2,k-3})$ where $N_\nu(f)$ is the number of vertices of the face f and $\pi_{2,k-3}$ has to be taken as 0 for $k \le 2$.

We point out that every $v \in V_{S,k}$ is still an element of $\widetilde{V}_k(f)$, and from its degrees of freedom (3.22)-(3.24) we are able to compute (through $\Pi_{k,f}$ and (3.21)) all its degrees of freedom in $\widetilde{V}_k(f)$, and in particular

$$\int_f v\, q_k \,\mathrm{d}f \quad \forall q_k \in \mathbb{P}_k(f) \text{ are computable for } v \in V_{S,k}(f). \tag{3.25}$$

Remark 3.5. The first degrees of freedom internal to faces appear in $V_{S,k}$ only for $k \ge 3$. Clearly, depending on the type of decomposition that we use and on the type of work that we have to perform, we might save more. For instance, if we have a decomposition that uses only quadrilateral faces (that do not degenerate into triangles) we can systematically use, in the spirit of Remark 3.3, polynomials of degree $k-4$ in (3.18) (and then in (3.24)). In that case, degrees of freedom internal to faces will appear only for $k \ge 4$. Similarly if our decomposition is made, say, of non degenerated hexagons, then the internal degrees of freedom will appear for $k \ge 6$. The spaces that we obtain in (3.21) will coincide with the usual polynomial spaces ($\mathbb{P}_k(f)$) whenever the face f is triangular. For faces with more than 3 edges, our number of internal degrees of freedom, as we saw, will be equal to the dimension of the space $\mathbb{P}_{k-3}(f)$. This is more, for instance, than what is done for (non degenerate!) quadrilaterals in [3], but we chose it here only to avoid the necessity to detect non degeneracy in our elements (that are allowed to have very general geometries). Similar observations can be done if we want to compare the elements here with the ones in [35]. Needless to say, if the same mesh is going to be used for many resolutions, then it would be worth to spend some additional effort on every element, and use the elements of [15] (that, in general, would be much slimmer) instead. All these choices will not affect in a significant way the theoretical treatment that follows in the present paper. □

3.3. Traces of the local spaces on ∂P

Having defined our spaces on every face f, for a given polyhedron P we can define the space of traces

$$\mathbb{B}_k(\partial\mathrm{P}) := \{\boldsymbol{v} \in (C^0(\partial\mathrm{P}))^3 \text{ such that } \boldsymbol{v}_{|f} \in (V_{S,k}(f))^3 \ \forall \text{ face } f \text{ in } \partial\mathrm{P}\}. \tag{3.26}$$

Proposition 3.6. *A unisolvent set of degrees of freedom for $\mathbb{B}_k(f)$ is given by*

- *value of $\boldsymbol{v}(\nu)$, for every vertex ν of P,* (3.27)
- *(for $k \ge 2$) value of $\int_e \boldsymbol{v} \cdot \boldsymbol{q}_{k-2}\,\mathrm{d}e$, $\forall \boldsymbol{q}_{k-2} \in (\mathbb{P}_{k-2}(e))^3$, for every edge e of P,* (3.28)
- *(for $k \ge 3$) value of $\int_f \boldsymbol{v} \cdot \boldsymbol{q}_{k-3}\,\mathrm{d}f$, $\forall \boldsymbol{q}_{k-3} \in (\mathbb{P}_{k-3}(f))^3$, for every face f of P.* (3.29)

Proof. The result follows immediately from the information on the degrees of freedom, taking into account the continuity requirements on edges and vertexes. ■

Remark 3.7. The degrees of freedom (3.13) and (3.28) could be substituted by the value of $\boldsymbol{v}$ at $k-1$ distinct points on each edge. □

3.4. Local spaces on a polyhedron

In order to define the spaces *inside* P we follow the basic ideas of [18], and we set

$$\mathcal{A}_k(\mathrm{P}) := \{\boldsymbol{v} \in (C^0(\overline{\mathrm{P}}))^3 \text{ such that } \boldsymbol{v}_{|\partial\mathrm{P}} \in \mathbb{B}_k(\partial\mathrm{P}),\ \mathbf{curl}(\Delta\boldsymbol{v}) \in (\mathbb{P}_{k-3}(\mathrm{P}))^3,\ \mathrm{div}\boldsymbol{v} \in \mathbb{P}_0(\mathrm{P})\}. \tag{3.30}$$

Following [18], we have the following properties.

