Mixed finite elements for Maxwell’s eigenproblem: the question of spurious modes

D. Boffi
Dipartimento di Matematica “F. Casorati”, Università di Pavia. e-mail: boffi@ian.pv.cnr.it

F. Brezzi
I.A.N.-C.N.R., Pavia and Dipartimento di Matematica “F. Casorati”, Università di Pavia. e-mail: brezzi@ian.pv.cnr.it

L. Gastaldi
Dipartimento di Elettronica per l’Automazione, Università di Brescia. e-mail: gastaldi@ian.pv.cnr.it

In this paper we study the finite element approximation of Maxwell’s cavity eigenproblem. It is well-known that several choices for the approximating space are available, but some of them give poor results. For example the discretization by means of piecewise linear finite elements gives rise to bad approximation of the zero frequency unless particular structured meshes are used. In this latter case, it has been observed in the numerical computations that the zero frequency is well approximated, as well as the nonzero ones. But the numerical scheme produces also “spurious” eigenvalues which converge to some finite values not belonging to the spectrum of the continuous problem. On the other hand the so-called edge elements produce a finite number of zero eigenvalues, quite good approximation of the nonzero ones and “no spurious eigenvalue”. Our aim will be to understand this behavior of the discrete spectrum giving sufficient conditions in order to ensure the absence of any spurious eigenvalue.

1 Setting of the problem

Let $\Omega$ be a polygon in $\mathbb{R}^2$ and $\mathbf{t}$ the counterclockwise oriented tangent versor to its boundary $\partial \Omega$. Consider the following eigenproblem:

$$\begin{cases}
\text{rot} \ (\text{rot} \ u) = \lambda^2 u & \text{in } \Omega \\
\text{div} \ u = 0 & \text{in } \Omega \\
u : \mathbf{t} = 0 & \text{on } \partial \Omega
\end{cases} \tag{1}$$

It is well known that problem (1) is obtained from Maxwell’s system for vector phasors when the magnetic field is eliminated. Our analysis could be extended to more general situations (see [4] for more details).

In the approximation of (1) it is classical to drop the divergence free con-
strain and to deal with the following variational formulation:

\[
\text{find } \lambda \in \mathbb{R} \text{ s.t. } \exists \mathbf{u} \in H_0(\text{rot}; \Omega), \mathbf{u} \neq 0 : \\
\text{(rot } \mathbf{u}; \text{ rot } \mathbf{v}) = \lambda^2 (\mathbf{u}; \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\text{rot}; \Omega).
\]  

(2)

The eigenpairs of (2) are the same as those of (1), except that the zero frequency belongs to the spectrum of (2) but not to that of (1). The corresponding infinite-dimensional eigenspace coincides with \( \sum H_0^2(\Omega) \).

Let \( \Sigma_h \) be a finite element subspace of \( H_0(\text{rot}; \Omega) \); thus the finite element discretization of (2) reads as follows:

\[
\text{find } \lambda_h \in \mathbb{R} \text{ s.t. } \exists \mathbf{u}_h \in \Sigma_h, \mathbf{u}_h \neq 0 : \\
\text{(rot } \mathbf{u}_h; \text{ rot } \mathbf{v}_h) = \lambda_h^2 (\mathbf{u}_h; \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \Sigma_h.
\]  

(3)

Several choices for \( \Sigma_h \) are available, but some of them give poor results. For example the discretization by means of piecewise linear finite elements gives rise to bad approximation of the zero frequency unless particular structured meshes are used. In this latter case, it has been observed in the numerical computations that both the zero and positive frequencies are well approximated. But the numerical scheme may produce also “spurious” eigenvalues which converge to some finite values not belonging to the spectrum of the continuous problem. On the other hand the so-called edge elements produce a finite number of zero eigenvalues, quite good approximation of the nonzero ones and “no spurious eigenvalue”.

