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VIRTUAL ELEMENTS FOR LINEAR ELASTICITY PROBLEMS*

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Abstract. We discuss the application of *virtual elements* to linear elasticity problems, for both the compressible and the nearly incompressible case. Virtual elements are very close to mimetic finite differences (see, for linear elasticity, [L. Beirão da Veiga, *M2AN Math. Model. Numer. Anal.*, 44 (2010), pp. 231–250]) and in particular to higher order mimetic finite differences. As such, they share the good features of being able to represent in an *exact* way certain physical properties (conservation, incompressibility, etc.) and of being applicable in very general geometries. The advantage of virtual elements is the ductility that easily allows high order accuracy and high order continuity.

Key words. mimetic finite differences, virtual elements, elasticity

AMS subject classifications. 65N30, 65N12, 65G99, 76R99

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1. Introduction.

In recent times the mimetic finite difference approach has been successfully applied to a great variety of problems, from diffusion problems to electromagnetism, on fairly irregular decompositions, including polygons with rather weird shapes, polyhedra in three dimensions with curved faces, hanging nodes, and so on. (For a partial list of citations we refer to [29, 25, 17, 24, 26, 13, 15, 8, 27, 1, 14, 6, 18, 21, 32, 11, 2, 10, 16].) Their use was limited to low order approximations until very recently, when people started to extend the methodology to include higher order approximations to gain better accuracy in the numerical results. See, e.g., [22, 5, 4, 3]. The first results in this direction were very encouraging, and people started to look more closely to these extensions, analyzing advantages and limitations and mostly looking for the key properties that could make things easier. This gave rise to a new interpretation of mimetic finite differences (see, to start with, [16]) and to a subsequent new approach, much closer to finite elements, that we call the virtual element method (VEM). Other methods that extend the Finite Element philosophy to polygonal meshes can be found in [30, 31, 28].

The basic idea of the new method can be described, roughly speaking, as follows. We start as we do for the classical finite elements of Lagrange or Hermite type, with one difference: in each element K, together with the usual polynomials, say, $\mathcal{S}$ (in general, all the polynomials up to a given degree k), some additional functions are also considered (typically solutions of PDEs within the element K) in order to get unisolvence. If things are properly done (good choice of the functions and of the degrees of freedom), the local stiffness matrix AE can be computed exactly *whenever one of the two entries is a polynomial of* $\mathcal{S}$, without solving the local PDE (virtual

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solution). For the other coefficients of AE it is enough to have numbers that are bounded from above and from below by the "right" ones with constants independent of h. The label *virtual* depends on the fact that some of the basis functions are not explicitly known.

In this paper we apply the virtual element approach to linear elasticity problems, compressible and nearly incompressible, in two dimensions. We shall prove "optimal" a priori estimates, meaning that if $\mathcal{S}$ contains all the polynomial of degree $\leq k$ (for some integer $k \geq 1$), then the error between the exact and the approximate solution, in the energy norm, behaves as $O(h^k)$ (times some $(k+1)$-norm of the exact solution) where h is the maximum diameter of the elements in the decomposition.

Finally, we shall also prove optimal L^2 estimates, that is, $O(h^{k+1})$.

As for the traditional mimetic finite difference approach the decomposition of the computational domain Ω can be done in a *very general* way as, for instance, in [11]. We point out that for a decomposition in triangles the VEM will reproduce, essentially, the classical finite element methods. This will not be the case for more general decompositions, including quadrilaterals. In particular, we stress the fact that general quadrilaterals do not require being treated as *isoparametric elements* (and, besides, we can allow aligned vertices and nonconvex elements).

Lowest degree virtual elements on quadrilaterals are indeed very close to nodal mimetic finite differences. See [19] for the use of nodal mimetic finite differences on the Laplace operator and for their many useful properties.

We also point out that small intrusions of a "practically arbitrary" shape, made of a material with different elastic properties, can be treated with a single element. In this respect, we have a rough imitation of the two-scale methods for composite materials (as, for instance, in [23]). But here the *fine scale* is just made by a single subelement (or a few, disconnected subelements) with constant material properties inside each of them. Instead, in most two-scale methods the fine grid problems involve more deeply the fine structure of the material and are a sort of preprocessor to be computed, element by element, in parallel.

Throughout the paper C will denote, as usual, a generic positive constant independent of the mesh size. For the definition of Sobolev spaces and their norms we refer to [20]. In particular we shall use the notation $(\cdot,\cdot)_{\mathcal{O},0}$ to denote the inner product in $L^2(\mathcal{O})$ or $(L^2(\mathcal{O}))^2$ and use simply $(\cdot,\cdot)_0$ whenever $\mathcal{O}=\Omega$. Moreover, for k integer ≥ 0, $\mathbb{P}_k$ will denote the space of polynomials of degree $\leq k$.

2. Compressible linear elasticity.

2.1. The problem. We consider the deformation problem of a linearly elastic body subjected to a volume load and with given boundary conditions, under the hypothesis of small deformations. Let Ω be a polygonal domain, and let Γ be its boundary. Let λ and μ be positive coefficients (Lamé coefficients) and let $\boldsymbol{f}$ be a vector valued function belonging to $(L^2(\Omega))^2$. For simplicity we will use (homogeneous) Dirichlet boundary conditions and hence consider the space

$$\mathbf{V} := (H^1_0(\Omega))^2 \qquad \text{with} \quad \|\boldsymbol{v}\|^2_{\mathbf{V}} := \|\nabla \boldsymbol{v}\|_{(L^2(\Omega))^4} \ \forall\, \boldsymbol{v}\in\mathbf{V}. \tag{2.1}$$

However, we will always write our bilinear form as

$$a(\boldsymbol{u},\boldsymbol{v}) := 2\mu\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\mathbf{x} + \lambda\int_\Omega \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{v}\,\mathrm{d}\mathbf{x} \equiv 2\mu a_\mu(\boldsymbol{u},\boldsymbol{v}) + \lambda a_\lambda(\boldsymbol{u},\boldsymbol{v}) \tag{2.2}$$

