A-posteriori error estimates for Discontinuous Galerkin approximations of second order elliptic problems

Carlo Lovadina [*], and L. Donatella Marini[†]

Abstract

Using the weighted residual formulation we derive a-posteriori estimates for Discontinuous Galerkin approximations of second order elliptic problems in mixed form. We show that our approach allows to include in a unified way all the methods presented so far in the literature.

Keywords: Discontinuous Finite Elements, A-posteriori error estimates, Weighted Residuals

1 Introduction

In this paper we study *a-posteriori* error estimates for the Discontinuous Galerkin (DG) approximations of the problem

$$\begin{cases} \mathbb{K}^{-1}\boldsymbol{\sigma} + \nabla u = 0 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\sigma} = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{1}$$

Above, $\mathbb{K}$ is a given permeability symmetric positive-definite tensor, f is a given source term, and $\Omega \subset \mathbb{R}^2$ is a simply connected polygon. Problem (1) is the mixed form of the second order problem

$$-\operatorname{div}(\mathbb{K}\nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{2}$$

[*]Dipartimento di Matematica, Università di Pavia, and IMATI-CNR, Via Ferrata 1, Pavia I-27100, Italy (carlo.lovadina@unipv.it)

[†]Dipartimento di Matematica, Università di Pavia, and IMATI-CNR, Via Ferrata 1, Pavia I-27100, Italy (marini@imati.cnr.it)

In recent times a-posteriori analysis for DG approximations of second order elliptic problems has received an increasing attention. For L^∞ estimates we refer to [23], and for energy norm estimations to, e.g., [7], [21], [24], [31]. For questions concerning convergence of adaptive schemes we refer to [25], [20], [8]. All the above papers deal with second order elliptic problems of the type (2), and they concentrate on one or two DG formulations, mostly on *interior penalty* type methods, symmetric or nonsymmetric. Results based on the mixed formulation (1) can be found, e.g., in [14] and [13] for the *LDG* method, in [22] for a method similar to the Bassi-Rebay formulation, and, more recently, in [26] for the classical *RT* and *BDM* mixed formulations and for the method by Hughes-Masud [27].

In the present paper, starting from the mixed problem (1), we apply the weighted residual approach of [9] and we carry out the a posteriori analysis in an abstract framework, without specifying the choice of the weighting operators. In such a way we identify the minimal approximation properties required on the operators to guarantee lower and upper bounds for the energy norm. We then show that our analysis applies to all the DG formulations presented so far in the literature.

The paper is organized as follows. In Section 2, after having briefly presented a suitable mixed variational formulation of the continuous problem, we introduce the DG discretizations using the approach of [9]. We remark that, under some assumptions on the mesh, we allow for the occurrence of hanging nodes. Section 3 deals with a unified a-posteriori error analysis. More precisely, we introduce the error estimator, and we prove, in an abstract setting, its effciency (section 3.1) and reliability (section 3.2). Finally, in Section 4 we detail how our analysis applies to most of the DG methods, so far presented in the literature.

Throughout the paper, we shall follow the usual notation for Sobolev spaces (see e.g. Ciarlet [16]). In particular, for any domain $D \subset \mathbb{R}^2$ we will denote by $||\cdot||_{s,D}$ (resp. $|\cdot|_{s,D}$) the usual norm (resp. seminorm) in $H^s(D)$. When $D = \Omega$, we will simply write $||\cdot||_s$ (resp. $|\cdot|_s$). Moreover, we shall use the following classical result [28]:

Theorem 1 Let $f \in L^2(\Omega)$, and let $\mathbb{K} \in L^\infty(\Omega)^4_s$ satisfying

$$0 < c_1||\boldsymbol{\xi}||^2 \leq \boldsymbol{\xi}^T\mathbb{K}(x)\boldsymbol{\xi} \leq c_2||\boldsymbol{\xi}||^2 \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^2 \quad \forall x \in \Omega. \tag{3}$$

Then problem (1) has a unique $(\boldsymbol{\sigma}, u)$ in $H(\mathrm{div}\,;\Omega) \times H^1_0(\Omega)$. Moreover, there exists $P > 2$, depending only on c_1 and c_2, such that

$$u \in W^{1,p}(\Omega) \qquad\qquad \forall\, p \in [2, P].$$

2 Mixed formulations and discretization

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of decompositions of Ω into triangular elements T; let h_T denote the diameter of T, and $h = \max_{T\in\mathcal{T}_h} h_T$. Let $\mathcal{E}_h$ be the set of edges of $\mathcal{T}_h$; given $e \in \mathcal{E}_h$, we denote by h_e its length. *Nonconforming meshes* are allowed (i.e., $\mathcal{T}_h$ may contain hanging nodes), provided they are nested refinements of an initial conforming triangulation. Therefore, by removing the hanging nodes, it is possible to identify an underlying *conforming* triangulation, as shown in Fig. 1. Instead, Fig. 2 displays an instance violating our assumption. Indeed, removing the hanging node does not result in a coarser *triangular* subdivision: a *quadrilateral* element arises.

If hanging nodes occur, we notice that the corresponding set $\mathcal{E}_h$ is formed by line segments e which may be part of an edge triangle. For example, in Fig. 1 the line segment e_4 is only a part of an edge for triangle T^1 (although it is a whole edge for triangle T^2). However, with a little abuse of terminology, in the sequel we shall call "edge" *any* element $e \in \mathcal{E}_h$. On $\mathcal{T}_h$ we make the following assumptions:

H1- $\mathcal{T}_h$ verifies the minimum angle condition: $\exists\theta_0 > 0$ such that $h_T/\rho_T \geq \theta_0$ $\forall T \in \mathcal{T}_h$, where ρ_T denotes the diameter of the inscribed circle to T;

H2- $\mathcal{T}_h$ is locally quasi-uniform, that is, there exists a constant $C_* > 0$, independent of h, such that, for any pair of adjacent elements T^1 and T^2, that is, such that the length $|\partial T^1 \cap \partial T^2| > 0$, it holds

$$C_*^{-1} h_{T^1} \leq h_{T^2} \leq C_* h_{T^1}, \qquad \text{i.e.,} \quad h_{T^1} \approx h_{T^2}.$$

We remark that our assumptions on $\mathcal{T}_h$ implies that (referring for instance to Fig. 1):

$$h_{e^i} \approx h_{T^1} \approx h_{T^2} \approx h_{T^3} \qquad i = 1, \dots, 7. \tag{4}$$

Finally, we define

$$\text{for } T \in \mathcal{T}_h, \qquad \omega_T = \bigcup T' \quad \text{with } T' \in \mathcal{T}_h \text{ adjacent to } T; \tag{5}$$

$$\text{for } e \in \mathcal{E}_h, \qquad \omega_e = \bigcup T \quad \text{with } T \in \mathcal{T}_h \text{ and } e \subset \partial T. \tag{6}$$

For an internal edge, ω_e will always be the union of two elements, while it will be reduced to one element for a boundary edge. For the sake of simplicity, we will only consider here the case of piecewise constant $\mathbb{K}$. However, we point out that in the case of a more general permeability coefficient we can always approximate it by means of a piecewise constant, substituting $\mathbb{K}$

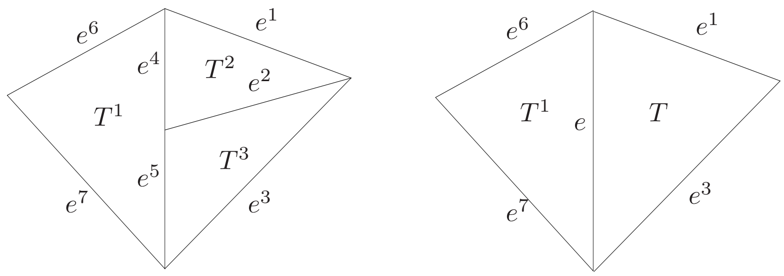

FIGURE 1
A hanging node (left) originated from a conforming triangular *mesh (right).*

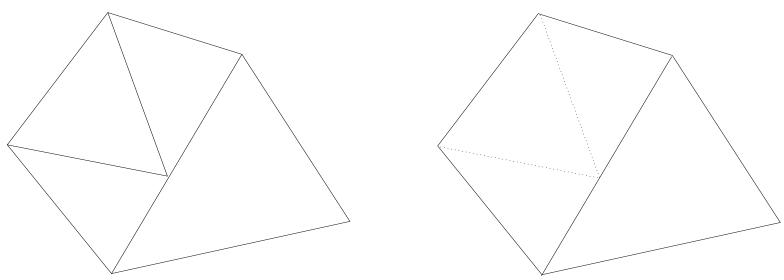

FIGURE 2
A hanging node (left) which is not *originated from a conforming* triangular *mesh (right).*

by its average in each element. Analogously, we shall suppose f piecewise polynomial.