Proposition 3.8. *A unisolvent set of degrees of freedom for $\mathcal{A}_k(P)$ is given by*

- *value of* $\boldsymbol{v}(\nu)$, *for every vertex* ν *of* P, (3.31)
- *(for* $k \geq 2$*) value of* $\int_e \boldsymbol{v} \cdot \boldsymbol{q}_{k-2}\,\mathrm{d}e,\ \forall \boldsymbol{q}_{k-2} \in (\mathbb{P}_{k-2}(e))^3$, *for every edge* e *of* P, (3.32)
- *(for* $k \geq 3$*) value of* $\int_f \boldsymbol{v} \cdot \boldsymbol{q}_{k-3}\,\mathrm{d}f,\ \forall \boldsymbol{q}_{k-3} \in (\mathbb{P}_{k-3}(f))^3$, *for every face* f *of* P, (3.33)
- *(for* $k \geq 3$*)value of* $\int_P \boldsymbol{v} \cdot (\boldsymbol{x} \wedge \boldsymbol{q}_{k-3})\,\mathrm{d}P,\ \forall \boldsymbol{q}_{k-3} \in (\mathbb{P}_{k-3}(P))^3$. (3.34)

Proof. First we recall that the values (3.31)-(3.33) determine uniquely the *boundary values* of a $\boldsymbol{v}$ in $\mathcal{A}_k(\mathrm{P})$. Consequently, the (constant) value of *the divergence* of $\boldsymbol{v}$ is also determined uniquely, using the mean value of $\boldsymbol{v} \cdot \boldsymbol{n}$ on $\partial\mathrm{P}$. Hence we just need to show that adding the degrees of freedom (3.34) we can determine uniquely $\boldsymbol{v} \in \mathcal{A}_k(\mathrm{P})$. For that it would be enough to restrict our attention to the elements of $\mathcal{A}_k(\mathrm{P})$ that belong to the subspace

$$\mathrm{AUX} := \{\boldsymbol{v} \in (H^1_0(\mathrm{P}))^3 \text{ such that } \mathrm{div}\boldsymbol{v} = 0\} \tag{3.35}$$

(meaning that their values in (3.31)-(3.33) are all zero), and show that the values of (3.34) would determine uniquely a $\boldsymbol{v}$ among them.

For this we check first that the number of conditions in (3.34) matches the dimension of $\mathcal{A}_k(\mathrm{P})\cap\mathrm{AUX}$. We observe that an element $\boldsymbol{v}$ of AUX belongs to $\mathcal{A}_k(\mathrm{P})$ *if and only if* $\mathbf{curl}\Delta\boldsymbol{v}$ is in $(\mathbb{P}_{k-3})^3$, and this amounts to $3\pi_{3,k-3} - \pi_{k-4}$ conditions: indeed, remember that a vector valued polynomial $\boldsymbol{q}$ of degree $k-3$, in order to be a **curl**, must have a zero divergence, which amounts to π_{k-4} conditions. On the other hand, (3.34) amounts to $3\pi_{3,k-3} - \pi_{k-4}$ conditions as well, since for all vectors $\boldsymbol{q}_{k-3}$ of the form $\boldsymbol{q}_{k-3} = \boldsymbol{x} q_{k-4}$ (with $q_{k-4} \in \mathbb{P}_{k-4}$) the product $\boldsymbol{x} \wedge \boldsymbol{q}_{k-3} \equiv \boldsymbol{x} \wedge \boldsymbol{x} q_{k-4}$ is identically zero.

Hence, we are reduced to prove that **if** $\boldsymbol{v} \in \mathcal{A}_K\cap\mathrm{AUX}$ has the values (3.34) all equal to zero **then** we must have $\boldsymbol{v} = 0$. We observe that $\mathbf{curl}(\Delta\boldsymbol{v})$ is in $(\mathbb{P}_{k-3}(\mathrm{P}))^3$, and, being a **curl**, has zero divergence; we deduce that $\mathbf{curl}(\Delta\boldsymbol{v})$ is equal to the **curl** of some polynomial vector in $(\mathbb{P}_{k-2}(\mathrm{P}))^3$. Using then (2.5) we have that there exists a $\boldsymbol{q}^*_{k-3} \in (\mathbb{P}_{k-3}(\mathrm{P}))^3$ such that

$$\mathbf{curl}(\Delta\boldsymbol{v}) = \mathbf{curl}(\boldsymbol{x} \wedge \boldsymbol{q}^*_{k-3}), \tag{3.36}$$

implying, since P is simply connected, that

$$\Delta\boldsymbol{v} = \boldsymbol{x} \wedge \boldsymbol{q}^*_{k-3} + \nabla s \tag{3.37}$$

for some $s \in H^1(\mathrm{P})$. Next, noting that for $\boldsymbol{v} \in AUX$ we have $\boldsymbol{v} = 0$ on $\partial\mathrm{P}$, integrating by parts and using (3.37) and then $\mathrm{div}\boldsymbol{v} = 0$ and (3.34) we have

$$\int_\mathrm{P} |\nabla\boldsymbol{v}|^2\,\mathrm{dP} = -\int_\mathrm{P} \boldsymbol{v} \cdot \Delta\boldsymbol{v}\,\mathrm{dP} = -\int_\mathrm{P} \boldsymbol{v} \cdot (\nabla s + \boldsymbol{x} \wedge \boldsymbol{q}^*_{k-3})\,\mathrm{dP} = 0 + 0 = 0 \tag{3.38}$$

and the proof is completed. ∎

Remark 3.9. Clearly, another (conceptually simpler) option wold be to take as $\mathcal{A}_k$ the space of triples of nodal VEMs. The advantage with the present choice is in the use of a **constant divergence**, that will allow to have a truly divergence-free solution, as well as a reduction of the number of degrees of freedom in P (that has nothing to see with the (possible) use of 3D Serendipity elements). □