(where $\boldsymbol{\varepsilon}$ represents as usual the *symmetric gradient operator*), and we will keep an eye on the set of *rigid body motions*

$$RM(\Omega) := \{\boldsymbol{v} \in (H^1(\Omega))^2 \text{ such that } \boldsymbol{\varepsilon}(\boldsymbol{v}) = 0\} = \{(c_1, c_2) + c_3(x_2, -x_1)\} \tag{2.3}$$

so that the extensions to more general cases will be immediate. To the bilinear form a we can associate, in the usual distributional sense, the linear elliptic operator $A_{\lambda,\mu}$ given by

$$A_{\lambda,\mu}\boldsymbol{u} := -\begin{pmatrix} 2\mu(u_{1,xx} + \frac{1}{2}(u_{1,yy} + u_{2,xy})) + \lambda(u_{1,xx} + u_{2,yx}) \\ 2\mu(\frac{1}{2}(u_{1,yx} + u_{2,xx}) + u_{2,yy}) + \lambda(u_{1,xy} + u_{2,yy}) \end{pmatrix}. \tag{2.4}$$

It is easy to see (possibly using the Korn inequality in the presence of more general boundary conditions) that there exist two constants, $M > 0$ and $\alpha > 0$, depending only on Ω, λ, and μ, such that

$$\alpha \, \|\boldsymbol{v}\|^2_{\mathbf{V}} \leq \, a(\boldsymbol{v}, \boldsymbol{v}) \leq M \, \|\boldsymbol{v}\|^2_{\mathbf{V}} \qquad \forall \, \boldsymbol{v} \in \mathbf{V}. \tag{2.5}$$

We note that we obviously have $\boldsymbol{f} \in \mathbf{V}'$, and we denote by $< \boldsymbol{f}, \boldsymbol{v} >$ the corresponding duality pairing (that here coincides with the usual $(L^2)^2$ inner product). We consider the problem: *find* $\boldsymbol{u} \in \mathbf{V}$ *such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) =< \boldsymbol{f}, \boldsymbol{v} > \quad \forall \, \boldsymbol{v} \in \mathbf{V} \tag{2.6}$$

that clearly has a unique solution that belongs at least to $(H^s(\Omega))^2$ for some $s > 3/2$ depending on the maximum angle in Γ. More generally we denote by $\mathbf{W}$ a space of vector valued functions that contains our solution $\boldsymbol{u}$ and where we are allowed to take point values. For instance, we can assume that $\mathbf{W} \subseteq (C^0(\overline{\Omega}))^2$.

Remark 2.1. It is clear that problem (2.6) is equivalent to

$$A_{\lambda,\mu}\boldsymbol{u} = \boldsymbol{f} \;\text{ in } \Omega \qquad \text{and} \qquad \boldsymbol{u} = 0 \;\text{ on } \Gamma. \qquad \Box \tag{2.7}$$

2.2. The decompositions and the discrete problems. In order to approximate the solution of (2.6) we consider a sequence $\{\mathcal{T}_h\}_h$ of decompositions of Ω into subpolygons, such that the following hold:

H0. There exists an integer N and a positive real number γ such that for every h and for every $K \in \mathcal{T}_h$,

- the number of edges of K is $\leq N$,
- the ratio between the shortest edge and the diameter h_K of K is bigger than γ, and
- K is star-shaped with respect to every point of a ball of radius γh_K. $\Box$

With obvious notation, we split the norm

$$\|\boldsymbol{v}\|^2_{\mathbf{V}} = \sum_{K \in \mathcal{T}_h} \|\boldsymbol{v}\|^2_{\mathbf{V},K} \quad \forall \, \boldsymbol{v} \, \in \mathbf{V} \tag{2.8}$$

and the bilinear form a as

$$a(\boldsymbol{u}, \boldsymbol{v}) = \sum_{K \in \mathcal{T}_h} a^K(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \, \boldsymbol{u}, \, \boldsymbol{v} \, \in \mathbf{V}. \tag{2.9}$$

H1. We fix an integer $k \geq 1$ (that will be our order of accuracy) and consider for each h

- a space $\mathbf{V}_h \subset \mathbf{V}$,
- a symmetric bilinear form a_h from $\mathbf{V}_h \times \mathbf{V}_h$ to $\mathbb{R}$, and
- an element $\boldsymbol{f}_h \in \mathbf{V}'_h$.

We also suppose that the approximate bilinear form a_h can also be split as we did for a in (2.9), namely,

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_K a_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h) \qquad \forall\, \boldsymbol{u}_h,\, \boldsymbol{v}_h \in \mathbf{V}_h. \qquad \square \tag{2.10}$$

2.3. An abstract convergence theorem. Together with H0 and H1 we further assume the following properties:

H2. For all h and for all K in $\mathcal{T}_h$

- $\forall \boldsymbol{p} \in (\mathbb{P}_k)^2$, $\forall \boldsymbol{v}_h \in \mathbf{V}_h$

$$a_h^K(\boldsymbol{p}, \boldsymbol{v}_h) = a^K(\boldsymbol{p}, \boldsymbol{v}_h). \tag{2.11}$$

- $\exists$ two positive constants α_* and α^*, independent of h and of K, such that

$$\forall \boldsymbol{v}_h \in \mathbf{V}_h \qquad \alpha_*\, a^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \le a_h^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \le \alpha^*\, a^K(\boldsymbol{v}_h, \boldsymbol{v}_h). \qquad \square \tag{2.12}$$

Note that the symmetry of a_h, (2.12), and (2.5) easily imply that

$$\begin{aligned} a_h^K(\boldsymbol{u}, \boldsymbol{v}) &\le \left(a_h^K(\boldsymbol{u}, \boldsymbol{u})\right)^{1/2} \left(a_h^K(\boldsymbol{v}, \boldsymbol{v})\right)^{1/2} \\ &\le \alpha^* \left(a^K(\boldsymbol{u}, \boldsymbol{u})\right)^{1/2} \left(a^K(\boldsymbol{v}, \boldsymbol{v})\right)^{1/2} \le \alpha^*\, M\, \|\boldsymbol{u}\|_{\mathbf{V},K}\, \|\boldsymbol{v}\|_{\mathbf{V},K} \end{aligned} \tag{2.13}$$

for all $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbf{V}_h$.