In order to write a discontinuous finite element approximation of problem (1) we first introduce the usual tools such as *jumps* and *averages* of scalar and vector valued functions across the edges of $\mathcal{T}_h$. Following the notation of [11], [12], [3], let e be an interior edge shared by elements T^1 and T^2. Define the unit normal vectors $\mathbf{n}^1$ and $\mathbf{n}^2$ on e pointing exterior to T^1 and T^2, respectively. For a function φ, piecewise smooth on $\mathcal{T}_h$, with $\varphi^i := \varphi|_{T^i}$ we set

$$\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2), \quad [\![\varphi]\!] = \varphi^1\mathbf{n}^1 + \varphi^2\mathbf{n}^2 \quad \text{on } e \in \mathcal{E}_h^\circ, \tag{7}$$

where $\mathcal{E}_h^\circ$ is the set of interior edges e. For a vector valued function $\boldsymbol{\tau}$, piecewise smooth on $\mathcal{T}_h$, we define $\boldsymbol{\tau}^1$ and $\boldsymbol{\tau}^2$ analogously, and set

$$\{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^1 + \boldsymbol{\tau}^2), \quad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^1 \cdot \mathbf{n}^1 + \boldsymbol{\tau}^2 \cdot \mathbf{n}^2 \quad \text{on } e \in \mathcal{E}_h^\circ. \tag{8}$$

For $e \in \mathcal{E}_h^\partial$, the set of boundary edges, we set

$$[\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau} \cdot \mathbf{n}, \quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}, \quad [\![\varphi]\!] = \varphi\mathbf{n}, \quad \{\varphi\} = \varphi \quad \text{on } e \in \mathcal{E}_h^\partial. \tag{9}$$

Throughout the paper we shall make extensive use of the following identity (see [3]):

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \boldsymbol{\tau} \cdot \mathbf{n}\varphi = \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\tau}\} \cdot [\![\varphi]\!] + \sum_{e \in \mathcal{E}_h^\circ} \int_e [\![\boldsymbol{\tau}]\!] \{\varphi\}, \tag{10}$$

and of the trace inequality (see, e.g., equation (2.4) of [2])

$$\|v\|_{0,e}^2 \leq C(h_e^{-1}\|v\|_{0,T}^2 + h_e|v|_{1,T}^2) \quad \forall v \in H^1(T), \quad \forall e \subset \partial T. \tag{11}$$

With the previous definitions, problem (1) is equivalent to

$$\begin{cases} \mathbb{K}^{-1}\boldsymbol{\sigma} + \nabla u &= 0 \quad \text{in each } T \in \mathcal{T}_h, \\ \operatorname{div} \boldsymbol{\sigma} &= f \quad \text{in each } T \in \mathcal{T}_h, \\ [\![u]\!] &= 0 \quad \text{on each } e \in \mathcal{E}_h, \\ [\![\boldsymbol{\sigma}]\!] &= 0 \quad \text{on each } e \in \mathcal{E}_h^\circ. \end{cases} \tag{12}$$

Following the weighted residual approach of [9], we shall introduce a variational formulation of (12) in which each of the equations above has the same relevance, and is therefore treated in the same fashion. To do so, we introduce the spaces

$$
\begin{aligned}
&\widetilde{V}(\mathcal{T}_h) := \{v \in L^2(\Omega) \text{ such that } v_{|T} \in H^2(T) \quad \forall T \in \mathcal{T}_h\} \equiv H^2(\mathcal{T}_h),\\
&\widetilde{\mathbf{\Sigma}}(\mathcal{T}_h) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega) \text{ such that } \boldsymbol{\tau}_{|T} \in \mathbf{H}^1(T) \quad \forall T \in \mathcal{T}_h\} \equiv \mathbf{H}^1(\mathcal{T}_h),
\end{aligned}
\tag{13}
$$

and, for $p \in [2, P]$ (see Theorem 1):

$$
\begin{aligned}
&V(\mathcal{T}_h) := \{v \in L^p(\Omega) \text{ such that } v_{|T} \in W^{1,p}(T) \quad \forall T \in \mathcal{T}_h\},\\
&\mathbf{\Sigma}(\mathcal{T}_h) := \{\boldsymbol{\tau} \in \boldsymbol{L}^p(\Omega) \text{ such that } (\operatorname{div} \boldsymbol{\tau})_{|T} \in L^2(T) \quad \forall T \in \mathcal{T}_h\}.
\end{aligned}
\tag{14}
$$

We then introduce three linear operators $\boldsymbol{B}_{00}$, $\boldsymbol{B}_{01}$, B_{02} from $\widetilde{\mathbf{\Sigma}}(\mathcal{T}_h)$ to $\mathbf{L}^2(\mathcal{T}_h)$, $\mathbf{L}^2(\mathcal{E}_h)$, $L^2(\mathcal{E}_h^\circ)$ respectively, and three linear operators B_{10}, $\boldsymbol{B}_{11}$, B_{12} from $\widetilde{V}(\mathcal{T}_h)$ to $L^2(\mathcal{T}_h)$, $\mathbf{L}^2(\mathcal{E}_h)$, $L^2(\mathcal{E}_h^\circ)$ respectively, and we consider the variational problem:

$$
\begin{cases}
\text{Find } (\boldsymbol{\sigma}, u) \in \mathbf{\Sigma}(\mathcal{T}_h) \times V(\mathcal{T}_h) \text{ such that :}\\
(\mathbb{K}^{-1}\boldsymbol{\sigma} + \nabla_h u, \boldsymbol{B}_{00}\boldsymbol{\tau})_{\mathcal{T}_h} + < [\![u]\!], \boldsymbol{B}_{01}\boldsymbol{\tau} >_{\mathcal{E}_h}\\
\qquad\qquad + < [\![\boldsymbol{\sigma}]\!], B_{02}\boldsymbol{\tau} >_{\mathcal{E}_h^\circ} = 0 \quad \forall \boldsymbol{\tau} \in \widetilde{\mathbf{\Sigma}}(\mathcal{T}_h)\\
(\operatorname{div} \boldsymbol{\sigma} - f, B_{10}v)_{\mathcal{T}_h} + < [\![u]\!], \boldsymbol{B}_{11}v >_{\mathcal{E}_h}\\
\qquad\qquad + < [\![\boldsymbol{\sigma}]\!], B_{12}v >_{\mathcal{E}_h^\circ} = 0 \quad \forall v \in \widetilde{V}(\mathcal{T}_h).
\end{cases}
\tag{15}
$$

In (15) ∇_h denotes the gradient operator element by element. Moreover, we set:

$$
(v, w)_{\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} \int_T vw \, \mathrm{d}\mathbf{x}, \quad < v, w >_{\mathcal{E}_h} := \sum_{e \in \mathcal{E}_h} \int_e vw \, \mathrm{d}s,
$$

for both scalar and vector valued functions; analogously, $< v, w >_{\mathcal{E}_h^\circ}$ denotes the L^2-scalar product on internal edges. The operators B must be properly chosen. Here we assume that the operators verify the conditions stated in Theorem 3 of [9]. Those assumptions, together with the regularity result of Theorem 1, are sufficient to ensure that problem (15) has a unique solution which coincides with the solution of (12). As shown in [9] (see also [17]), all the methods appeared so far in the literature correspond to take $\boldsymbol{B}_{00} = s\,\mathbf{Id}$, with $s > 0$, and $B_{10} = \mathrm{Id}$. Accordingly, in the sequel we shall make this choice. Therefore, problem (15) becomes

$$\begin{cases} \text{Find } (\boldsymbol{\sigma}, u) \in \boldsymbol{\Sigma}(\mathcal{T}_h) \times V(\mathcal{T}_h) \text{ such that :} \\ s(\mathbb{K}^{-1}\boldsymbol{\sigma} + \nabla_h u, \boldsymbol{\tau})_{\mathcal{T}_h} + < [\![u]\!], \boldsymbol{B}_{01}\boldsymbol{\tau} >_{\mathcal{E}_h} \\ \qquad\qquad + < [\![\boldsymbol{\sigma}]\!], B_{02}\boldsymbol{\tau} >_{\mathcal{E}_h^\circ} = 0 \quad \forall \boldsymbol{\tau} \in \widetilde{\boldsymbol{\Sigma}}(\mathcal{T}_h) \\ -(\boldsymbol{\sigma}, \nabla_h v)_{\mathcal{T}_h} + < [\![u]\!], \boldsymbol{B}_{11} v >_{\mathcal{E}_h} + < [\![\boldsymbol{\sigma}]\!], B_{12} v + \{v\} >_{\mathcal{E}_h^\circ} \\ \qquad\qquad + < \{\boldsymbol{\sigma}\}, [\![v]\!] >_{\mathcal{E}_h} = (f, v) \quad \forall v \in \widetilde{V}(\mathcal{T}_h), \end{cases} \tag{16}$$

where an integration by parts and (10) have been used in the second equation.

Next, for $k \geq 1$, we define the finite element spaces:

$$\begin{aligned} V_h^k &= \{v \in L^2(\Omega) : \ v_{|T} \in P_k(T) \quad \forall T \in \mathcal{T}_h\}, \\ \boldsymbol{\Sigma}_h^k &= \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega) : \ \boldsymbol{\tau}_{|T} \in [P_k(T)]^2 \quad \forall T \in \mathcal{T}_h\}, \end{aligned} \tag{17}$$

and the norm

$$|\!|\!|(\boldsymbol{\tau}, v)|\!|\!|^2 := |\!|\!|v|\!|\!|^2 + \|\mathbb{K}^{-1}\boldsymbol{\tau}\|_0^2 \quad v \in V(\mathcal{T}_h), \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\mathcal{T}_h), \tag{18}$$

with $|\!|\!|v|\!|\!|$ defined by

$$|\!|\!|v|\!|\!| := \left(\sum_{T \in \mathcal{T}_h} ||\nabla v||_{0,T}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1/2} ||[\![v]\!]||_{0,e}^2 \right)^{1/2}. \tag{19}$$

In (18) we used $\|\mathbb{K}^{-1}\boldsymbol{\tau}\|_0^2$ in order to match the physical dimensions of $|\!|\!|v|\!|\!|^2$. However, due to assumption (3), we have

$$\|\mathbb{K}^{-1}\boldsymbol{\tau}\|_0^2 \approx \|\mathbb{K}^{-1/2}\boldsymbol{\tau}\|_0^2 \approx \|\boldsymbol{\tau}\|_0^2 \approx \|\mathbb{K}^{1/2}\boldsymbol{\tau}\|_0^2 \approx \|\mathbb{K}\boldsymbol{\tau}\|_0^2. \tag{20}$$