3.5. Quantities that are computable in $\mathcal{A}_k(\mathrm{P})$

Assume now that we are given a polyhedron P and, for an integer $k \geq 1$, the VEM nodal space $\mathcal{A}_k(\mathrm{P})$ as defined in (3.30). Assume moreover that we are given the degrees of freedom (3.31)-(3.34) of an element $\boldsymbol{v} \in \mathcal{A}_k(\mathrm{P})$. The question is: what are the quantities, related to $\boldsymbol{v}$, that we can actually calculate on a computer, without solving a (system of) PDE's in P? As a general set-up of the problem, we assume that we can compute: the integral over edges, faces, and P of all polynomials of degree $\leq k$. But the elements of $\mathcal{A}_k(\mathrm{P})$ are **not** polynomials, in general, apart from very special cases (e.g., if P is a tetrahedron and $k \leq 2$). Or, to be more precise, all polynomials of $(\mathbb{P}_k)^3$ with constant divergence will belong to $\mathcal{A}_k(\mathrm{P})$, that however will contain other, non polynomial, functions.

To start with, using (3.31) and (3.32) we see that:

$$\bullet \text{ The values of each component of } \boldsymbol{v} \text{ on every edge of P are computable.} \tag{3.39}$$

Then, on each face f we can use (3.25) to see that:

$$\bullet \text{ For every face } f, \forall \boldsymbol{q} \in (\mathbb{P}_k(f))^3 \text{ the moments } \int_f \boldsymbol{v} \cdot \boldsymbol{q} \,\mathrm{d}f \text{ are computable.} \tag{3.40}$$

In particular, on every face f we will be able to compute

$$\int_f \boldsymbol{v} \cdot \boldsymbol{n}_{\mathrm{P}} \,\mathrm{d}f, \tag{3.41}$$

where, on each f, $\boldsymbol{n}_{\mathrm{P}}$ is the (3-dimensional) unit vector normal to the face f. As the divergence of $\boldsymbol{v}$ is constant in P (see (3.30)), from (3.41) we immediately see that:

$$\bullet \text{ The value of } \mathrm{div}\boldsymbol{v} \text{ in P is computable.} \tag{3.42}$$

We can also compute the moments of $\boldsymbol{v}$ against all (vector valued) polynomials of degree $\leq k-2$ in P. Indeed, given a $\boldsymbol{p}_{k-2} \in (\mathbb{P}_{k-2}(\mathrm{P}))^3$ we can use (2.4) and write it as

$$\boldsymbol{p}_{k-2} = \nabla q_{k-1} + \boldsymbol{x} \wedge \boldsymbol{q}_{k-3}$$

with $q_{k-1} \in \mathbb{P}_{k-1}(\mathrm{P})$ and $\boldsymbol{q}_{k-3} \in (\mathbb{P}_{k-3}(\mathrm{P}))^3$. Hence:

$$\begin{aligned} \int_{\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{p}_{k-2} \,\mathrm{dP} &= \int_{\mathrm{P}} \boldsymbol{v} \cdot (\nabla q_{k-1} + \boldsymbol{x} \wedge \boldsymbol{q}_{k-3}) \,\mathrm{dP} = \int_{\mathrm{P}} \boldsymbol{v} \cdot \nabla q_{k-1} \,\mathrm{dP} + \int_{\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{x} \wedge \boldsymbol{q}_{k-3} \,\mathrm{dP} \\ &= -\int_{\mathrm{P}} \mathrm{div}\boldsymbol{v}\, q_{k-1} \,\mathrm{dP} + \int_{\partial\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{n}_{\mathrm{P}}\, q_{k-1} \,\mathrm{d}S + \int_{\mathrm{P}} \boldsymbol{v} \cdot (\boldsymbol{x} \wedge \boldsymbol{q}_{k-3}) \,\mathrm{dP} \end{aligned} \tag{3.43}$$

and all the three terms of the last line are computable (the third using (3.34)). Hence:

$$\bullet \text{ The values of } \int_{\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{q}_{k-2} \,\mathrm{dP} \;\; \forall \boldsymbol{q}_{k-2} \in (\mathbb{P}_{k-2}(\mathrm{P}))^3 \text{ are computable.} \tag{3.44}$$

For $k = 1$, using $\boldsymbol{p}_0 = \nabla \boldsymbol{q}_1$ and proceeding as in (3.43) we obtain that

$$\bullet \int_{\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{q}_0 \,\mathrm{dP} \text{ is computable.} \tag{3.45}$$

The moments of $\mathbf{grad}\boldsymbol{v}$ against all tensor valued polynomials of degree $\leq k-1$ are also computable. To see this, let $\boldsymbol{\tau}_{k-1} \in (\mathbb{P}_{k-1}(\mathrm{P}))^{3\times 3}$ and consider

$$\int_{\mathrm{P}} (\mathbf{grad}\boldsymbol{v}) : \boldsymbol{\tau}_{k-1} \,\mathrm{dP} = -\int_{\mathrm{P}} \boldsymbol{v} \cdot (\mathbf{div}(\boldsymbol{\tau}_{k-1})) \,\mathrm{dP} + \int_{\partial\mathrm{P}} \boldsymbol{v} \cdot (\boldsymbol{\tau}_{k-1} \cdot \boldsymbol{n}_{\mathrm{P}}) \,\mathrm{d}S. \tag{3.46}$$