Our aim will be to understand this behavior of the discrete spectrum giving sufficient conditions in order to ensure the absence of any spurious eigenvalue. As a first step we consider another discrete problem, in mixed form, whose eigenvalues (all greater than zero) are exactly all the nonzero eigenvalues of (3), see [4].

\[
\text{Find } \lambda_h \in \mathbb{R} \text{ s.t. } \exists (\mathbf{u}_h, p_h) \in \Sigma_h \times W_h, (\mathbf{u}_h, p_h) \neq (0,0) : \\
\begin{aligned}
(\mathbf{u}_h, \mathbf{v}_h) - (\text{rot } \mathbf{u}_h, \mathbf{v}_h) &= 0 \forall \mathbf{v}_h \in \Sigma_h, \\
(\text{rot } \mathbf{v}_h, \mathbf{u}_h) &= \lambda_h^2 (p_h, \mathbf{v}_h) \forall \mathbf{v}_h \in W_h := \text{rot } \Sigma_h.
\end{aligned}
\]  

(4)

Hence we are lead to analyze the approximation of the eigenproblem (4). Surprisingly enough, the well-posedness of the source problem associated to (4) is not sufficient to provide a good behavior of the discrete eigenvalues of problem (4). In particular we shall show that the usual \textit{inf-sup} and \textit{ellipticity in the kernel} conditions do not imply the convergence of the discrete spectrum to the continuous one. We will then introduce a suitable additional abstract condition and prove that this will be sufficient for optimal approximation properties of the spectrum. Examples and counterexamples will be given to illustrate the abstract results.
2 Approximation of Laplace eigenproblem in mixed form

2.1 The pointwise convergence of the resolvent operator

Problem (4) is nothing else than a mixed approximation of the eigenproblem associated with a Laplace-type problem. More precisely, given \( f \in L^2_0(\Omega) \), where \( L^2_0(\Omega) \) is the subset of \( L^2(\Omega) \) of functions with zero mean-value, let \( p \in L^2(\Omega) \) be the solution of:

\[
- \Delta p = f \text{ in } \Omega \\
\frac{\partial p}{\partial n} = 0 \text{ on } \partial \Omega. 
\]

A mixed variational formulation of (5) reads as follows:

find \((\alpha, \beta) \in H_0(\text{rot}; \Omega) \times L^2_0(\Omega)\) s.t.

\[
\begin{cases}
(\alpha, \rot \beta) - (\rot \beta, \beta) = 0 & \forall \beta \in H_0(\text{rot}; \Omega) \\
(\rot \alpha, q) = (f, q) & \forall q \in L^2_0(\Omega),
\end{cases}
\]  

where \( H_0(\text{rot}; \Omega) \) is defined as the set of square integrable vectors \( \mathbf{z} \) such that \( \text{rot} \mathbf{z} \in L^2(\Omega) \) and \( \mathbf{z} \cdot \mathbf{n} \) vanishes on \( \partial \Omega \). Given finite element subspaces \( \Sigma_h \subset H_0(\text{rot}; \Omega) \) and \( W_h \subset L^2_0(\Omega) \), the approximation of (6) is given by:

find \((\alpha_h, \beta_h) \in \Sigma_h \times W_h\) s.t.

\[
\begin{cases}
(\alpha_h, \rot \beta_h) - (\rot \beta_h, \beta_h) = 0 & \forall \beta_h \in \Sigma_h \\
(\rot \alpha_h, q_h) = (f, q_h) & \forall q_h \in W_h.
\end{cases}
\]

Following [3] we introduce the resolvent operator \( T : L^2_0(\Omega) \rightarrow L^2_0(\Omega) \) defined as follows (see (6))

\[
T(f) = p. 
\]

In a similar way the discrete operator \( T_h : L^2_0(\Omega) \rightarrow W_h \) is defined using the approximated problem (7).

First of all we observe that it is not difficult to see that problem (6) (whence the operator \( T \)) is well-posed.

Let us recall (see [5]) the two main hypotheses for the well-posedness of problem (7).