We also notice that (cf. (13) and (14))

$$V_h^k \subset \widetilde{V}(\mathcal{T}_h) \subset V(\mathcal{T}_h); \qquad \boldsymbol{\Sigma}_h^k \subset \widetilde{\boldsymbol{\Sigma}}(\mathcal{T}_h) \subset \boldsymbol{\Sigma}(\mathcal{T}_h). \tag{21}$$

The discrete problem is

$$\begin{cases} \text{Find } (\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h^k \times V_h^k \text{ such that } \forall (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_h^k \times V_h^k : \\ s(\mathbb{K}^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, \boldsymbol{\tau})_{\mathcal{T}_h} + < [\![u_h]\!], \boldsymbol{B}_{01}\boldsymbol{\tau} >_{\mathcal{E}_h} \\ \qquad\qquad + < [\![\boldsymbol{\sigma}_h]\!], B_{02}\boldsymbol{\tau} >_{\mathcal{E}_h^\circ} = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k \\ -(\boldsymbol{\sigma}_h, \nabla_h v)_{\mathcal{T}_h} + < [\![u_h]\!], \boldsymbol{B}_{11} v >_{\mathcal{E}_h} + < [\![\boldsymbol{\sigma}_h]\!], B_{12} v + \{v\} >_{\mathcal{E}_h^\circ} \\ \qquad\qquad + < \{\boldsymbol{\sigma}_h\}, [\![v]\!] >_{\mathcal{E}_h} = (f, v) \quad \forall v \in V_h^k, \end{cases} \tag{22}$$

which, in view of (21), turns out to be a *conforming* approximation of the variational problem (16).

3 A-posteriori error bounds

Given $T \in \mathcal{T}_h$ and $\epsilon \in \mathcal{E}_h$, we introduce the following error indicators:

$$\begin{aligned} &\eta_e^u := h_e^{-1/2} \| [\![\, u_h \,]\!] \|_{0,e} \quad \forall e \in \mathcal{E}_h, \quad \eta_e^\sigma := h_e^{1/2} \| [\![\, \boldsymbol{\sigma}_h \,]\!] \|_{0,e} \quad \forall e \in \mathcal{E}_h^\circ, \\ &\eta_{T,0}^\sigma := \| \mathbb{K}^{-1} \boldsymbol{\sigma}_h + \nabla u_h \|_{0,T}, \quad \eta_{T,1}^\sigma := h_T \| \mathrm{div}\, \boldsymbol{\sigma}_h - f \|_{0,T} \quad \forall T \in \mathcal{T}_h. \end{aligned} \tag{23}$$

Then, for every $T \in \mathcal{T}_h$ we set

$$\eta_T := \eta_{T,0}^\sigma + \eta_{T,1}^\sigma + \sum_{e \subset \partial T} \eta_e^u + \sum_{e \subset (\partial T \setminus \partial \Omega)} \eta_e^\sigma . \tag{24}$$

3.1 Lower bounds

As far as the efficiency of the error indicator η_T is concerned, we have the following main result.

Theorem 2 Let $(\boldsymbol{\sigma}, u)$ ($(\boldsymbol{\sigma}_h, u_h)$ resp.) be the solution of (15) ((22) resp.). For every $T \in \mathcal{T}_h$ the following estimate holds:

$$\begin{aligned} \eta_T \leq C \Big(&||\mathbb{K}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})||^2_{0,\omega_T} + ||\nabla(u_h - u)||^2_{0,T} \\ &+ \sum_{e \subset \partial T} h_e^{-1} || [\![\, u_h - u \,]\!] ||^2_{0,e} \Big)^{1/2}, \end{aligned} \tag{25}$$

where η_T and ω_T are defined in (24) and (5), respectively, and C is a positive constant independent of h_T.

We postpone the proof of Theorem 2 after some useful intermediate Lemmata.

Lemma 1 Let $T \in \mathcal{T}_h$, and let $p \in P_k(T)$. The following inverse inequality holds:

$$h_T \|p\|_{0,T} \leq C_1 \|p\|_{-1,T}, \tag{26}$$

with $C_1 > 0$ independent of h_T. Moreover, for $e \in \mathcal{E}_h$ with $e \subset \partial T$, defining the space

$$S = H_{00}^{1/2}(e) = \{ v \in H^{1/2}(\partial T) \text{ such that } v \equiv 0 \text{ on } \partial T \setminus e \} \tag{27}$$

it holds:

$$h_e^{1/2} \|p\|_{0,e} \leq C_2 \|p\|_{S'}, \tag{28}$$

with $C_2 > 0$ independent of h_T, and S' being the dual space of S.

Proof. Using a scaling argument and the definition of the norm in $H^{-1}(T)$ we have:

$$
\begin{aligned}
h_T\|p\|_{0,T} = h_T^2\|\hat{p}\|_{0,\hat{T}} \le \hat{c}h_T^2\|\hat{p}\|_{-1,\hat{T}} = \hat{c}h_T^2 \sup_{\hat{\varphi}\in H_0^1(\hat{T})} \frac{\int_{\hat{T}} \hat{p}\hat{\varphi}\mathrm{d}\hat{\mathbf{x}}}{|\hat{\varphi}|_{1,\hat{T}}} \\
= \hat{c}h_T^2 \sup_{\varphi\in H_0^1(T)} \frac{h_T^{-2}\int_T p\varphi\,\mathrm{d}\mathbf{x}}{|\varphi|_{1,T}} = \hat{c}\|p\|_{-1,T}.
\end{aligned}
\tag{29}
$$

To prove (28) we first observe that, denoting by $\widetilde{\varphi}$ the harmonic extension of $\varphi \in S$ to T we have

$$
\|\varphi\|_S := \|\varphi\|_{1/2,\partial T} \approx |\widetilde{\varphi}|_{1,T}. \tag{30}
$$

Using again a scaling argument, and the definition of the norm in S' we obtain

$$
\begin{aligned}
h_e^{1/2}\|p\|_{0,e} = h_e\|\hat{p}\|_{0,\hat{e}} \le \hat{c}h_e\|\hat{p}\|_{\hat{S}'} = \hat{c}h_e \sup_{\hat{\varphi}\in\hat{S}} \frac{\int_{\hat{e}} \hat{p}\hat{\varphi}\mathrm{d}\hat{s}}{\|\hat{\varphi}\|_{\hat{S}}} \\
= \hat{c}h_e \sup_{\varphi\in S} \frac{h_e^{-1}\int_e p\varphi\mathrm{d}s}{\|\varphi\|_S} = \hat{c}\|p\|_{S'}.
\end{aligned}
\tag{31}
$$

□

Corollary 1 As a consequence of (26) we immediately deduce that

$$
h_T\|\mathrm{div}\,\mathbf{v}\|_{0,T} \le C_1||\mathbf{v}||_{0,T} \quad \forall \mathbf{v} \in H(\mathrm{div}\,;T) \text{ with } \mathrm{div}\,\mathbf{v} \text{ polynomial.} \tag{32}
$$

Indeed, the definition of norm in H^{-1} and integration by parts give

$$
\|\mathrm{div}\,\mathbf{v}\|_{-1,T} = \sup_{\psi\in H_0^1(T)} \frac{\int_T \mathrm{div}\,\mathbf{v}\psi\,\mathrm{d}\mathbf{x}}{|\psi|_{1,T}} = \sup_{\psi\in H_0^1(T)} \frac{\int_T \mathbf{v}\cdot\nabla\psi\,\mathrm{d}\mathbf{x}}{|\psi|_{1,T}} \le ||\mathbf{v}||_{0,T}. \tag{33}
$$

□

The following result can also be found in [26] (Lemma 3.1).

Lemma 2 Let $\mathbf{v} \in H(\mathrm{div}\,;T)$, and let $e \in \mathcal{E}_h$ with $e \subset \partial T$. Then

$$
\|\mathbf{v}\cdot\mathbf{n}\|_{S'} \le C(||\mathbf{v}||_{0,T} + h_T\|\mathrm{div}\,\mathbf{v}\|_{0,T}), \tag{34}
$$

with $C > 0$ independent of h_T.

Proof. We first note that, if $\widetilde{\varphi}$ is the harmonic extension to T of $\varphi \in S$ we have:

$$
\int_e (\mathbf{v}\cdot\mathbf{n})\varphi\mathrm{d}s \equiv \int_{\partial T} (\mathbf{v}\cdot\mathbf{n})\widetilde{\varphi}\mathrm{d}s = \int_T (\mathbf{v}\cdot\nabla\widetilde{\varphi} + \mathrm{div}\,\mathbf{v}\,\widetilde{\varphi})\,\mathrm{d}\mathbf{x}. \tag{35}
$$

By Poincaré inequality $\|\widetilde{\varphi}\|_{0,T} \leq Ch_T|\widetilde{\varphi}|_{1,T}$ we then obtain

$$\int_e (\mathbf{v}\cdot\mathbf{n})\varphi \mathrm{d}s \leq C\left(||\mathbf{v}||_{0,T} + h_T\|\mathrm{div}\,\mathbf{v}\|_{0,T}\right)|\widetilde{\varphi}|_{1,T}. \tag{36}$$

Using this in the definition of the norm in S', and recalling (30), we obtain:

$$\|\mathbf{v}\cdot\mathbf{n}\|_{S'} = \sup_{\varphi\in S}\frac{\int_e(\mathbf{v}\cdot\mathbf{n})\varphi\mathrm{d}s}{||\varphi||_S} \leq C\left(||\mathbf{v}||_{0,T} + h_T\|\mathrm{div}\,\mathbf{v}\|_{0,T}\right). \tag{37}$$

□

We are now ready to prove Theorem 2.