In (3.46) $\mathbf{div}(\boldsymbol{\tau}_{k-1})$ is a vector in $(\mathbb{P}_{k-2}(\mathrm{P}))^3$, so that the first term is computable from (3.43). Similarly, $\boldsymbol{\tau}_{k-1} \cdot \boldsymbol{n}_{\mathrm{P}}$ is in $(\mathbb{P}_{k-1}(f))^3$ on each face, so that remembering (3.40) the second term is computable as well. Hence:

$$\bullet \text{ The value of } \int_{\mathrm{P}} \mathbf{grad}\boldsymbol{v} : \boldsymbol{\tau}_{k-1} \,\mathrm{dP} \text{ is computable } \forall \boldsymbol{\tau}_{k-1} \in (\mathbb{P}_{k-1}(\mathrm{P}))^{3\times 3}. \tag{3.47}$$

Note that (3.47) implies that for every $\boldsymbol{v} \in \mathcal{A}_k$, for every component v_i, $(i = 1, 2, 3)$ and for every index j, $(j = 1, 2, 3)$:

- the $L^2(\mathrm{P})$-projection of $\dfrac{\partial v_i}{\partial x_j}$ onto $\mathbb{P}_{k-1}(\mathrm{P})$ is computable. (3.48)

Hence we can also compute, for $\boldsymbol{v} \in \mathcal{A}_k(\mathrm{P})$ and $\boldsymbol{q} \in (\mathbb{P}_k(\mathrm{P}))^3$, the quantities

$$\int_{\partial \mathrm{P}} (\boldsymbol{v} \wedge \boldsymbol{n}) \cdot (\boldsymbol{q} \wedge \boldsymbol{n}) \, \mathrm{d}S, \tag{3.49}$$

$$\int_{\mathrm{P}} (\mathrm{div}\boldsymbol{v}) \, (\mathrm{div}\boldsymbol{q}) \, \mathrm{dP}, \tag{3.50}$$

$$\int_{\mathrm{P}} \mathbf{curl}\boldsymbol{v} \cdot \mathbf{curl}\boldsymbol{q} \, \mathrm{dP}. \tag{3.51}$$

Introducing the restriction of the bilinear form a to P, as natural

$$a^{\mathrm{P}}(\boldsymbol{u}, \boldsymbol{v}) := \int_{\mathrm{P}} \mu^{-1} \mathbf{curl}\boldsymbol{u} \cdot \mathbf{curl}\boldsymbol{v} \, \mathrm{dP} + \int_{\mathrm{P}} \mathrm{div}\boldsymbol{u} \, \mathrm{div}\boldsymbol{v} \, \mathrm{dP} \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}_k(\mathrm{P}), \tag{3.52}$$

we also have as an immediate consequence that:

- $\forall \boldsymbol{v} \in \mathcal{A}_k(\mathrm{P})$ and $\forall \boldsymbol{q} \in (\mathbb{P}_k(\mathrm{P}))^3 : \qquad a^{\mathrm{P}}(\boldsymbol{v}, \boldsymbol{q})$ is computable. (3.53)

All this will allow us to compute a projection operator Π^A_k from smooth-enough vector valued functions onto $(\mathbb{P}_k(\mathrm{P}))^3$. For this we first introduce the space

$$\mathbb{H}_k := \{\boldsymbol{q}_k \in (\mathbb{P}_k(\mathrm{P}))^3 \text{ such that } \exists \varphi \in \mathbb{P}_{k+1}(\mathrm{P}) \text{ with } \Delta\varphi = 0 \text{ and } \boldsymbol{q}_k = \nabla\varphi\} \tag{3.54}$$

of the gradients of the harmonic polynomials in $\mathbb{P}_{k+1}(\mathrm{P})$. We note that, as it can be easily checked:

$$\mathbb{H}_k \equiv \{\boldsymbol{q}_k \in (\mathbb{P}_k(\mathrm{P}))^3 \text{ such that } a_P(\boldsymbol{q}_k, \boldsymbol{q}_k) = 0\}. \tag{3.55}$$

We also note that:

$$\forall \boldsymbol{q} \in \mathbb{H}_k : \qquad \{\boldsymbol{q} \wedge \boldsymbol{n}_{\mathrm{P}} = 0 \text{ on } \partial\mathrm{P}\} \Leftrightarrow \{\boldsymbol{q} \equiv \boldsymbol{0}\}. \tag{3.56}$$

Then we can introduce the following definition.