Remark 2.2. It is clear that in the classical terminology, assumption (2.11) is meant to ensure *consistency* of the bilinear form a_h, and (2.12) is meant to ensure *stability.* $\square$

From now on, since we will deal also with functions that belong to $\prod_K (H^1(K))^2$, but are not globally in $(H^1(\Omega))^2$, we will use the broken H^1 norm:

$$\|\boldsymbol{v}\|_{h,\mathbf{V}} := \left(\sum_{K \in \mathcal{T}_h} \|\boldsymbol{v}\|^2_{\mathbf{V},K} \right)^{1/2}. \tag{2.14}$$

THEOREM 2.1. *Under the assumptions* H0*,* H1*, and* H2 *the following discrete problem has a unique solution* $\boldsymbol{u}_h$*: Find* $\boldsymbol{u}_h$ *in* $\mathbf{V}_h$ *such that*

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \langle \boldsymbol{f}_h \boldsymbol{v}_h \rangle \qquad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h. \tag{2.15}$$

Moreover, for every approximation $\boldsymbol{u}_I$ *of* $\boldsymbol{u}$ *in* $\mathbf{V}_h$ *and for every approximation* $\boldsymbol{u}_\pi$ *of* $\boldsymbol{u}$ *that is piecewise in* $(\mathbb{P}_k)^2$*, we have*

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathbf{V}} \le C(\|\boldsymbol{u} - \boldsymbol{u}_I\|_{\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \mathfrak{F}),$$

where C *is a constant depending only on* λ*,* μ*,* α_**,* α^* *and where* $\mathfrak{F}$ *is the smallest constant such that*

$$|\langle \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h \rangle| \le \mathfrak{F} \|\boldsymbol{v}_h\|_{\mathbf{V}} \quad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h. \tag{2.16}$$

Remark 2.3. Note that we cannot write $\|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}}$ as $\|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{\mathbf{V}}$ since, in general, $\boldsymbol{u}_\pi$ will not be an element of $\mathbf{V}_h$ (lacking the necessary continuity). $\square$

Proof. Existence and uniqueness of the solution of (2.15) follow immediately from (2.12) and (2.5). Next, setting $\boldsymbol{\delta}_h := \boldsymbol{u}_h - \boldsymbol{u}_I$ we have

(2.17)

$$\begin{aligned}
\alpha_* \, \alpha \|\boldsymbol{\delta}_h\|_{\mathbf{V}}^2 &\le \alpha_* \, a(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \le a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \\
&= a_h(\boldsymbol{u}_h, \boldsymbol{\delta}_h) - a_h(\boldsymbol{u}_I, \boldsymbol{\delta}_h) \text{ (use (2.15) and (2.10))} \\
&=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K a_h^K(\boldsymbol{u}_I, \boldsymbol{\delta}_h) \text{ (use } \pm \boldsymbol{u}_\pi) \\
&=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K \big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a_h^K(\boldsymbol{u}_\pi, \boldsymbol{\delta}_h)\big) \text{ (use (2.11))} \\
&=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K \big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K(\boldsymbol{u}_\pi, \boldsymbol{\delta}_h)\big) \text{ (use } \pm u \text{ and (2.9))} \\
&=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K \big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{\delta}_h)\big) - a(\boldsymbol{u}, \boldsymbol{\delta}_h) \text{ (use (2.6))} \\
&=< \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K \big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{\delta}_h)\big) - < \boldsymbol{f}, \boldsymbol{\delta}_h > \\
&= < \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{\delta}_h > - \sum_K \big(a_h^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K(\boldsymbol{u}_\pi - u, \boldsymbol{\delta}_h)\big).
\end{aligned}$$

Now we use (2.16), (2.13), and the continuity of each a^K in (2.17) in order to obtain

$$\|\boldsymbol{\delta}_h\|_{\mathbf{V}}^2 \le C\big(\mathfrak{F} + \|\boldsymbol{u}_I - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}}\big) \|\boldsymbol{\delta}_h\|_{\mathbf{V}} \tag{2.18}$$

for some constant C depending only on λ, μ, α_*, α^*. Then the result follows easily by the triangle inequality. ∎

2.4. Construction of $\mathbf{V}_h$. We can now tackle the second part of our construction, showing how, given the sequence of decompositions $\{\mathcal{T}_h\}_h$ and the integer k, we can construct (and use!) the corresponding spaces $\mathbf{V}_h$ and the bilinear forms a_h such that the assumptions (2.10)–(2.12) of our theorem are satisfied.

DEFINITION 2.2. *For every decomposition $\mathcal{T}_h$ and for every $k \ge 1$ we define $\mathbf{V}_h$ as the space of (vector valued) continuous functions $\boldsymbol{v}_h$ that vanish on the boundary of Ω, that are polynomials of degree $\le k$ on each edge e of $\mathcal{T}_h$, and such that in each element K we have $(A_{\lambda,\mu}\boldsymbol{v}_h)_{|K} \in (\mathbb{P}_{k-2})^2$ when $k \ge 2$ and $(A_{\lambda,\mu}\boldsymbol{v}_h)_{|K} = 0$ when $k = 1$.*

It is not difficult to see that for an element $K \in \mathcal{T}_h$ having n edges, the set of all (vector valued) continuous functions on ∂K that are polynomials of degree $\le k$ on each edge of ∂K is a linear space of dimension $2n + 2n(k-1)$. Indeed, a continuous *scalar function* which is a polynomial of degree $\le k$ on each edge is uniquely determined by its values at the vertices (n conditions) plus, for $k > 1$, by the moments up to the order $k-2$ ($k-1$ conditions) on each edge (hence $n + n(k-1)$ conditions in total). And for a *vector valued* function we have two scalar functions. Therefore, denoting by $\mathbf{V}_h^K$ the restriction of $\mathbf{V}_h$ to K we can check easily that, say, for an internal element K with n vertices the dimension of $\mathbf{V}_h^K$ is given by

$$\mathrm{N}^K \equiv dim \mathbf{V}_h^K = 2n + 2n(k-1) + k(k-1), \tag{2.19}$$

where the last term corresponds to the dimension of vector valued polynomials of degree $\le k-2$ in two dimensions (and hence to the number of *internal* degrees of freedom). Similarly, we can easily see that the dimension of the whole space $\mathbf{V}_h$ is given by

$$\mathrm{N}^{tot} \equiv \dim \mathbf{V}_h = 2N_V + 2N_E(k-1) + N_P k(k-1), \tag{2.20}$$

where N_V, N_E, and N_P are, respectively, the total number of *internal vertices*, *internal edges*, and *elements (polygons)* in $\mathcal{T}_h$.