(18) The inf-sup condition

\[
\exists \beta > 0 : \inf_{q \in W_h} \sup_{\beta \in \Sigma_h} \frac{(\rot \alpha, q)}{||\beta||_{H_0(\text{rot}; \Omega)} ||q||_{L^2_0(\Omega)}} \geq \beta. 
\]
(EK) The ellipticity in the kernel

\[ \exists \alpha > 0 : \|\mathbf{z}\|_{L^2(\Omega)^2}^2 \geq \alpha \|\mathbf{z}\|_{H_0^1(\Omega,\Omega)}^2, \quad \forall \mathbf{z} \in \Sigma_h \text{ with } (\text{rot } \mathbf{z}, \mathbf{q}) = 0 \quad \forall \mathbf{q} \in W_h. \]  

(10)

It is well-known that (IS) and (EK) imply that \( T_h \) is well-defined and that it converges pointwise to \( T \), that is

\[ \forall f \in L_0^2(\Omega) \quad \|T(f) - T_h(f)\|_{L_0^2(\Omega)} \to 0. \]  

(11)

On the other hand, in order to have good eigenvalues approximation, the natural condition is the following uniform convergence

\[ \|T(f) - T_h(f)\|_{L_0^2(\Omega)} \leq \rho_1(h)\|f\|_{L_0^2(\Omega)} \quad \forall f \in L_0^2(\Omega), \]  

(12)

with \( \rho_1(h) \) going to zero as \( h \) decreases.

The following theorem has been proved in [3]

**Theorem 2.1** The \( \inf\)-\( \sup \) (IS) and ellipticity in the kernel (EK) conditions are not sufficient in order to have the uniform convergence (12).

The proof of this theorem is based on the following counterexample. Let \( \Omega \) be a square, subdivided into equal subsquares, each of them partitioned into four triangles by its diagonals. Then let \( \Sigma_h \) be the space of continuous piecewise linear vector fields on such mesh and let \( W_h \) be the space \( \text{rot}(\Sigma_h) \). In [3] it has been proved that both (IS) and (EK) are satisfied, but (12) is not verified.

In the following section we present numerical examples showing the following behavior of the discrete eigenproblem (3) relative to this discretization. The correct eigenvalues are well approximated (this is due to the pointwise convergence) and moreover the discrete eigenvalues associated to the null-space are correctly confined to the zero frequency (this is due to the \( \inf\)-\( \sup \) condition), but some additional spurious mode can appear converging to values which do not belong to the spectrum of the continuous problem (this is due to the lack of the uniform convergence).

### 2.2 The uniform convergence of the resolvent operator

Following [3, 2], we recall here some sufficient hypotheses in order to have the uniform convergence (12) and hence a good eigenvalues convergence. We introduce some notation.

Let \( \Sigma_0 = T(L_0^2(\Omega)) \). If \( \Omega \) is smooth enough (e.g. convex polygon), it is then classical to obtain \( \Sigma_0 \subset H^1(\Omega)^2 \).
Let us recall the so-called Fortin operator $\Pi_h : \Sigma_0 \rightarrow \Sigma_h$ (see [5]) which verifies

\[
\begin{align}
\langle \text{rot}(\Pi_h \mathbf{z} - \mathbf{z}), q \rangle &= 0 \quad \forall q \in \Sigma_h, \\
\|\Pi_h \mathbf{z}\|_{H_0(\text{rot}; \Omega)} &\leq C\|\mathbf{z}\|_{\Sigma_0}.
\end{align}
\] (13)

It is well-known that the existence of $\Pi_h$ verifying (13) implies the validity of the inf-sup condition (IS).

Let us consider the following additional assumption.

(FORT-ID) The Fortin operator converges uniformly to the identity, that is:

\[
\|\Pi_h \mathbf{z} - \mathbf{z}\|_{L^2(\Omega)} \leq \rho_2(h)\|\mathbf{z}\|_{\Sigma_0} \quad \forall \mathbf{z} \in \Sigma_0.
\] (14)

with $\rho_2(h)$ going to zero as $h$ decreases.