Definition 3.10. Given $\boldsymbol{v}$, for instance, in $(H^1(\mathrm{P})$ we define its projection $\Pi^A_k \boldsymbol{v}$ onto $(\mathbb{P}_k(\mathrm{P}))^3$ as follows:

$$a^{\mathrm{P}}(\Pi^A_k \boldsymbol{v} - \boldsymbol{v}, \boldsymbol{q}_k) = 0 \qquad \forall \boldsymbol{q}_k \in (\mathbb{P}_k(\mathrm{P}))^3, \tag{3.57}$$

$$\int_{\partial\mathrm{P}} [(\Pi^A_k \boldsymbol{v} - \boldsymbol{v}) \wedge \boldsymbol{n}] \cdot [\boldsymbol{q}_k \wedge \boldsymbol{n}] \, \mathrm{d}S = 0 \qquad \forall \boldsymbol{q}_k \in \mathbb{H}_k. \tag{3.58}$$

Note that, due to (3.55) and (3.56), the solution of (3.57) -(3.58) is unique in $(\mathbb{P}_k(\mathrm{P}))^3$. □

Remark 3.11. Clearly, the projection operator Π^A_k is not $(L^2(\mathrm{P}))^3$-orthogonal, but, in some sense, is a_P-orthogonal. This, however, will not be a problem in what follows. □

Remark 3.12. The space $\mathcal{A}_k(\mathrm{P})$, as presented in (3.30), **does not** contain all polynomials in $(\mathbb{P}_k)^3$, but only the subspace made of those with constant divergence. In order to keep all polynomials of $(\mathbb{P}_k)^3$ inside, we should (obviously) take instead

$$\widetilde{A}_k(\mathrm{P}) := \{\boldsymbol{v} \in (C^0(\overline{\mathrm{P}}))^3 \text{ s. t. } \boldsymbol{v}_{|\partial\mathrm{P}} \in \mathbb{B}_k(\partial\mathrm{P}), \ \mathbf{curl}(\Delta\boldsymbol{v}) \in (\mathbb{P}_{k-3}(\mathrm{P}))^3, \ \mathrm{div}\boldsymbol{v} \in \mathbb{P}_{k-1}(\mathrm{P})\}, \tag{3.59}$$

and, as degrees of freedom, add to (3.31)-(3.34) the natural ones

- (for $k \geq 2$) value of $\displaystyle\int_{\mathrm{P}} \mathrm{div}\boldsymbol{v} \, q^0_{k-1} \quad \forall q^0_{k-1} \in \mathbb{P}^0_{k-1}(\mathrm{P})$ (3.60)

as in [18]. Note that this space, for higher order degree k, would be decidedly bigger than $\mathcal{A}_k(\mathrm{P})$: e.g. for, say, $k = 4$ it would have 34 additional internal degrees of freedom. □

At this point we are able to re-enter the more classical path of VEMs. In particular, we can define *the contribution of the element P* to the *global approximated bilinear form*: for $\boldsymbol{u}$ and $\boldsymbol{v} \in (\mathcal{A}_k(\mathrm{P}) + (\mathbb{P}_k)^3)$ we set

$$a_h^{\mathrm{P}}(\boldsymbol{u},\boldsymbol{v}) := a^{\mathrm{P}}(\Pi_k^A \boldsymbol{u}, \Pi_k^A \boldsymbol{v}) + S_h^{\mathrm{P}}((\mathrm{I} - \Pi_k^A)\boldsymbol{u}, (\mathrm{I} - \Pi_k^A)\boldsymbol{v}), \tag{3.61}$$

where $S_h^{\mathrm{P}}(\cdot\,,\cdot)$, as usual, is a symmetric bilinear form on $(\mathcal{A}_k(\mathrm{P}) + (\mathbb{P}_k)^3)$ such that there exist two constants α_* and α^*, independent of P, with

$$\alpha_* a^{\mathrm{P}}(\boldsymbol{v},\boldsymbol{v}) \le a_h^{\mathrm{P}}(\boldsymbol{v},\boldsymbol{v}) \le \alpha^* |\boldsymbol{v}|^2_{1,\mathrm{P}} \quad \forall \boldsymbol{v} \in \mathcal{A}_k(\mathrm{P}). \tag{3.62}$$

Needless to say, (3.57) and (3.61) easily imply that

$$a_h^{\mathrm{P}}(\boldsymbol{u},\boldsymbol{v}) \equiv a^{\mathrm{P}}(\boldsymbol{u},\boldsymbol{v}) \qquad \text{whenever either } \boldsymbol{u} \text{ or } \boldsymbol{v} \text{ is in } (\mathbb{P}_k(\mathrm{P}))^3. \tag{3.63}$$

Remark 3.13. It is easy to check that, in order for (3.62) to hold, it is sufficient for S_h^{P} to satisfy $c_\star |\boldsymbol{v}|^2_{1,\mathrm{P}} \le S_h^{\mathrm{P}}(\boldsymbol{v},\boldsymbol{v}) \le c^\star |\boldsymbol{v}|^2_{1,\mathrm{P}}$ for some uniform constants $c_\star, c^\star$. A classical choice for S_h^{P} consists in taking

$$S_h^{\mathrm{P}}(\boldsymbol{u},\boldsymbol{v}) := \sum_i \delta_i(\boldsymbol{u})\delta_i(\boldsymbol{v}) \tag{3.64}$$

where the $\delta_i(\boldsymbol{v})$ are the degrees of freedom of $\boldsymbol{v}$ in P, properly scaled. See e.g. [9]. □