Proof of Theorem 2. Fix $T \in \mathcal{T}_h$ and recall that we assumed $f(= \mathrm{div}\,\boldsymbol{\sigma})$ polynomial in T. Using (32) with $\mathbf{v} = \boldsymbol{\sigma}_h - \boldsymbol{\sigma}$ we have:

$$h_T\|\mathrm{div}\,\boldsymbol{\sigma}_h - f\|_{0,T} = h_T\|\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|_{0,T} \leq C_1||\boldsymbol{\sigma}_h - \boldsymbol{\sigma}||_{0,T}. \tag{38}$$

Since $\mathbb{K}^{-1}\boldsymbol{\sigma} + \nabla u = 0$ in T, we have

$$\begin{aligned}\|\mathbb{K}^{-1}\boldsymbol{\sigma}_h + \nabla u_h\|_{0,T} &= \|\mathbb{K}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) + \nabla(u_h - u)\|_{0,T}\\ &\leq ||\mathbb{K}^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})||_{0,T} + ||\nabla(u_h - u)||_{0,T}.\end{aligned} \tag{39}$$

Let $e \subset \partial T$. Since $[\![\, u \,]\!]_{|e} = 0$, it holds

$$h_e^{-1/2}||[\![\, u_h \,]\!]||_{0,e} = h_e^{-1/2}||[\![\, u_h - u \,]\!]||_{0,e}. \tag{40}$$

Let now $e \subset (\partial T \setminus \partial\Omega)$. Use first (28) with $p = [\![\, \boldsymbol{\sigma}_h \,]\!]_{|e}$, then $[\![\, \boldsymbol{\sigma} \,]\!]_{|e} = 0$, then (34) with $\mathbf{v} = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)_{|T}$ ($T \subseteq \omega_e$, see (6)), to get:

$$\begin{aligned}h_e^{1/2}\|[\![\, \boldsymbol{\sigma}_h \,]\!]\|_{0,e} &\leq C_2\|[\![\, \boldsymbol{\sigma}_h \,]\!]\|_{S'} = C_2\|[\![\, \boldsymbol{\sigma}_h - \boldsymbol{\sigma} \,]\!]\|_{S'}\\ &\leq C\left(||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{0,\omega_e} + \sum_{T\subseteq\omega_e} h_T\|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,T}\right).\end{aligned} \tag{41}$$

Noting that $\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = f - \mathrm{div}\,\boldsymbol{\sigma}_h$ is a polynomial, from (41) and (32) we obtain, recalling (20):

$$h_e^{1/2}\|[\![\, \boldsymbol{\sigma}_h \,]\!]\|_{0,e} \leq C\,||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{0,\omega_e} \leq C\,||\mathbb{K}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||_{0,\omega_e}. \tag{42}$$

Joining estimates (38), (39), (40), and (42) we easily get (25). □

3.2 Upper bounds

We make use of the Helmholtz-type decomposition of $\boldsymbol{\sigma}_h$:

$$\boldsymbol{\sigma}_h = -\mathbb{K}\nabla\varphi + \mathbf{curl}\, p. \tag{43}$$

Above, $\varphi \in H_0^1(\Omega)$ is the solution of the elliptic problem:

$$\begin{cases} \operatorname{div}(\mathbb{K}\nabla\varphi) = -\operatorname{div}\boldsymbol{\sigma}_h & \text{in } \Omega, \\ \varphi_{|\partial\Omega} = 0, \end{cases} \tag{44}$$

which has to be understood in the sense of distributions, since $\operatorname{div}\boldsymbol{\sigma}_h \in H^{-1}(\Omega)$, and $p \in H^1(\Omega)$ is determined, up to a constant, by solving

$$\mathbf{curl}\, p = \boldsymbol{\sigma}_h + \mathbb{K}\nabla\varphi, \qquad \text{with } \mathbf{curl}\, p := \left(\frac{\partial p}{\partial y}, -\frac{\partial p}{\partial x}\right)^t. \tag{45}$$

We notice that such an equation is solvable since Ω is simply connected and $\operatorname{div}(\boldsymbol{\sigma}_h + \mathbb{K}\nabla\varphi) = 0$ (see (44)). Recalling that $\boldsymbol{\sigma} = -\mathbb{K}\nabla u$ we have

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \mathbb{K}\nabla w - \mathbf{curl}\, p \qquad \text{where } w := \varphi - u. \tag{46}$$

From (46) we get

$$\mathbb{K}^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = \mathbb{K}^{1/2}\nabla w - \mathbb{K}^{-1/2}\mathbf{curl}\, p, \tag{47}$$

and therefore

$$\begin{aligned} ||\mathbb{K}^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||_0^2 &= ||\mathbb{K}^{1/2}\nabla w - \mathbb{K}^{-1/2}\mathbf{curl}\, p||_0^2 \\ &= ||\mathbb{K}^{1/2}\nabla w||_0^2 + ||\mathbb{K}^{-1/2}\mathbf{curl}\, p||_0^2, \end{aligned} \tag{48}$$

since ($\mathbb{K}$ being symmetric) $(\mathbb{K}^{1/2}\nabla w, \mathbb{K}^{-1/2}\mathbf{curl}\, p) \equiv (\nabla w, \mathbf{curl}\, p) = 0$.

We shall estimate the terms $||\mathbb{K}^{1/2}\nabla w||_0$ and $||\mathbb{K}^{-1/2}\mathbf{curl}\, p||_0$ separately. In the sequel, given the mesh $\mathcal{T}_h$ which possibly contains hanging nodes, we denote by $\mathcal{T}_h^c$ the *finest conforming* mesh such that $\mathcal{T}_h^c \subseteq \mathcal{T}_h$. For any $T \in \mathcal{T}_h^c$, we set

$$T \in \mathcal{T}_h^c \longrightarrow \pi(T) = \{T' \in \mathcal{T}_h \,:\, T' \subseteq T\}\,. \tag{49}$$

We shall need to introduce suitable interpolants for w and p. Hence, let w_I (resp. p_I) be the usual piecewise linear Clément interpolant of w (resp. p), defined on the *conforming mesh* $\mathcal{T}_h^c$. It is well-known that it holds:

$$\begin{aligned} \left(\sum_{T\in\mathcal{T}_h^c} h_T^{2r-2}|w - w_I|_{r,T}^2\right)^{1/2} &\leq C|w|_1 \qquad r = 0,1 \\ \left(\sum_{T\in\mathcal{T}_h^c} h_T^{2r-2}|p - p_I|_{r,T}^2\right)^{1/2} &\leq C|p|_1 \qquad r = 0,1. \end{aligned} \tag{50}$$

We will use the following Lemma, which somehow establishes a connection between the mesh $\mathcal{T}_h$ (through $\mathcal{E}_h$) and the corresponding *conforming* mesh $\mathcal{T}_h^c$.

Lemma 3 Given the mesh $\mathcal{T}_h$ with edge set $\mathcal{E}_h$, let $\mathcal{T}_h^c$ the *finest conforming* mesh such that $\mathcal{T}_h^c \subseteq \mathcal{T}_h$. Let $\varphi \in H^1(\Omega)$. Then, it holds

$$(\sum_{e\in\mathcal{E}_h} h_e^{-1}||\varphi||_{0,e}^2)^{1/2} \le C\left(\sum_{T\in\mathcal{T}_h^c}(h_T^{-2}||\varphi||_{0,T}^2 + |\varphi|_{1,T}^2)\right)^{1/2}. \tag{51}$$

Proof. Fix $e \in \mathcal{E}_h$. Using (11), and $h_e^{-2} \le Ch_T^{-2}$ $\forall T \subseteq \omega_e$ (see (4) and (6)), we get

$$h_e^{-1}||\varphi||_{0,e}^2 \le C(\sum_{T\subseteq\omega_e} h_T^{-2}||\varphi||_{0,T}^2 + |\varphi|_{1,T}^2). \tag{52}$$

Therefore, we have

$$(\sum_{e\in\mathcal{E}_h} h_e^{-1}||\varphi||_{0,e}^2)^{1/2} \le C\left(\sum_{T\in\mathcal{T}_h}(h_T^{-2}||\varphi||_{0,T}^2 + |\varphi|_{1,T}^2)\right)^{1/2}. \tag{53}$$

Rearranging the terms in the right-hand side of (53), and recalling that $h_{T'}^{-2} \le C_* h_T^{-2}$, $\forall T' \in \pi(T)$, we obtain

$$\begin{aligned}
&\left(\sum_{T\in\mathcal{T}_h}(h_T^{-2}||\varphi||_{0,T}^2 + |\varphi|_{1,T}^2)\right)^{1/2} \\
&\quad = \left(\sum_{T\in\mathcal{T}_h^c}\sum_{T'\in\pi(T)}(h_{T'}^{-2}||\varphi||_{0,T'}^2 + |\varphi|_{1,T'}^2)\right)^{1/2} \qquad (54)\\
&\quad \le C\left(\sum_{T\in\mathcal{T}_h^c}(h_T^{-2}||\varphi||_{0,T}^2 + |\varphi|_{1,T}^2)\right)^{1/2}.
\end{aligned}$$