In [3] the following theorem has been proved.

**Theorem 2.2** The conditions (FORT-ID) and (EK) are sufficient to ensure the uniform convergence (12).

**Remark 2.1** A natural choice for $\Sigma_h$ consists in the so-called “edge element” space (see [7]). The analysis of such element is essentially contained in [1, 6], where it is proved that the hypotheses of Theorem 2.2 are fulfilled (see also [5], p. 132).

**Remark 2.2** Theorem 2.2 has been generalized in [2]. In that paper a more abstract analysis has been developed, which can be applied to other situations. Moreover, particular emphasis has been devoted to the characterization of necessary conditions for the uniform convergence (12).

3 Numerical results

In our examples $\Omega$ is the square $[0, \pi] \times [0, \pi]$. In this case it is easy to find the exact eigenvalues of the problem, which are given by $0 \neq \lambda^2 = n^2 + m^2$, with $n, m = 0, 1, \ldots$

Let $\Sigma_h$ be the space of continuous piecewise linear vector fields. We present results obtained with both unstructured and structured meshes. In the former case the inf-sup condition is not verified and we obtain a poorly approximated spectrum in which it is quite impossible to pick out the physical eigenvalues among the others. In the latter case the zero frequencies are exactly individuated, but spurious eigenvalues are generated by the numerical scheme.

In the case of an unstructured mesh, the first seventy eigenvalues are plotted in Fig. 1. In this plot one cannot pick out the approximations of the physical eigenvalues from the nonphysical modes. In Fig. 3 we present the
Figure 1: Nodal approximation on an unstructured mesh: the filled dots correspond to the correct approximations.

Figure 2: Nodal approximation on a criss-cross mesh: the filled dots are spurious eigenvalues.

Figure 3: Nodal approximation.
Table 1: Nodal approximation on criss-cross mesh

<table>
<thead>
<tr>
<th>exact</th>
<th>computed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000</td>
<td>1.00428</td>
</tr>
<tr>
<td>1.00000</td>
<td>1.00428</td>
</tr>
<tr>
<td>2.00000</td>
<td>2.01711</td>
</tr>
<tr>
<td>4.00000</td>
<td>4.06804</td>
</tr>
<tr>
<td>4.00000</td>
<td>4.06804</td>
</tr>
<tr>
<td>5.00000</td>
<td>5.10634</td>
</tr>
<tr>
<td>5.00000</td>
<td>5.10634</td>
</tr>
<tr>
<td>5.92293</td>
<td>5.96578</td>
</tr>
<tr>
<td>8.00000</td>
<td>8.27128</td>
</tr>
<tr>
<td>d.o.f.</td>
<td>254</td>
</tr>
<tr>
<td># zeros</td>
<td>63</td>
</tr>
</tbody>
</table>

Figure 4: Nodal approximation on a criss-cross mesh
picture of four eigenfunctions corresponding to eigenvalues close to the value 1. It is apparent that the 49-th and 50-th eigenfunctions well approximate the correct ones associated to the double eigenvalue 1. The remaining ones are evidently associated to nonphysical eigenvalues. This behavior is due to the fact that on this mesh the inf-sup constant goes to zero with $h$.

The second test involves the structured criss-cross meshes described above. In this case the results are quite different. One can see in Table 1 that the zero mode is exactly approximated by a number of zero discrete eigenvalues. The first ten nonvanishing eigenvalues are listed in Table 1. We point out that the physical modes are well approximated. However in the table there is a numerical spurious mode which seems to converge to 6. This is motivated by the fact that the nodal elements do not verify condition (FORT-ID) although they fulfill both (EK) and (IS) (see [3]).

Figs 2 and 4 refer to the criss-cross $16 \times 16$ mesh. In the former the eigenvalues from 240 to 280 are plotted. The marked eigenvalues do not approximate any point in the continuous spectrum and their associated eigenfunctions are shown in the latter.

References