4. The global spaces and the discretized problem

4.1. The global spaces

From the *local* Virtual Element spaces, defined in each $\mathrm{P} \in \mathcal{T}_h$, we can now construct easily the *global* spaces in Ω. We set

$$\mathcal{A}_h \equiv \mathcal{A}_h(\Omega) := \{\boldsymbol{v} \in \mathcal{A} \text{ such that } \boldsymbol{v} \in \mathcal{A}_k(\mathrm{P}) \text{ for all element } \mathrm{P} \in \mathcal{T}_h\}. \tag{4.1}$$

On $\mathcal{A}_h$ we can define the *global* bilinear form a_h simply setting

$$a_h(\boldsymbol{u},\boldsymbol{v}) := \sum_{\mathrm{P}\in\mathcal{T}_h} a_h^{\mathrm{P}}(\boldsymbol{u},\boldsymbol{v}). \tag{4.2}$$

Finally, we also define, in each element P

$$\left(\boldsymbol{j}_h\right)_{|\mathrm{P}} := \begin{cases} (L^2(\mathrm{P}))^3\text{-orthogonal projection of } \boldsymbol{j} \text{ onto } (\mathbb{P}_0(\mathrm{P}))^3 \text{ for } k = 1 \\ (L^2(\mathrm{P}))^3\text{-orthogonal projection of } \boldsymbol{j} \text{ onto } (\mathbb{P}_{k-2}(\mathrm{P}))^3 \text{ for } k \ge 2, \end{cases} \tag{4.3}$$

and we note that the integral

$$\int_\Omega \boldsymbol{j}_h \cdot \boldsymbol{v} \,\mathrm{d}\Omega$$

is computable for every $\boldsymbol{v} \in \mathcal{A}_h$ (due to (3.44) and (3.45)).

4.2. The discretized problem

The discretized version of (3.6) will now read

$$\begin{cases} \text{find } \boldsymbol{A}_h \in \mathcal{A}_h \text{ such that:} \\ a_h(\boldsymbol{A}_h,\boldsymbol{v}) = \displaystyle\int_\Omega \boldsymbol{j}_h \cdot \boldsymbol{v} \,\mathrm{d}\Omega. \quad \forall \boldsymbol{v} \in \mathcal{A}_h. \end{cases} \tag{4.4}$$

It is very easy to see that a_h is symmetric, and satisfies the two fundamental properties of VEM approximations of linear elliptic problems, namely:

$$a(\boldsymbol{v},\boldsymbol{q}) = a_h(\boldsymbol{v},\boldsymbol{q}) \quad \text{ for all } \boldsymbol{v} \in \mathcal{A}_h, \text{ and for all } \boldsymbol{q} \text{ piecewise in } (\mathbb{P}_k)^3 \tag{4.5}$$

and

$$\exists\,\alpha_* \text{ and } \alpha^* \text{ in } \mathbb{R} \text{ such that:} \qquad \alpha_* a(\boldsymbol{v},\boldsymbol{v}) \le a_h(\boldsymbol{v},\boldsymbol{v}) \le \alpha^* a(\boldsymbol{v},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{A}_h. \tag{4.6}$$

Note that the constants α_* and α^* will also depend on μ_0 and μ_1. We point out that in deriving (4.6) from the local (3.62) we are now able to use (3.5) (that needs the boundary conditions on $\partial\Omega$ but not on ∂P) and use $a(\boldsymbol{v},\boldsymbol{v})$ on the right-hand side instead of $|\boldsymbol{v}|_1^2$ as we had in (3.62). Indeed using (3.62) and (3.5) we have

$$a_h(\boldsymbol{v},\boldsymbol{v}) = \sum_{\mathrm{P}} a_h^{\mathrm{P}}(\boldsymbol{v},\boldsymbol{v}) \le \alpha^* \sum_{\mathrm{P}} |\boldsymbol{v}|^2_{1,\mathrm{P}} = \alpha^* \|\boldsymbol{v}\|^2_{1,\Omega} \le \alpha^*\, c_2^2\, \|\boldsymbol{v}\|^2_{\mathcal{A}}. \tag{4.7}$$

We also note that the symmetry of a_h and (4.6) easily imply the continuity of a_h with

$$\begin{aligned} a_h(\boldsymbol{u},\boldsymbol{v}) &\le \Big(a_h(\boldsymbol{u},\boldsymbol{u})\Big)^{1/2}\Big(a_h(\boldsymbol{v},\boldsymbol{v})\Big)^{1/2} \\ &\le \alpha^* \Big(a(\boldsymbol{u},\boldsymbol{u})\Big)^{1/2}\Big(a(\boldsymbol{v},\boldsymbol{v})\Big)^{1/2} \le \alpha^*\, \|\boldsymbol{u}\|_{\mathcal{A}}\, \|\boldsymbol{v}\|_{\mathcal{A}} \end{aligned} \tag{4.8}$$

for all $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathcal{A}_h$.