In $\mathbf{V}_h$ we choose the following *degrees of freedom*:

- $\mathcal{V}$, the values of $\boldsymbol{v}_h$ at the *internal vertices*;
- $\mathcal{E}$, for $k > 1$, the moments $\int_e \boldsymbol{q}(t)\boldsymbol{v}_h(t)\mathrm{d}t$ for $\boldsymbol{q} \in (\mathbb{P}_{k-2}(e))^2$ on each *internal edge* e;
- $\mathcal{P}$, for $k > 1$, the moments $\int_K \boldsymbol{q}(\mathbf{x})\boldsymbol{v}_h(\mathbf{x})\mathrm{d}\mathbf{x}$ for $\boldsymbol{q} \in (\mathbb{P}_{k-2}(K))^2$ in each *element* K.

It is not difficult to check that the dimension N^{tot} of $\mathbf{V}_h$, computed in (2.20), equals the total number of degrees of freedom $\mathcal{V}$ plus $\mathcal{E}$ plus $\mathcal{P}$. For future use, it will also be convenient to define the interpolation operator χ as the operator that to each smooth enough vector valued function $\boldsymbol{\varphi}$ associates the N^{tot} values $\chi_1(\boldsymbol{\varphi}), \ldots, \chi_{N^{tot}}(\boldsymbol{\varphi})$ of its degrees of freedom and, for each element K, its restriction χ^K to the degrees of freedom related to $\overline{K}$.

Remark 2.4. We notice that in each element K, the degrees of freedom $\mathcal{V}$ plus $\mathcal{E}$ uniquely determine a polynomial of degree $\leq k$ on each edge of K, that is, $\mathcal{V}$ plus $\mathcal{E}$ are equivalent to prescribing $\boldsymbol{v}_h$ on ∂K. On the other hand, the degrees of freedom $\mathcal{P}$ are equivalent to prescribing $\mathbf{\Pi}^K_{k-2}\boldsymbol{v}_h$ in K (where $\mathbf{\Pi}^K_{k-2}$ is the projection operator, in the $(L^2(K))^2$ norm, onto the space $(\mathbb{P}_{k-2})^2$).

PROPOSITION 2.3. *The degrees of freedom given by $\mathcal{V}$ plus $\mathcal{E}$ plus $\mathcal{P}$ are unisolvent in $\mathbf{V}_h$.*

Proof. According to Remark 2.4, to prove the proposition it is enough to see that for each $K \in \mathcal{T}_h$, a vector valued function $\boldsymbol{v}_h$ such that

$$\boldsymbol{v}_h = 0 \qquad \text{on } \partial K \quad \forall K \in \mathcal{T}_h \tag{2.21}$$

and

$$\mathbf{\Pi}^K_{k-2}\boldsymbol{v}_h = 0 \qquad \text{in } K \quad \forall K \in \mathcal{T}_h \tag{2.22}$$

is actually identically zero in K. In order to prove this, we will prove that $A_{\lambda,\mu}\boldsymbol{v}_h = 0$ in K (that joined with (2.21) gives $\boldsymbol{v}_h \equiv 0$). In order to see this, we first solve, for every $\boldsymbol{q} \in (\mathbb{P}_{k-2}(K))^2$, the following auxiliary problem: Find $\boldsymbol{w} \in (H^1_0(K))^2$ such that

$$a^K(\boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{q}, \boldsymbol{v})_{0,K} \qquad \forall\, \boldsymbol{v} \in (H^1_0(K))^2; \tag{2.23}$$

this, in agreement with Remark 2.1, could also be written

$$A^K_{\lambda,\mu}\boldsymbol{w} = \boldsymbol{q} \;\text{ in } K \qquad \text{and} \qquad \boldsymbol{w} = 0 \;\text{ on } \partial K. \tag{2.24}$$

Next, we consider the map R, from $(\mathbb{P}_{k-2}(K))^2$ into itself, defined by

$$R\boldsymbol{q} := \mathbf{\Pi}^K_{k-2}(A^K_{\lambda,\mu})^{-1}(\boldsymbol{q}) \equiv \mathbf{\Pi}^K_{k-2}\boldsymbol{w}. \tag{2.25}$$

We claim that R with this definition is an isomorphism. Indeed, from (2.25) and the definition of $\mathbf{\Pi}^K_{k-2}$ we have for every $\boldsymbol{q} \in (\mathbb{P}_{k-2})^2$,

$$(R(\boldsymbol{q}), \boldsymbol{q})_{0,K} = (\mathbf{\Pi}^K_{k-2}(A^K_{\lambda,\mu})^{-1}(\boldsymbol{q}), \boldsymbol{q})_{0,K} = (\mathbf{\Pi}^K_{k-2}\boldsymbol{w}, \boldsymbol{q})_{0,K} = (\boldsymbol{w}, \boldsymbol{q})_{0,K} = a^K(\boldsymbol{w}, \boldsymbol{w}). \tag{2.26}$$

Since $\boldsymbol{w}$ is in $(H^1_0(K))^2$ we have then that

$$\{R(\boldsymbol{q})=0\} \Leftrightarrow \{a^K(\boldsymbol{w},\boldsymbol{w})=0\} \Leftrightarrow \{\boldsymbol{w}=0\} \Leftrightarrow \{\boldsymbol{q}=0\}. \tag{2.27}$$

We notice that if $\boldsymbol{v}_h=0$ on ∂K, then

$$\mathbf{\Pi}^K_{k-2}\boldsymbol{v}_h = R(A^K_{\lambda,\mu}\boldsymbol{v}_h).$$

Hence, $\mathbf{\Pi}^K_{k-2}\boldsymbol{v}_h = 0 \implies R(A^K_{\lambda,\mu}\boldsymbol{v}_h) = 0 \implies A^K_{\lambda,\mu}\boldsymbol{v}_h = 0$, and the proof is concluded. □

Remark 2.5. It follows easily from the above construction that for every smooth enough $\boldsymbol{w}\in\mathbf{W}$ there exists a unique element $\boldsymbol{w}_I$ of V_h such that

$$\chi(\boldsymbol{w}-\boldsymbol{w}_I)=0. \tag{2.28}$$

Moreover, by the usual Bramble–Hilbert/Deny–Lions technique (see, e.g., [20]) and using a scaling argument to get around the variability in the geometry (see, e.g., [12]) it is not difficult to see that

$$\|\boldsymbol{w}-\boldsymbol{w}_I\|_{r,\Omega} \le C\,h^{s-r}\,|w|_{s,\Omega}, \qquad r=0,1,\ r\le s\le k+1 \tag{2.29}$$