From (53) and (54) we have (51). □

Theorem 3 Let $(\boldsymbol{\sigma}, u)$ ($(\boldsymbol{\sigma}_h, u_h)$ resp.) be the solution of (15) ((22) resp.). Let $w \in H_0^1(\Omega)$ and $p \in H^1(\Omega)$ as in (43)–(46), with piecewise linear Clément interpolants w_I and p_I, respectively, defined on $\mathcal{T}_h^c$. Then it holds:

$$||\mathbb{K}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||_0 \le C\left(\sum_{T\in\mathcal{T}_h}\eta_T^2\right)^{1/2} + T_0(u_h, \boldsymbol{\sigma}_h; p) + T_1(u_h, \boldsymbol{\sigma}_h; w), \tag{55}$$

where η_T is defined in (24), and $T_0(u_h, \boldsymbol{\sigma}_h; p)$ and $T_1(u_h, \boldsymbol{\sigma}_h; w)$ are given by

$$\begin{aligned} T_0(u_h, \boldsymbol{\sigma}_h; p) &:= -\frac{< [\![u_h]\!], s^{-1}\boldsymbol{B}_{01}(\mathbf{curl}\, p_I) + \{\mathbf{curl}\, p_I\} >_{\mathcal{E}_h}}{|p|_1} \\ &\quad - \frac{< [\![\boldsymbol{\sigma}_h]\!], s^{-1}B_{02}(\mathbf{curl}\, p_I) >_{\mathcal{E}_h^\circ}}{|p|_1} \qquad (56) \\ T_1(u_h, \boldsymbol{\sigma}_h; w) &:= -\frac{< [\![u_h]\!], \boldsymbol{B}_{11} w_I >_{\mathcal{E}_h}}{|w|_1} - \frac{< [\![\boldsymbol{\sigma}_h]\!], B_{12} w_I + \{w_I\} >_{\mathcal{E}_h^\circ}}{|w|_1}. \end{aligned}$$

Moreover, it holds:

$$|\!|\!| u - u_h |\!|\!| \le \sqrt{2}\left(||\mathbb{K}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||_0^2 + \sum_{T\in\mathcal{T}_h} (\eta^{\sigma}_{T,0})^2 + \sum_{e\in\mathcal{E}_h} (\eta^u_e)^2 \right)^{1/2}, \quad (57)$$

where $|\!|\!| \cdot |\!|\!|$, η^u_e and $\eta^\sigma_{T,0}$ are defined in (19) and (23).

Proof. We proceed in three steps.
<u>*First Step*</u> *– Estimate for* $||\mathbb{K}^{-1/2}\mathbf{curl}\, p||_0$.
Testing the first equation of (22) with $\boldsymbol{\tau} = \mathbf{curl}\, p_I \in \boldsymbol{\Sigma}_h^k$, we have

$$\begin{aligned} s(\mathbb{K}^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, \mathbf{curl}\, p_I)_{\mathcal{T}_h} &= - < [\![u_h]\!], \boldsymbol{B}_{01}(\mathbf{curl}\, p_I) >_{\mathcal{E}_h} \\ &\quad - < [\![\boldsymbol{\sigma}_h]\!], B_{02}(\mathbf{curl}\, p_I) >_{\mathcal{E}_h^\circ}. \end{aligned} \quad (58)$$

Integrating by parts the term $(\nabla_h u_h, \mathbf{curl}\, p_I)_{\mathcal{T}_h}$, using (10), and noting that $\operatorname{div} \mathbf{curl}\, p_I = 0$ and $[\![\mathbf{curl}\, p_I]\!]_{|e} = 0 \ \forall e \in \mathcal{E}_h^\circ$, we get

$$\begin{aligned} (\mathbb{K}^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\, p_I)_{\mathcal{T}_h} &= - < [\![u_h]\!], s^{-1}\boldsymbol{B}_{01}(\mathbf{curl}\, p_I) + \{\mathbf{curl}\, p_I\} >_{\mathcal{E}_h} \\ &\quad - < [\![\boldsymbol{\sigma}_h]\!], s^{-1}B_{02}(\mathbf{curl}\, p_I) >_{\mathcal{E}_h^\circ}. \end{aligned} \quad (59)$$

On the other hand, we also have from (43)

$$\begin{aligned} (\mathbb{K}^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\, p)_{\mathcal{T}_h} &= (-\nabla\varphi + \mathbb{K}^{-1}\mathbf{curl}\, p, \mathbf{curl}\, p)_{\mathcal{T}_h} \\ &= (\mathbb{K}^{-1}\mathbf{curl}\, p, \mathbf{curl}\, p)_{\mathcal{T}_h} = ||\mathbb{K}^{-1/2}\mathbf{curl}\, p||_0^2. \end{aligned} \quad (60)$$

Hence, from (60), adding and subtracting first $(\mathbb{K}^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\, p_I)_{\mathcal{T}_h}$, then $(\nabla_h u_h, \mathbf{curl}\,(p - p_I))_{\mathcal{T}_h}$, and using (59), we get

$$\begin{aligned} ||\mathbb{K}^{-1/2}\mathbf{curl}\, p||_0^2 &= (\mathbb{K}^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\,(p - p_I))_{\mathcal{T}_h} + (\mathbb{K}^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\, p_I)_{\mathcal{T}_h} \\ &= (\mathbb{K}^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, \mathbf{curl}\,(p - p_I))_{\mathcal{T}_h} - (\nabla_h u_h, \mathbf{curl}\,(p - p_I))_{\mathcal{T}_h} \\ &\quad - < [\![u_h]\!], s^{-1}\boldsymbol{B}_{01}(\mathbf{curl}\, p_I) + \{\mathbf{curl}\, p_I\} >_{\mathcal{E}_h} \\ &\quad - < [\![\boldsymbol{\sigma}_h]\!], s^{-1}B_{02}(\mathbf{curl}\, p_I) >_{\mathcal{E}_h^\circ}. \end{aligned} \quad (61)$$

Using the second estimate in (50) with $r = 1$, we easily get

$$(\mathbb{K}^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h, \mathbf{curl}\,(p - p_I))_{\mathcal{T}_h} \leq C\Big(\sum_{T\in\mathcal{T}_h} ||\mathbb{K}^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h||^2_{0,T}\Big)^{1/2}|p|_1. \tag{62}$$

After integrating by parts the term $(\nabla_h u_h, \mathbf{curl}\,(p - p_I))_{\mathcal{T}_h}$, recalling that $\nabla_h u_h \cdot \mathbf{t}_T = \mathbf{curl}\, u_h \cdot \mathbf{n}_T$, using (10), $[\![\, p - p_I \,]\!]_{|e} = 0 \ \forall e \in \mathcal{E}_h^\circ$, and (9) we obtain:

$$\begin{aligned}(\nabla_h u_h, \mathbf{curl}\,(p - p_I))_{\mathcal{T}_h} &= -\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla_h u_h \cdot \mathbf{t}_T)(p - p_I)\,\mathrm{d}s \\ &= -\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\mathbf{curl}\, u_h \cdot \mathbf{n}_T)(p - p_I)\,\mathrm{d}s \\ &= -< [\![\,\mathbf{curl}\, u_h\,]\!], \{p - p_I\} >_{\mathcal{E}_h^\circ} \\ &\quad - \sum_{e\in\mathcal{E}_h^\partial}\int_e(\mathbf{curl}\, u_h \cdot \mathbf{n}_T)(p - p_I)\,\mathrm{d}s \\ &= -< [\![\,\mathbf{curl}\, u_h\,]\!], \{p - p_I\} >_{\mathcal{E}_h} \\ &\leq C(\sum_{e\in\mathcal{E}_h} h_e||[\![\,\mathbf{curl}\, u_h\,]\!]||^2_{0,e})^{1/2}(\sum_{e\in\mathcal{E}_h} h_e^{-1}||\{p - p_I\}||^2_{0,e})^{1/2}.\end{aligned} \tag{63}$$

An inverse inequality gives:

$$(\sum_{e\in\mathcal{E}_h} h_e||[\![\,\mathbf{curl}\, u_h\,]\!]||^2_{0,e})^{1/2} \leq C(\sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![\, u_h \,]\!]||^2_{0,e})^{1/2}. \tag{64}$$

Furthermore, from Lemma 3 with $\varphi = p - p_I$, and (50) we get:

$$\sum_{e\in\mathcal{E}_h} h_e^{-1}||\{p-p_I\}||^2_{0,e} \leq C\sum_{T\in\mathcal{T}_h^c}(h_T^{-2}||p-p_I||^2_{0,T} + |p-p_I|^2_{1,T}) \leq C|p|^2_1. \tag{65}$$

Inserting (64) and (65) in estimate (63) we deduce:

$$(\nabla_h u_h, \mathbf{curl}\,(p - p_I))_{\mathcal{T}_h} \leq C(\sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![\, u_h \,]\!]||^2_{0,e})^{1/2}|p|_1. \tag{66}$$

Hence, combining (62) with (66) we infer in (61)

$$
\begin{aligned}
&||\mathbb{K}^{-1/2}\mathbf{curl}\, p||_0^2 \\
&\leq C\Big(\sum_{T\in\mathcal{T}_h} ||\mathbb{K}^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h||_{0,T}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![u_h]\!]||_{0,e}^2\Big)^{1/2}|p|_1 \\
&\quad - < [\![u_h]\!], s^{-1}\boldsymbol{B}_{01}(\mathbf{curl}\, p_I) + \{\mathbf{curl}\, p_I\} >_{\mathcal{E}_h} \\
&\quad - < [\![\boldsymbol{\sigma}_h]\!], s^{-1}B_{02}(\mathbf{curl}\, p_I) >_{\mathcal{E}_h^\circ} .
\end{aligned}
\tag{67}
$$