5. Error Estimates

5.1. The convergence theorem

We start our discussion with an abstract convergence result, that bounds the error $|u - u_h|$ in terms of suitable interpolation errors for u and in terms of the error $|\boldsymbol{j} - \boldsymbol{j}_h|$ in the right-hand side.

Theorem 5.1. *The discrete problem* (4.4) *has a unique solution* $\boldsymbol{A}_h$. *Moreover, for every approximation* $\boldsymbol{A}_I$ *of* $\boldsymbol{A}$ *in* $\mathcal{A}_h$ *and for every approximation* $\boldsymbol{A}_\pi$ *of* $\boldsymbol{A}$ *that is piecewise in* $\mathbb{P}_k$, *we have*

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\mathcal{A}} \le C\Big(\|\boldsymbol{A} - \boldsymbol{A}_I\|_{1,h} + \|\boldsymbol{A} - \boldsymbol{A}_\pi\|_{1,h} + \|\boldsymbol{j} - \boldsymbol{j}_h\|_{\mathcal{A}_h'}\Big), \tag{5.1}$$

where:

- C *is a constant depending only on* α_*, α^*, μ_0, μ_1,
- $\|\boldsymbol{v}\|_{1,h} := \Big(\sum_{P\in\mathcal{T}_h} \|\boldsymbol{v}\|^2_{1,P}\Big)^{1/2}$
- $\|\boldsymbol{j} - \boldsymbol{j}_h\|_{\mathcal{A}_h'}$ *is defined as the smallest constant* $\mathfrak{C}$ *such that*

$$(\boldsymbol{j},\boldsymbol{v}) - (\boldsymbol{j}_h,\boldsymbol{v}) \le \mathfrak{C}\,|\boldsymbol{v}|_{\mathcal{A}} \quad \forall \boldsymbol{v} \in \mathcal{A}_h. \tag{5.2}$$

Proof. The proof follows exactly the same lines as the original one in [9]. Existence and uniqueness of the solution of (4.4) are a consequence of (4.6) and (3.7). Next, setting $\boldsymbol{\delta}_h := \boldsymbol{A}_h - \boldsymbol{A}_I$ we have, starting from (4.6) and then $\boldsymbol{\delta}_h = \boldsymbol{A}_h - \boldsymbol{A}_I$,

$$\begin{aligned}
&\alpha_* \|\boldsymbol{\delta}_h\|^2_{\mathcal{A}} = \alpha_*\, a(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h) \le a_h(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h) = a_h(\boldsymbol{A}_h,\boldsymbol{\delta}_h) - a_h(\boldsymbol{A}_I,\boldsymbol{\delta}_h)\\
&\text{[use (4.4) and (4.2))]} \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} a_h^{\mathrm{P}}(\boldsymbol{A}_I,\boldsymbol{\delta}_h)\\
&\text{[use } \pm \boldsymbol{A}_\pi] \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I - \boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a_h^{\mathrm{P}}(\boldsymbol{A}_\pi,\boldsymbol{\delta}_h)\Big)\\
&\text{[use (3.63)]} \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I - \boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a^{\mathrm{P}}(\boldsymbol{A}_\pi,\boldsymbol{\delta}_h)\Big)\\
&\text{[use } \pm a(\boldsymbol{A},\boldsymbol{\delta}_h)] \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I - \boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a^{\mathrm{P}}(\boldsymbol{A}_\pi - \boldsymbol{A},\boldsymbol{\delta}_h)\Big) - a(\boldsymbol{A},\boldsymbol{\delta}_h)\\
&\text{[use (3.6)]} \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I - \boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a^{\mathrm{P}}(\boldsymbol{A}_\pi - \mathcal{A},\boldsymbol{\delta}_h)\Big) - (\boldsymbol{j},\boldsymbol{\delta}_h)\\
&\text{[re-order]} \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - (\boldsymbol{j},\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I - \boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a^{\mathrm{P}}(\boldsymbol{A}_\pi - \boldsymbol{A},\boldsymbol{\delta}_h)\Big).
\end{aligned}$$

Now use (5.2), (4.8), and the continuity of each a^{P} to obtain

$$\|\boldsymbol{\delta}_h\|^2_{\mathcal{A}} \le C\Big(\|\boldsymbol{j}-\boldsymbol{j}_h\|_{\mathcal{A}'_h} + \|\boldsymbol{A}_I-\boldsymbol{A}_\pi\|_{1,h} + \|\boldsymbol{A}-\boldsymbol{A}_\pi\|_{1,h}\Big)\,\|\boldsymbol{\delta}_h\|_{\mathcal{A}} \tag{5.3}$$

for some constant C depending only on α_*, α^*, μ_0 and μ_1. Then the result follows easily by the triangle inequality. ∎

From the estimate (5.1), given the sequence of decompositions $\{\mathcal{T}_h\}_h$ one can then deduce an error estimate in terms of powers of $|h|$ (as defined in (3.9)), of some regularity constant for the polyhedrons, and of the regularity of the solution $\boldsymbol{u}$. For this we need suitable interpolation estimates.