(with a constant C depending only on the constants N and γ in H0) as in the usual finite element framework. Similarly, following for instance [9], one gets easily that for every element K and for every smooth enough $\boldsymbol{w}$ on K there exists a $\boldsymbol{w}_\pi$ in $\mathbb{P}_k$ such that

$$\|\boldsymbol{w}-\boldsymbol{w}_\pi\|_{r,K} \le C\,h_K^{s-r}\,|w|_{s,K}, \qquad r=0,1,\ r\le s\le k+1 \tag{2.30}$$

(where, again, C is a constant depending only on N and γ in H0). □

We finally note that the operator $A_{\lambda,\mu}$ appearing in Definition 2.2 is the most natural choice, but it could be replaced by any second order elliptic operator such as, for instance, a componentwise laplacian.

2.5. Construction of a_h.

We are left to show how to construct a (computable!) a_h that satisfies (2.11) and (2.12).

First we observe that for every $K\in\mathcal{T}_h$ and for every $\boldsymbol{v}_h\in\mathbf{V}^K_h$, knowing the degrees of freedom that identify $\boldsymbol{v}_h$ we can compute

- the value of $\boldsymbol{v}_h$ on ∂K (that can be computed explicitly on every edge e),
- the value of $\mathbf{\Pi}^K_{k-2}\boldsymbol{v}_h$ (that comes immediately out of the degrees of freedom).

Then we observe that on any $K\in\mathcal{T}_h$, if $\boldsymbol{p}\in(\mathbb{P}_k)^2$ and $\boldsymbol{v}_h$ is in $\mathbf{V}^K_h$, then

$$a^K_\mu(\boldsymbol{p},\boldsymbol{v}_h) = \int_K \boldsymbol{\varepsilon}(\boldsymbol{p}):\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\,\mathrm{d}\mathbf{x} = -\int_K \mathbf{div}(\boldsymbol{\varepsilon}(\boldsymbol{p}))\cdot\boldsymbol{v}_h\,\mathrm{d}\mathbf{x} + \int_{\partial K}(\boldsymbol{\varepsilon}(\boldsymbol{p})\cdot\mathbf{n})\cdot\boldsymbol{v}_h\,\mathrm{d}s. \tag{2.31}$$

We note then that $\mathbf{div}(\boldsymbol{\varepsilon}(\boldsymbol{p}))$ belongs to $(\mathbb{P}_{k-2})^2$ so that

$$\int_K \mathbf{div}(\boldsymbol{\varepsilon}(\boldsymbol{p}))\cdot\boldsymbol{v}_h\,\mathrm{d}\mathbf{x} = \int_K \mathbf{div}(\boldsymbol{\varepsilon}(\boldsymbol{p}))\cdot\mathbf{\Pi}^K_{k-2}\boldsymbol{v}_h\,\mathrm{d}\mathbf{x} \tag{2.32}$$

is computable using only the values of $\mathbf{\Pi}^K_{k-2}\boldsymbol{v}_h$. On the other hand, the boundary integral in (2.31) is also easily computable knowing $\boldsymbol{v}_h$ on ∂K. Similarly,

(2.33)
$$a_\lambda^K(\boldsymbol{p},\boldsymbol{v}_h)=\int_K \mathrm{div}(\boldsymbol{p})\mathrm{div}(\boldsymbol{v}_h)\mathrm{d}\mathbf{x}=-\int_K \nabla(\mathrm{div}(\boldsymbol{p}))\cdot\boldsymbol{v}_h\mathrm{d}\mathbf{x}+\int_{\partial K}(\mathrm{div}(\boldsymbol{p})\,\boldsymbol{v}_h\cdot\mathbf{n}\,\mathrm{d}s$$

is computable knowing only $\boldsymbol{v}_h$ on ∂K and $\mathbf{\Pi}_{k-2}^K\boldsymbol{v}_h$. Hence we can *define*

$$a_h^K(\boldsymbol{p},\boldsymbol{v}_h):=a^K(\boldsymbol{p},\boldsymbol{v}_h)\text{ and }a_h^K(\boldsymbol{v}_h,\boldsymbol{p}):=a^K(\boldsymbol{v}_h,\boldsymbol{p})\tag{2.34}$$

that come out to be *computable* whenever $\boldsymbol{p}$ is an element of $(\mathbb{P}_k)^2$ and $\boldsymbol{v}_h$ is any element of $\mathbf{V}_h^K$. Note that by this choice we already took care of (2.11), and we have only (2.12) to take care of. At this point, however, we are also able to compute a new basis for $\mathbf{V}_h^K$ by taking first all the elements of $(\mathbb{P}_k)^2$ (whose dimension is $(k+1)(k+2)$) and then completing it with

$$N^K-(k+1)(k+2)\equiv 2n+2n(k-1)+k(k-1)-(k+1)(k+2)\equiv 2nk-4k-2$$

independent elements $\widetilde{\boldsymbol{v}}_h$ such that

$$a_h^K(\boldsymbol{p},\widetilde{\boldsymbol{v}}_h)=a_h^K(\widetilde{\boldsymbol{v}}_h,\boldsymbol{p})\ (=a^K(\boldsymbol{p},\widetilde{\boldsymbol{v}}_h)=a^K(\widetilde{\boldsymbol{v}}_h,\boldsymbol{p})\,)=0\quad\forall\,\boldsymbol{p}\in(\mathbb{P}_k)^2.\tag{2.35}$$

Denoting by $\widetilde{\mathbf{V}}_h^K$ the set of elements of $\mathbf{V}_h^K$ that satisfy (2.35), we have that in the new basis, the local stiffness matrices corresponding to a^K and to a_h^K (respectively) are both 2×2 block diagonal, one block being made of

$$a^K(\boldsymbol{p},\boldsymbol{q})\equiv a_h^K(\boldsymbol{p},\boldsymbol{q})\qquad\text{for }\boldsymbol{p}\text{ and }\boldsymbol{q}\text{ both in }(\mathbb{P}_k)^2\tag{2.36}$$

and the other block concerning, respectively,

$$a^K(\widetilde{\boldsymbol{v}}_h,\widetilde{\boldsymbol{w}}_h)\text{ and }a_h^K(\widetilde{\boldsymbol{v}}_h,\widetilde{\boldsymbol{w}}_h)\qquad\text{for }\widetilde{\boldsymbol{v}}_h\text{ and }\widetilde{\boldsymbol{w}}_h\text{ both in }\widetilde{\mathbf{V}}_h^K.\tag{2.37}$$

Note, however, that the choice of this second part for a_h cannot jeopardize the property (2.11) that has been already taken care of in (2.36) and (2.35). On the other hand, it is easy to check that the bilinear form corresponding to the block (2.37), for the form a^K, has a maximum and a minimum *positive* eigenvalue that depend (continuously) on the geometry of K but not on its size (note that, in particular, all the rigid body motions are already considered in the first block). Hence, by simply taking for the form a_h^K the block corresponding to (2.37) as the identity matrix (or, if you prefer, the identity matrix multiplied by the trace of the first block) we will have that the last property, (2.12), is also satisfied.