Recalling (56), and noting that $|p|_1 \approx ||\mathbb{K}^{-1/2}\mathbf{curl}\, p||_0$ (see (20)), we get:

$$
\begin{aligned}
||\mathbb{K}^{-1/2}\mathbf{curl}\, p||_0 &\leq C\Big(\sum_{T\in\mathcal{T}_h} ||\mathbb{K}^{-1}\boldsymbol{\sigma}_h + \nabla_h u_h||_{0,T}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![u_h]\!]||_{0,e}^2\Big)^{1/2} \\
&\quad + T_0(u_h, \boldsymbol{\sigma}_h; p).
\end{aligned}
\tag{68}
$$

Second Step – Estimate for $||\mathbb{K}^{1/2}\nabla w||_0^2$.
From the second equations of (16) and (22) we obtain the error equation, that we test with $v_h = w_I$:

$$
\begin{aligned}
(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \nabla w_I)_{\mathcal{T}_h} &=< [\![u - u_h]\!], \boldsymbol{B}_{11} w_I >_{\mathcal{E}_h} \\
&+ < [\![\boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\!], B_{12} w_I + \{w_I\} >_{\mathcal{E}_h^\circ} + < \{\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\}, [\![w_I]\!] >_{\mathcal{E}_h} .
\end{aligned}
\tag{69}
$$

Since $[\![u]\!]_{|e} = [\![w_I]\!]_{|e} = 0\ \forall e \in \mathcal{E}_h$, and $[\![\boldsymbol{\sigma}]\!]_{|e} = 0\ \forall e \in \mathcal{E}_h^\circ$, we obtain

$$
(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla w_I)_{\mathcal{T}_h} = - < [\![u_h]\!], \boldsymbol{B}_{11} w_I >_{\mathcal{E}_h} - < [\![\boldsymbol{\sigma}_h]\!], B_{12} w_I + \{w_I\} >_{\mathcal{E}_h^\circ} .
\tag{70}
$$

On the other hand, we have (cf. (46)):

$$
(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla w)_{\mathcal{T}_h} = (\mathbb{K}\nabla w - \mathbf{curl}\, p, \nabla w)_{\mathcal{T}_h} = ||\mathbb{K}^{1/2}\nabla w||_0^2.
\tag{71}
$$

Hence, from (71), adding and subtracting ∇w_I, and using (70), it holds

$$
\begin{aligned}
||\mathbb{K}^{1/2}\nabla w||_0^2 &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla(w - w_I))_{\mathcal{T}_h} + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla w_I)_{\mathcal{T}_h} \\
&= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla(w - w_I))_{\mathcal{T}_h} - < [\![u_h]\!], \boldsymbol{B}_{11} w_I >_{\mathcal{E}_h} \\
&\quad - < [\![\boldsymbol{\sigma}_h]\!], B_{12} w_I + \{w_I\} >_{\mathcal{E}_h^\circ} .
\end{aligned}
\tag{72}
$$

An integration by parts, (10) and the equations $[\![\boldsymbol{\sigma}]\!]_{|e} = 0\ \forall e \in \mathcal{E}_h^\circ$, and $[\![w - w_I]\!]_{|e} = 0\ \forall e \in \mathcal{E}_h$ give

$$
\begin{aligned}
||\mathbb{K}^{1/2}\nabla w||_0^2 &= -(\mathrm{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), w - w_I)_{\mathcal{T}_h} - < [\![u_h]\!], \boldsymbol{B}_{11} w_I >_{\mathcal{E}_h} \\
&\quad - < [\![\boldsymbol{\sigma}_h]\!], B_{12} w_I + \{w_I\} >_{\mathcal{E}_h^\circ} - < [\![\boldsymbol{\sigma}_h]\!], \{w - w_I\} >_{\mathcal{E}_h^\circ} \\
&= (\mathrm{div}_h \boldsymbol{\sigma}_h - f, w - w_I)_{\mathcal{T}_h} - < [\![u_h]\!], \boldsymbol{B}_{11} w_I >_{\mathcal{E}_h} \\
&\quad - < [\![\boldsymbol{\sigma}_h]\!], B_{12} w_I + \{w_I\} >_{\mathcal{E}_h^\circ} - < [\![\boldsymbol{\sigma}_h]\!], \{w - w_I\} >_{\mathcal{E}_h^\circ} .
\end{aligned}
\tag{73}
$$

Above and in the sequel, div_h denotes the divergence operator element by element. We have

$$(\mathrm{div}_h\boldsymbol{\sigma}_h-f,w-w_I)_{\mathcal{T}_h}\leq(\sum_{T\in\mathcal{T}_h}h_T^2\|\mathrm{div}\,\boldsymbol{\sigma}_h-f\|_{0,T}^2)^{1/2}(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|w-w_I\|_{0,T}^2)^{1/2}. \tag{74}$$

A rearrangement gives (see (49))

$$\sum_{T\in\mathcal{T}_h}h_T^{-2}\|w-w_I\|_{0,T}^2=\sum_{T\in\mathcal{T}_h^c}\sum_{T'\in\pi(T)}h_{T'}^{-2}\|w-w_I\|_{0,T'}^2. \tag{75}$$

Since $h_{T'}^{-2}\leq Ch_T^{-2}$ (cf. (4)), using (50), from (75) we obtain:

$$(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|w-w_I\|_{0,T}^2)^{1/2}\leq C|w|_1. \tag{76}$$

Therefore, from (74) and (76) we get

$$(\mathrm{div}_h\boldsymbol{\sigma}_h-f,w-w_I)_{\mathcal{T}_h}\leq C(\sum_{T\in\mathcal{T}_h}h_T^2\|\mathrm{div}\,\boldsymbol{\sigma}_h-f\|_{0,T}^2)^{1/2}|w|_1. \tag{77}$$

Next, we have

$$<[\![\boldsymbol{\sigma}_h]\!],\{w-w_I\}>_{\mathcal{E}_h^\circ}\leq(\sum_{e\in\mathcal{E}_h^\circ}h_e\|[\![\boldsymbol{\sigma}_h]\!]\|_{0,e}^2)^{1/2}(\sum_{e\in\mathcal{E}_h^\circ}h_e^{-1}\|\{w-w_I\}\|_{0,e}^2)^{1/2}. \tag{78}$$

Lemma 3 with $\varphi=w-w_I$, and estimates (50) yield:

$$(\sum_{e\in\mathcal{E}_h^\circ}h_e^{-1}\|\{w-w_I\}\|_{0,e}^2)^{1/2}\leq C|w|_1. \tag{79}$$

Hence,

$$<[\![\boldsymbol{\sigma}_h]\!],\{w-w_I\}>_{\mathcal{E}_h^\circ}\leq C(\sum_{e\in\mathcal{E}_h^\circ}h_e\|[\![\boldsymbol{\sigma}_h]\!]\|_{0,e}^2)^{1/2}|w|_1. \tag{80}$$

Therefore, from (73), (77), and (80) we get

$$\begin{aligned}||\mathbb{K}^{1/2}\nabla w||_0^2\leq C(&\sum_{T\in\mathcal{T}_h}h_T^2\|\mathrm{div}\,\boldsymbol{\sigma}_h-f\|_{0,T}^2+\sum_{e\in\mathcal{E}_h^\circ}h_e\|[\![\boldsymbol{\sigma}_h]\!]\|_{0,e}^2)^{1/2}|w|_1\\ &-<[\![u_h]\!],\boldsymbol{B}_{11}w_I>_{\mathcal{E}_h}-<[\![\boldsymbol{\sigma}_h]\!],B_{12}w_I+\{w_I\}>_{\mathcal{E}_h^\circ},\end{aligned}$$

by which we obtain (see also (56), and use $|w|_1\approx||\mathbb{K}^{1/2}\nabla w||_0$, cf. (20)):

$$\begin{aligned}||\mathbb{K}^{1/2}\nabla w||_0\leq C&\left(\sum_{T\in\mathcal{T}_h}h_T^2\|\mathrm{div}\,\boldsymbol{\sigma}_h-f\|_{0,T}^2+\sum_{e\in\mathcal{E}_h^\circ}h_e\|[\![\boldsymbol{\sigma}_h]\!]\|_{0,e}^2\right)^{1/2}\\ &+T_1(u_h,\boldsymbol{\sigma}_h;w).\end{aligned} \tag{81}$$

Recalling (48), a combination of (68) and (81) gives

$$\begin{aligned} ||\mathbb{K}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)||_0 &\le C||\mathbb{K}^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)||_0 \\ &\le C\left(\sum_{T\in\mathcal{T}_h}\eta_T\right)^{1/2} + T_0(u_h\boldsymbol{\sigma}_h;p) + T_1(u_h\boldsymbol{\sigma}_h;w), \end{aligned} \tag{82}$$

i.e. estimate (55).

Third Step

Since $\mathbb{K}^{-1}\boldsymbol{\sigma} = -\nabla u$ in T $\forall T \in \mathcal{T}_h$, it holds

$$\begin{aligned} \sum_{T\in\mathcal{T}_h}||\nabla(u_h-u)||^2_{0,T} &= \sum_{T\in\mathcal{T}_h}||\nabla u_h + \mathbb{K}^{-1}\boldsymbol{\sigma}||^2_{0,T} \\ &\le 2\sum_{T\in\mathcal{T}_h}\left(||\nabla u_h + \mathbb{K}^{-1}\boldsymbol{\sigma}_h||^2_{0,T} + ||\mathbb{K}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)||^2_{0,T}\right) \\ &= 2\sum_{T\in\mathcal{T}_h}\left((\eta^{\sigma}_{T,0})^2 + ||\mathbb{K}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)||^2_{0,T}\right). \end{aligned} \tag{83}$$

Furthermore, we have

$$\sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![\, u_h - u\,]\!]||^2_{0,e} = \sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![\, u_h\,]\!]||^2_{0,e} = \sum_{e\in\mathcal{E}_h}(\eta^u_e)^2. \tag{84}$$

Summing (83) with (84), and taking the square root, we infer (cf. (19))

$$|\!|\!| u-u_h |\!|\!| \le \sqrt{2}\left(||\mathbb{K}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)||^2_0 + \sum_{T\in\mathcal{T}_h}(\eta^{\sigma}_{T,0})^2 + \sum_{e\in\mathcal{E}_h}(\eta^u_e)^2\right)^{1/2}, \tag{85}$$

i.e. estimate (57). The proof is complete. □

Immediate consequences of Theorem 3 are the following.