5.2. Interpolation estimates

Theorem 5.2. *Assume that the sequence of decompositions $\{\mathcal{T}_h\}_h$ satisfies the following assumptions (that are quite standard in the VEM literature): There exists a positive constant γ, independent of h, such that for every h all polyhedrons P of $\mathcal{T}_h$ satisfy:*

- **D1**) *P is star-shaped with respect to a sphere of radius bigger than γh_P;*
- **D2**) *every face $f \in \partial P$ is star-shaped with respect to a disk of radius bigger than γh_P, and every edge of P has length bigger than γh_P.*

Then if the spaces $\mathcal{A}_h$ are defined as in (4.1) *for some integer $k \ge 1$ we have*

$$\|\boldsymbol{A}-\boldsymbol{A}_I\|_{1,h} + \|\boldsymbol{A}-\boldsymbol{A}_\pi\|_{1,h} \le C_1\,|h|^k \tag{5.4}$$

and

$$(\boldsymbol{j}-\boldsymbol{j}_h,\boldsymbol{v}) = \le C_2\,|h|^k \tag{5.5}$$

where C_1 and C_2 are constants that depend only on $\gamma, \alpha_*, \alpha^*, \mu_0, \mu_1$ and on the regularity of $\boldsymbol{A}$ and $\boldsymbol{j}$, respectively.

Proof.
From known results on polynomial approximation (see e.g. [26]), one can first get easily

$$\|\boldsymbol{A}-\boldsymbol{A}_\pi\|_{1,h} \le c_{ext}\,|h|^k\,\|\boldsymbol{A}\|_{k,\Omega} \tag{5.6}$$

and

$$(\boldsymbol{j}-\boldsymbol{j}_h,\boldsymbol{v}) = \le c_{ext}\,|h|^k \|\boldsymbol{j}\|_{k,\Omega} \tag{5.7}$$

for some constant c_{ext} depending on k and on the maximum (over the polygons P) of the constants that bound the extension of a function φ from P to a sphere of diameter $2\,h_{\mathrm{P}}$ containing P. Note that these constants, themselves, can also be uniformly bounded in terms of the γ appearing in **D1** and **D2**. Then we define $\boldsymbol{A}_I$ as the interpolant of $\boldsymbol{A}$, locally, in $\widetilde{\mathcal{A}}_k(\mathrm{P})$ as defined in (3.59). At first sight, such an $\boldsymbol{A}_I$ might fail to belong to $\mathcal{A}_k(\mathrm{P})$: indeed, $\mathcal{A}_k(\mathrm{P})$, being made of vectors with constant divergence, is *smaller* than $\widetilde{\mathcal{A}}_k(\mathrm{P})$ which is made of vectors having divergence in $\mathbb{P}_{k-1}$. But we recall that $\boldsymbol{A}$ has *zero* divergence, and it is easy to see that the degrees of freedom of $\widetilde{\mathcal{A}}_k(\mathrm{P})$ are such that the interpolant of a solenoidal vector is itself solenoidal. Now me make profit of the fact that $\widetilde{\mathcal{A}}_k(\mathrm{P})$ contains *all* vector polynomials of degree $\le k$, and with the (nowadays) classical instruments of Virtual Element approximation theory (see e.g. [17, 19, 25, 27, 32, 49]) it is not difficult to see that we also have

$$\|\boldsymbol{A}-\boldsymbol{A}_I|_{1,h} \le c\,|h|^k\,\|\boldsymbol{A}\|_{k,\Omega} \tag{5.8}$$

for some constant c that depends on k and on the constant γ in **D1** and **D2**.

In realtà sarebbe facile far vedere che

$$\|\boldsymbol{A} - \boldsymbol{A}_I|_{L^\infty} \leq c\,|h|^k\,\|\boldsymbol{A}\|_{k,\infty,\Omega} \tag{5.9}$$

e poi maggiorare la norma in H^1 con la norma euclidea dei gradi di libertà (Lorenzo's style). Facciamo finta di niente? ■

Setting now $\boldsymbol{B}_h := \mathbf{curl}\,\boldsymbol{A}_h$ and $\boldsymbol{H}_h := \mu^{-1}\boldsymbol{B}_h$, and inserting estimates (5.4) and (5.5) into (5.1) we finally obtain:

$$\|\boldsymbol{H} - \boldsymbol{H}_h\|_{0,\Omega} + \|\boldsymbol{B} - \boldsymbol{B}_h\|_{0,\Omega} \leq C\,|h|^k(\|\boldsymbol{A}\|_{k+1,\Omega} + \|\boldsymbol{j}\|_{k-1,\Omega}) \tag{5.10}$$

for a constant C that depends only on $\gamma, \alpha_*, \alpha^*, \mu_0, \mu_1$ and k.

Forse anche questa stronzata andrebbe travestita da "teorema finale" : richiamando chi sono $\boldsymbol{A}$ ed $\boldsymbol{A}_h$ e le ipotesi sulla decomposizione, e definendo $\boldsymbol{B}_h$ e $\boldsymbol{H}_h$ si ha BLA BLA. Magari pi' tardi...

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