2.6. Construction of the loading term. In order to build the loading term $<\boldsymbol{f}_h,\boldsymbol{v}_h>$ for $\boldsymbol{v}_h\in\mathbf{V}_h$, we define $\boldsymbol{f}_h$ on each element K as the $(L^2(K)^2)$ projection of the load $\boldsymbol{f}$ on the space of piecewise polynomials of degree $\overline{k}$, where $\overline{k}:=\max\{k-2,0\}$, that is,

$$\boldsymbol{f}_h=\mathbf{\Pi}_{\overline{k}}^K\boldsymbol{f}\ ,\qquad \overline{k}=\max\{k-2,0\},\qquad\text{on each }K\in\mathcal{T}_h.$$

Then if $k\geq 2$, the associated loading term

$$<\boldsymbol{f}_h,\boldsymbol{v}_h>=\sum_{K\in\mathcal{T}_h}\int_K\boldsymbol{f}_h\cdot\boldsymbol{v}_h\mathrm{d}\mathbf{x}\equiv\sum_{K\in\mathcal{T}_h}\int_K\mathbf{\Pi}_{k-2}^K\boldsymbol{f}\cdot\boldsymbol{v}_h\mathrm{d}\mathbf{x}=\sum_{K\in\mathcal{T}_h}\int_K\boldsymbol{f}\cdot\mathbf{\Pi}_{k-2}^K\boldsymbol{v}_h\mathrm{d}\mathbf{x}$$

can be exactly computed using the degrees of freedom for $\mathbf{V}_h$ that represent the internal moments; see section 2.4. In such a case, standard L^2 orthogonality and approximation estimates on star-shaped domains yield

$$
\begin{aligned}
(2.38)\qquad < \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h > &= \sum_{K\in\mathcal{T}_h} \int_K (\boldsymbol{\Pi}^K_{k-2}\boldsymbol{f} - \boldsymbol{f})\cdot(\boldsymbol{v}_h - \boldsymbol{\Pi}_0\boldsymbol{v}_h)\mathrm{d}\mathbf{x} \\
&\le C \sum_{K\in\mathcal{T}_h} h_K^{k-1}|\boldsymbol{f}|_{k-1,K}\; h_K\|\boldsymbol{v}_h\|_{1,K} \le C\, h^k \left(\sum_{K\in\mathcal{T}_h}|\boldsymbol{f}|^2_{k-1,K}\right)^{1/2} \|\boldsymbol{v}_h\|_V
\end{aligned}
$$

and thus, recalling (2.16),

$$
\mathfrak{F} \le Ch^k \left(\sum_{K\in\mathcal{T}_h}|\boldsymbol{f}|^2_{k-1,K}\right)^{1/2}. \tag{2.39}
$$

If $k = 1$, an integration rule based on the vertex values of $\boldsymbol{v}_h$ needs to be used in order to compute $\int_K \boldsymbol{f}_h\cdot\boldsymbol{v}_h\mathrm{d}\mathbf{x} = \int_K \boldsymbol{\Pi}^K_0\boldsymbol{f}\cdot\boldsymbol{v}_h\mathrm{d}\mathbf{x}$. In this case the same procedure as in (2.38)–(2.39) gives again

$$
\mathfrak{F} \le Ch|\boldsymbol{f}|_{0,\Omega}. \tag{2.40}
$$

Remark 2.6. The estimate (2.39) could become meaningless when $\boldsymbol{f}$ does not have enough regularity, since the right-hand side becomes $+\infty$. However, with a quite similar procedure one could easily get

$$
\mathfrak{F} \le Ch^s \left(\sum_{K\in\mathcal{T}_h}|\boldsymbol{f}|^2_{s-1,K}\right)^{1/2} \qquad \text{for } 1\le s\le k. \qquad \square \tag{2.41}
$$

2.7. Estimates in the L^2 norm. In the present section we derive error estimates in the L^2 norm. We have the following result.

LEMMA 2.4. *Assume that the domain Ω is convex and $k\ge 2$. Under the same assumptions and notation of Theorem* 2.1 *the following holds. For every approximation $\boldsymbol{u}_\pi$ of $\boldsymbol{u}$ that is piecewise in $(\mathbb{P}_k)^2$, we have*

$$
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega} \le Ch\Big(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{V}} + \|\boldsymbol{u}-\boldsymbol{u}_\pi\|_{h,\mathbf{V}} + h^{\widehat{k}}\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\Omega}\Big),
$$

where $\widehat{k} = 0$ if the polynomial degree $k = 2$ and $\widehat{k} = 1$ otherwise. The constant C depends only on Ω, λ, μ, α^* and on the constants N and γ in H0.

Proof. We consider the usual auxiliary problem: Find $\boldsymbol{\psi}\in\mathbf{V}$ such that

$$
a(\boldsymbol{\psi},\boldsymbol{v}) = (\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}) \qquad \forall\boldsymbol{v}\in\mathbf{V}.
$$

Since Ω is a convex polygonal domain, the regularity of the problem guarantees that $||\boldsymbol{\psi}||_{2,\Omega} \le C||\boldsymbol{u}-\boldsymbol{u}_h||_{0,\Omega}$ with $C = C(\Omega,\mu,\lambda)$.