Corollary 2 With the notation of Theorem 3, one has (cf. also (18)–(19)):

$$|\!|\!|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-u_h)|\!|\!| \le C\left(\sum_{T\in\mathcal{T}_h}\eta_T^2\right)^{1/2} + T_0(u_h,\boldsymbol{\sigma}_h;p) + T_1(u_h,\boldsymbol{\sigma}_h;w). \tag{86}$$

Corollary 3 Suppose that:

$$\Big(\sum_{e\in\mathcal{E}_h} h_e||s^{-1}\boldsymbol{B}_{01}(\mathbf{curl}\,p_I)+\{\mathbf{curl}\,p_I\}||^2_{0,e} + \sum_{e\in\mathcal{E}_h^\circ} h_e^{-1}||s^{-1}B_{02}(\mathbf{curl}\,p_I)||^2_{0,e}\Big)^{1/2} \le C|p|_1 \tag{87}$$

$$\Big(\sum_{e\in\mathcal{E}_h} h_e||\boldsymbol{B}_{11}w_I||^2_{0,e} + \sum_{e\in\mathcal{E}_h^\circ} h_e^{-1}||B_{12}w_I+\{w_I\}||^2_{0,e}\Big)^{1/2} \le C|w|_1.$$

Then it holds

$$|||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-u_h)||| \le C\left(\sum_{T\in\mathcal{T}_h}\eta_T^2\right)^{1/2}. \tag{88}$$

4 Various methods

We first notice that the lower bounds of Theorem 2 in Section 3.1 do not depend on the B operators. As a consequence, they hold true for any scheme one selects. Therefore, in the following we focus on the upper bounds for the various methods.

Let us introduce, for each piecewise smooth function v, the lifting of its jumps, $\mathcal{L}([\![v]\!]) \in \boldsymbol{\Sigma}_h^k$, as the unique solution in $\boldsymbol{\Sigma}_h^k$ of

$$(\mathcal{L}([\![v]\!]),\boldsymbol{\tau})_{\mathcal{T}_h} = < [\![v]\!], \boldsymbol{B}_{01}\boldsymbol{\tau} >_{\mathcal{E}_h} \ \forall \boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k. \tag{89}$$

We will also use the lift of the jumps on each edge, $\ell_e([\![v]\!]) \in \boldsymbol{\Sigma}_h^k$, defined as

$$(\ell_e([\![v]\!]),\boldsymbol{\tau})_{\mathcal{T}_h} = < [\![v]\!], \boldsymbol{B}_{01}\boldsymbol{\tau} >_{e} \ \forall \boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k \quad\Longrightarrow\quad \mathcal{L}([\![v]\!]) = \sum_{e\in\mathcal{E}_h}\ell_e([\![v]\!]). \tag{90}$$

Using the above definitions, we set

$$\begin{aligned} \mathcal{S}_L(u,v) &:= (\mathcal{L}([\![u]\!]),\mathbb{K}\mathcal{L}([\![v]\!]))_{\mathcal{T}_h},\\ \mathcal{S}_J(u,v) &:= \sum_{e\in\mathcal{E}_h} h_e^{-1} < [\![u]\!], \{\mathbb{K}[\![v]\!]\} >_{\mathcal{E}_h},\\ \mathcal{S}_\ell(u,v) &:= \sum_{e\in\mathcal{E}_h}(\ell_e([\![u]\!]),\mathbb{K}\ell_e([\![v]\!]))_{\mathcal{T}_h}. \end{aligned} \tag{91}$$

As we shall see, these terms are associated with different choices of the operators B, and give rise to different stabilizing terms.

4.1 First set of methods

The first set of methods we present is characterized by the following choice

$$\boldsymbol{B}_{00}\boldsymbol{\tau} = \boldsymbol{\tau}, \quad \boldsymbol{B}_{01}\boldsymbol{\tau} = -\{\boldsymbol{\tau}\}, \quad B_{02}\boldsymbol{\tau} = 0, \quad B_{12}v = -\{v\}. \tag{92}$$

In particular, we notice that $s = 1$. We also recall that $B_{10}v = v$, as we have assumed from the beginning. With the choice (92) for $\boldsymbol{B}_{01}$, the lifting operator (89) is given by

$$(\mathcal{L}([\![v]\!]), \boldsymbol{\tau})_{\mathcal{T}_h} = - < [\![v]\!], \{\boldsymbol{\tau}\} >_{\mathcal{E}_h} \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k, \tag{93}$$

and the following estimate holds (see, e.g.,[10]):

$$C_1 \|\mathcal{L}([\![v]\!])\|_0^2 \leq \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![v]\!]\|_{0,e}^2 \leq C_2(\|\mathcal{L}([\![v]\!])\|_0^2 + |v|_{1,h}^2). \tag{94}$$

In particular, (94) shows that all the terms (91) are equivalent. According with (92), different schemes are obtained by varying $\boldsymbol{B}_{11}$ only. We show the correspondence between the choice of $\boldsymbol{B}_{11}$ and the resulting method in Table 1. In the table, $s_1 : \mathcal{E}_h \to \mathbb{R}$ is a function defined by

$$s_{1|e} = \frac{\eta_e}{h_e} \qquad \forall\, e \in \mathcal{E}_h, \tag{95}$$

where the η_e's are suitable positive constants, uniformly bounded and bounded away from zero (see [3]).

TABLE 1
The operator B_{11} for some DG methods, with the corresponding stability term and references

$\boldsymbol{B}_{11}v$	Stab.	Method
$\boldsymbol{B}_{11}v = 0$	–	Original BR [5]
$\boldsymbol{B}_{11}v = \{\mathbb{K}\mathcal{L}([\![v]\!])\} - \{\mathbb{K}\ell_e([\![v]\!])\}$	$\mathcal{S}_\ell(u,v)$	BR1 [6]
$\boldsymbol{B}_{11}v = -\{\mathbb{K}\ell_e([\![v]\!])\}$	$\mathcal{S}_\ell(u,v)$	Brezzi et al [12]
$\boldsymbol{B}_{11}v = s_1\{\mathbb{K}[\![v]\!]\} + \{\mathbb{K}\mathcal{L}([\![v]\!])\}$	$\mathcal{S}_J(u,v)$	IP [4, 33, 2]
$\boldsymbol{B}_{11}v = 2\{\mathbb{K}\nabla_h v\} + \{\mathbb{K}\mathcal{L}([\![v]\!])\}$	–	BO [29]
$\boldsymbol{B}_{11}v = 2\{\mathbb{K}\nabla_h v\} + \{\mathbb{K}\mathcal{L}([\![v]\!])\} + s_1\{\mathbb{K}[\![v]\!]\}$	$\mathcal{S}_J(u,v)$	NIPG [30]
$\boldsymbol{B}_{11}v = \{\mathbb{K}\nabla_h v\} + \{\mathbb{K}\mathcal{L}([\![v]\!])\} + s_1\{\mathbb{K}[\![v]\!]\}$	$\mathcal{S}_J(u,v)$	IIP [19, 32]

We remark that the original format of the various methods can be recovered by performing the following two steps.

1. Use (89) (or (90)) in the first equation of (22). Recalling that one has $B_{02}\boldsymbol{\tau} = 0$, $\mathbb{K}$ is piecewise constant, and $\nabla_h(V_h^k) \subseteq \boldsymbol{\Sigma}_h^k$, one obtains

$$\boldsymbol{\sigma}_h = -\mathbb{K}\,(\nabla_h u_h + \mathcal{L}([\![u_h]\!]))\,. \tag{96}$$

2. Substitute the above expression in the second equation of (22) to get a variational formulation involving only the unknown u_h.