Let $\boldsymbol{\psi}_I$ and $\boldsymbol{\psi}_\pi$ denote approximations of ψ with $\boldsymbol{\psi}_I\in\mathbf{V}_h$ and $\boldsymbol{\psi}_\pi$ piecewise in $(\mathbb{P}_k)^2$. Standard approximation estimates combined with the above regularity result yield immediately

$$
\|\boldsymbol{\psi}-\boldsymbol{\psi}_I\|_{0,\Omega} + h\|\boldsymbol{\psi}-\boldsymbol{\psi}_I\|_{\mathbf{V}} + h\|\boldsymbol{\psi}-\boldsymbol{\psi}_\pi\|_{h,\mathbf{V}} \le Ch^2\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}. \tag{2.42}
$$

Simple manipulations, first adding and subtracting $\boldsymbol{\psi}_I$, then using (2.42), give

$$
\begin{aligned}
||\boldsymbol{u}-\boldsymbol{u}_h||^2_{0,\Omega} &= a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\psi}-\boldsymbol{\psi}_I) + a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\psi}_I) \\
&\le Ch\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{V}}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega} + a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\psi}_I).
\end{aligned} \tag{2.43}
$$

For the second term above, we note that $\boldsymbol{\psi}_I \in \mathbf{V}$ and use (2.6), add and subtract $< \boldsymbol{f}_h, \boldsymbol{\psi}_I >$, and finally apply (2.15). We get

$$a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{\psi}_I) =< \boldsymbol{f}-\boldsymbol{f}_h, \boldsymbol{\psi}_I > + \Big(a_h(\boldsymbol{u}_h, \boldsymbol{\psi}_I) - a(\boldsymbol{u}_h, \boldsymbol{\psi}_I) \Big) =: T_1 + T_2. \tag{2.44}$$

In order to bound the term T_1, we first add and subtract $\boldsymbol{\psi}$, then use the orthogonality property of $\boldsymbol{f}-\boldsymbol{f}_h$. Following the same notation as in section 2.6 we obtain

$$T_1 = \sum_{K\in\mathcal{T}_h} \left(\int_K (\boldsymbol{f}-\boldsymbol{f}_h)(\boldsymbol{\psi}_I - \boldsymbol{\psi})\mathrm{d}\mathbf{x} + \int_K (\boldsymbol{f}-\boldsymbol{f}_h)(\boldsymbol{\psi} - \boldsymbol{\Pi}^K_{\overline{k}}\boldsymbol{\psi})\mathrm{d}\mathbf{x} \right),$$

where we recall that $\overline{k} = \max\{k-2, 0\}$.

Cauchy–Schwarz inequalities, bound (2.42), standard approximation estimates for the operator $\boldsymbol{\Pi}^K_{\overline{k}}$, and the regularity result for $\boldsymbol{\psi}$ yield

$$\begin{aligned} T_1 &\le \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\Omega}\big(\|\boldsymbol{\psi}-\boldsymbol{\psi}_I\|_{0,\Omega} + \|\boldsymbol{\psi}-\boldsymbol{\Pi}^K_{\overline{k}}\boldsymbol{\psi}\|_{0,\Omega}\big) \\ &\le C(h^2 + h^{\min(2,\overline{k}+1)})\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\Omega}||\boldsymbol{u}-\boldsymbol{u}_h||_{0,\Omega} \\ &\le C(h^{\widehat{k}+1})\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\Omega}||\boldsymbol{u}-\boldsymbol{u}_h||_{0,\Omega}. \end{aligned} \tag{2.45}$$

For the second term in (2.44) we use the consistency property H2 twice, thus obtaining

$$\begin{aligned} T_2 &= \sum_{K\in\mathcal{T}_h} \big(a_h^K(\boldsymbol{u}_h, \boldsymbol{\psi}_I) - a^K(\boldsymbol{u}_h, \boldsymbol{\psi}_I)\big) \\ &= \sum_{K\in\mathcal{T}_h} \big(a_h^K(\boldsymbol{u}_h, \boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi) - a^K(\boldsymbol{u}_h, \boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi)\big) \\ &= \sum_{K\in\mathcal{T}_h} \big(a_h^K(\boldsymbol{u}_h - \boldsymbol{u}_\pi, \boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi) - a^K(\boldsymbol{u}_h - \boldsymbol{u}_\pi, \boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi)\big). \end{aligned}$$

From the above bound, the continuity of the bilinear form a and (2.13) yield

$$T_2 \le C \left(\sum_{K\in\mathcal{T}_h} |\boldsymbol{u}_h - \boldsymbol{u}_\pi|^2_{1,K} \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} |\boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi|^2_{1,K} \right)^{1/2}.$$

By a triangle inequality it now follows that

$$\left(\sum_{K\in\mathcal{T}_h} |\boldsymbol{u}_h - \boldsymbol{u}_\pi|^2_{1,K} \right)^{1/2} \le C \left(\sum_{K\in\mathcal{T}_h} |\boldsymbol{u} - \boldsymbol{u}_h|^2_{1,K} + |\boldsymbol{u} - \boldsymbol{u}_\pi|^2_{1,K} \right)^{1/2},$$

and, by the same argument and using (2.42),

$$\left(\sum_{K\in\mathcal{T}_h} |\boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi|^2_{1,K} \right)^{1/2} \le Ch\, |\boldsymbol{\psi}|_{2,\Omega} \le C\, h\, ||\boldsymbol{u}-\boldsymbol{u}_h||_{0,\Omega}.$$

Combining the above bounds we obtain for T_2

$$T_2 \le Ch\, \big(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{V}} + \|\boldsymbol{u}-\boldsymbol{u}_\pi\|_{h,\mathbf{V}}\big)\, ||\boldsymbol{u}-\boldsymbol{u}_h||_{0,\Omega}. \tag{2.46}$$

The final bound follows applying bounds (2.43), (2.44), (2.45), and (2.46). ∎

By application of standard approximation estimates and Theorem 2.1, if $k \geq 3$ from Lemma 2.4 it immediately follows that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq Ch^{k+1}|\boldsymbol{u}|_{k+1,\Omega}, \tag{2.47}$$

provided the solution $\boldsymbol{u}$ is sufficiently regular.