Proposition 1 Suppose to choose $\boldsymbol{B}_{00}$, $\boldsymbol{B}_{01}$, B_{02}, B_{10}, and B_{12} as in (92). For all the choices detailed Table 1 it holds

$$|\!|\!|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-u_h)|\!|\!| \le C\left(\sum_{T\in\mathcal{T}_h}\eta_T^2\right)^{1/2}. \tag{97}$$

Proof. We simply apply Corollary 3. We first notice that (92) implies

$$\begin{aligned} &s^{-1}\boldsymbol{B}_{01}(\mathbf{curl}\,p_I) + \{\mathbf{curl}\,p_I\} = 0 \quad ; \quad s^{-1}B_{02}(\mathbf{curl}\,p_I) = 0 \\ &B_{12}w_I + \{w_I\} = 0. \end{aligned} \tag{98}$$

Therefore, we only need to estimate the term $\sum_{e\in\mathcal{E}_h} h_e||\boldsymbol{B}_{11}w_I||^2_{0,e}$ in (87). Since $[\![w_I]\!]_{|e} = 0\ \forall e\in\mathcal{E}_h$, for the first four choices in Table 1 we have $\boldsymbol{B}_{11}w_I \equiv 0$, and estimate (97) follows, while for the last three choices in Table 1 we have $\boldsymbol{B}_{11}w_I = \alpha\{\mathbb{K}\nabla_h w_I\}$ ($\alpha = 1,2$). Then:

$$\sum_{e\in\mathcal{E}_h} h_e||\boldsymbol{B}_{11}w_I||^2_{0,e} = \alpha^2\sum_{e\in\mathcal{E}_h} h_e||\{\mathbb{K}\nabla_h w_I\}||^2_{0,e}. \tag{99}$$

Using the trace inequality (11), the equivalence of norms (20), $|\nabla w_I|_{1,T} = 0\ \forall T\in\mathcal{T}_h$, the summation properties of norms, and finally estimates (50) we deduce:

$$\begin{aligned} \sum_{e\in\mathcal{E}_h} h_e||\boldsymbol{B}_{11}w_I||^2_{0,e} &= \alpha^2\sum_{e\in\mathcal{E}_h} h_e||\{\mathbb{K}\nabla_h w_I\}||^2_{0,e} \le C\sum_{T\in\mathcal{T}_h}||\nabla w_I||^2_{0,T} \\ &= C\sum_{T\in\mathcal{T}_h^c}||\nabla w_I||^2_{0,T} \le C|w|_1^2, \end{aligned} \tag{100}$$

which leads to (87). Then, estimate (97) follows by invoking Corollary 3.
□

4.2 LDG methods

Another example arises from the following choices:

$$\begin{aligned} &\boldsymbol{B}_{00}\boldsymbol{\tau} = \boldsymbol{\tau}, \quad \boldsymbol{B}_{01}\boldsymbol{\tau} = -\{\boldsymbol{\tau}\} + \boldsymbol{\beta}[\![\boldsymbol{\tau}]\!], \quad B_{02}\boldsymbol{\tau} = 0 \\ &\boldsymbol{B}_{11}v = s_1\{\mathbb{K}[\![v]\!]\}, \quad B_{12}v = -\{v\} - \boldsymbol{\beta}\cdot[\![v]\!], \end{aligned} \tag{101}$$

with $\boldsymbol{\beta}$ a suitable vector, and s_1 as in (95). Still, we have $s = 1$ and $B_{10}v = v$. The definition of the lifting operator (89) is now

$$(\mathcal{L}([\![v]\!]), \boldsymbol{\tau})_{\mathcal{T}_h} = - < [\![v]\!], \{\boldsymbol{\tau}\} >_{\mathcal{E}_h} + < \boldsymbol{\beta} \cdot [\![v]\!], [\![\boldsymbol{\tau}]\!] >_{\mathcal{E}_h^\circ} \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k, \quad (102)$$

and estimate (94) still holds. These choices lead to the so-called *LDG* method of Cockburn and Shu (see [18]), for which we prove the following Proposition.

Proposition 2 With the choice (101) it holds

$$|||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)||| \leq C \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2}. \qquad (103)$$

Proof. Again, we apply Corollary 3. To do so, we simply note that we have

$$\begin{aligned} &s^{-1}\boldsymbol{B}_{01}(\mathbf{curl}\, p_I) + \{\mathbf{curl}\, p_I\} = \boldsymbol{\beta} [\![\mathbf{curl}\, p_I]\!] = 0 \\ &s^{-1}B_{02}(\mathbf{curl}\, p_I) = 0 \\ &B_{12}w_I + \{w_I\} = -\boldsymbol{\beta} \cdot [\![w_I]\!] = 0 \\ &\boldsymbol{B}_{11}w_I = s_1\{\mathbb{K}[\![w_I]\!]\} = 0. \end{aligned} \qquad (104)$$

Therefore, Corollary 3 straightforwardly applies. □

We now detail a variant of the methods above, which accounts for choosing all the operators as in (101), but

$$B_{02}\boldsymbol{\tau} = s_2 [\![\mathbb{K}^{-1}\boldsymbol{\tau}]\!]. \qquad (105)$$

Above, $s_2 : \mathcal{E}_h \to \mathbb{R}$ is a function defined by

$$s_{2|e} = \tau_e h_e \qquad \forall\, e \in \mathcal{E}_h, \qquad (106)$$

where the τ_e's are suitable positive constants, uniformly bounded and bounded away from zero (see [15]). We remark that in this case the elimination of $\boldsymbol{\sigma}_h$ cannot be done as in (96). However, (105)-(106) imply

$$\begin{aligned} \sum_{e \in \mathcal{E}_h} h_e^{-1} ||s^{-1}B_{02}(\mathbf{curl}\, p_I)||_{0,e}^2 &= \sum_{e \in \mathcal{E}_h} h_e^{-1} s_2^2 ||[\![\mathbb{K}^{-1}\mathbf{curl}\, p_I]\!]||_{0,e}^2 \\ &\leq C \big(\sum_{e \in \mathcal{E}_h} h_e ||[\![\mathbb{K}^{-1}\mathbf{curl}\, p_I]\!]||_{0,e}^2 \big). \end{aligned} \qquad (107)$$

With the same arguments used for proving (100), the trace inequality (11), the equivalence of norms (20), $|\mathbf{curl}\, p_I|_{1,T} = 0\ \forall T \in \mathcal{T}_h$, the summation properties of norms, and finally estimates (50) lead to:

$$\sum_{e \in \mathcal{E}_h} h_e ||[\![\mathbb{K}^{-1}\mathbf{curl}\, p_I]\!]||_{0,e}^2 \leq C \big(\sum_{T \in \mathcal{T}_h^c} ||\mathbf{curl}\, p_I||_{0,T}^2 \big) \leq C|p|_1^2. \qquad (108)$$

Therefore, Corollary 3 applies (cf. also (104)).

4.3 Hughes-Masud methods

With this terminology we indicate a collection of methods introduced and analyzed in [10], based on the original method proposed by Hughes and Masud in [27]. (See also [1] for numerical results on all the formulations). These methods are obtained with the following choice:

$$\begin{aligned} &\boldsymbol{B}_{00}\boldsymbol{\tau} = s\boldsymbol{\tau}, \quad \boldsymbol{B}_{01}\boldsymbol{\tau} = -\{\boldsymbol{\tau}\}, \quad B_{02}\boldsymbol{\tau} = 0 \\ &B_{10}v = v, \quad B_{12}v = -\{v\} \end{aligned} \tag{109}$$

with s a positive parameter to be chosen to get stability. Varying $\boldsymbol{B}_{11}$ produces various schemes, as detailed in Table 2. In the table, the first choice is stable and robust for $s \in [s_0, s_1]$, with $[s_0, s_1] \subset]0, 1[$; the second and the third ones are so for $s \in [s_0, s_1]$, with $s_0 > 0$. Finally, the fourth choice is stable and robust for $s \in [s_0, s_1]$, with $[s_0, s_1] \subset]0, 4[$.

TABLE 2
The operator B_{11} for the Hughes-Masud methods and related references.

$\boldsymbol{B}_{11}v$	Method	Refs.
$\boldsymbol{B}_{11}v = \frac{1-s}{s}\{\mathbb{K}\nabla_h v\}$	$\text{IP} + \frac{1}{s}\mathcal{S}_L(u,v)$	[33, 2, 10]
$\boldsymbol{B}_{11}v = -\frac{1-s}{s}\{\mathbb{K}\nabla_h v\}$	$\frac{1}{s}\text{BR} - \frac{1-s}{s}\text{BO}$	[10]
$\boldsymbol{B}_{11}v = \frac{1+s}{s}\{\mathbb{K}\nabla_h v\}$	$\text{BO} + \frac{1}{s}\mathcal{S}_L(u,v)$	[10, 29]
$\boldsymbol{B}_{11}v = \frac{1}{s}\{\mathbb{K}\nabla_h v\}$	$\text{IIP} + \frac{1}{s}\mathcal{S}_L(u,v)$	[19, 1]

We remark that the original format of the various methods can be recovered by performing the two steps detailed in section 4.1. The only difference is that equation (96) now becomes

$$\boldsymbol{\sigma}_h = -\mathbb{K}\left(\nabla_h u_h + s^{-1}\mathcal{L}([\![u_h]\!])\right) \qquad \text{(see first equation of (22))}. \tag{110}$$

We now prove the following result.

Proposition 3 It holds

$$|\!|\!|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)|\!|\!| \leq C \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2}. \tag{111}$$

Proof. We notice that we have

$$
\begin{aligned}
&s^{-1}\boldsymbol{B}_{01}(\mathbf{curl}\, p_I) + \{\mathbf{curl}\, p_I\} = (1 - s^{-1})\{\mathbf{curl}\, p_I\} \\
&s^{-1}B_{02}(\mathbf{curl}\, p_I) = 0 \\
&B_{12}w_I + \{w_I\} = 0 \\
&\boldsymbol{B}_{11}w_I = c(s)\{\mathbb{K}\nabla_h w_I\},
\end{aligned}
\tag{112}
$$

where $c(s)$ is a constant, which is defined accordingly with the above choices of 2. Notice that the two nonzero terms are of the same type as (108) and (99), and can be estimated exactly in the same way. Thus:

$$
\begin{aligned}
&\left(\sum_{e\in\mathcal{E}_h} h_e||s^{-1}\boldsymbol{B}_{01}(\mathbf{curl}\, p_I) + \{\mathbf{curl}\, p_I\}||^2_{0,e}\right)^{1/2} \leq C|p|_1 \\
&\left(\sum_{e\in\mathcal{E}_h} h_e||\boldsymbol{B}_{11}w_I||^2_{0,e}\right)^{1/2} \leq C|w|_1.
\end{aligned}
\tag{113}
$$

Estimate (111) now follows from Corollary 3. □

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