In the case $k = 1$ an analogous result can be derived. The steps are essentially identical to the ones in the proof of Lemma 2.4, the only difference being in the treatment of the loading term T_1. Assuming that the integration rule used for the evaluation of the loading is at least of first order (see section 2.6), one finally obtains

$$\begin{aligned}\|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega} &\leq Ch\big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega} + h\|\boldsymbol{f}\|_{0,\Omega}\big) \\ &\leq Ch^2 \left(|\boldsymbol{u}|^2_{2,\Omega} + \sum_{K\in\mathcal{T}_h} |\boldsymbol{f}|^2_{1,K}\right)^{1/2},\end{aligned}$$

where the last bound holds if $\boldsymbol{u}$ is sufficiently regular.

In the case $k = 2$, the proposed method does not guarantee a $O(h^{k+1})$ convergence rate in the L^2 norm. Such result is sharp and is not related to some theoretical limitation in the proof of Lemma 2.4. Indeed, the reason for this nonoptimal behavior is related to the load approximation, as is clear from the proof above, and can be cured in the following way.

It is immediate to check that for any $\boldsymbol{v}_h \in \mathbf{V}_h$ and $K \in \mathcal{T}_h$, the (componentwise) averages

$$\overline{\boldsymbol{v}}_h = \left(\int_K \boldsymbol{v}_h \mathrm{d}\mathbf{x}\right) \Big/ |K| \ , \quad \overline{\boldsymbol{\nabla} \boldsymbol{v}_h} = \left(\int_K \boldsymbol{\nabla} \boldsymbol{v}_h \mathrm{d}\mathbf{x}\right) \Big/ |K|$$

can be exactly computed using the degree of freedom of $\mathbf{V}_h$, where in the case of the gradient an integration by parts needs to be used.

Therefore, denoting by $\mathbf{x}_B$ the coordinates of the barycenter of K, we can define

$$\begin{aligned}&\widetilde{\boldsymbol{v}}_h = \overline{\boldsymbol{v}}_h + \overline{\boldsymbol{\nabla} \boldsymbol{v}_h} \cdot (\mathbf{x} - \mathbf{x}_B), \\ &< \boldsymbol{f}_h, \boldsymbol{v}_h > = \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\Pi}_1^K \boldsymbol{f} \cdot \widetilde{\boldsymbol{v}}_h \mathrm{d}\mathbf{x}.\end{aligned} \tag{2.48}$$

Then we have

(2.49)

$$\begin{aligned}< \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h > &= \sum_{K\in\mathcal{T}_h} \int_K \big(\boldsymbol{\Pi}_1^K \boldsymbol{f} \cdot \widetilde{\boldsymbol{v}}_h - \boldsymbol{f} \cdot \boldsymbol{v}_h\big) \mathrm{d}\mathbf{x} \\ &= \sum_{K\in\mathcal{T}_h} \int_K \big(\boldsymbol{\Pi}_1^K \boldsymbol{f} \cdot (\widetilde{\boldsymbol{v}}_h - \boldsymbol{v}_h) + (\boldsymbol{\Pi}_1^K \boldsymbol{f} - \boldsymbol{f}) \cdot \boldsymbol{v}_h\big) \mathrm{d}\mathbf{x} \\ &= \sum_{K\in\mathcal{T}_h} \int_K \big((\boldsymbol{\Pi}_1^K \boldsymbol{f} - \boldsymbol{\Pi}_0^K \boldsymbol{f}) \cdot (\widetilde{\boldsymbol{v}}_h - \boldsymbol{v}_h) + (\boldsymbol{\Pi}_1^K \boldsymbol{f} - \boldsymbol{f}) \cdot (\boldsymbol{v}_h - \widetilde{\boldsymbol{v}}_h)\big) \mathrm{d}\mathbf{x} \\ &\leq C \sum_{K\in\mathcal{T}_h} \big(\|\boldsymbol{\Pi}_1^K \boldsymbol{f} - \boldsymbol{\Pi}_0^K \boldsymbol{f}\|_{0,K} + \|\boldsymbol{\Pi}_1^K \boldsymbol{f} - \boldsymbol{f}\|_{0,K}\big) \|\widetilde{\boldsymbol{v}}_h - \boldsymbol{v}_h\|_{0,K} \\ &\leq C\, h^3 \left(\sum_{K\in\mathcal{T}_h} |\boldsymbol{f}|^2_{1,K}\right)^{1/2} |\boldsymbol{v}_h|_{2,K}.\end{aligned}$$

Clearly, $O(h^3)$ is more than we need for the error estimate in $\mathbf{V}$, and this is why we did not use it in section 2.6. Moreover, although easily computable, the integral in (2.48) is not immediate as the one proposed in section 2.6. Now, proceeding as in (2.49) we have

$$\begin{aligned} T_1 &= \sum_{K\in\mathcal{T}_h} \int_K \big(\mathbf{\Pi}_1^K \boldsymbol{f}\cdot\widetilde{\boldsymbol{\psi}}_I - \boldsymbol{f}\cdot\boldsymbol{\psi}_I\big)\mathrm{d}\mathbf{x} \\ &\le C\,h^3\left(\sum_{K\in\mathcal{T}_h}|\boldsymbol{f}|^2_{1,K}\right)^{1/2}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}. \end{aligned} \tag{2.50}$$

3. Nearly incompressible materials. As is well known, we say that the material is *nearly incompressible* when $\lambda \gg \mu$. It is equally well known that in such a case it is more convenient (for the analysis and the discretization of the problem) to relax the incompressibility constraint introducing a projection operator or, equivalently, to shift to the so-called $(\boldsymbol{u},p)$-formulation. In the present section we shall follow the first option.

In order to understand what happens in the nearly incompressible case we will assume that $\lambda \ge \mu$ and we shall derive error bounds which do not explode when $\lambda \to +\infty$. Let hereafter Π_{k-1} denote the L^2 projection operator on the auxiliary space

$$Q_h := \{q \in L^2(\Omega)/\mathbb{R} \text{ such that } q_{|K} \in \mathbb{P}_{k-1}(K) \quad \forall\, K\in\mathcal{T}_h\}. \tag{3.1}$$

We define, for all $\boldsymbol{u}_h, \boldsymbol{v}_h \in \mathbf{V}_h$,

$$\begin{aligned} a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) &= \sum_{K\in\mathcal{T}_h} a_h^K(\boldsymbol{u}_h,\boldsymbol{v}_h), \\ a_h^K(\boldsymbol{u}_h,\boldsymbol{v}_h) &= 2\mu a_{\mu,h}^K(\boldsymbol{u}_h,\boldsymbol{v}_h) + \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}_h, \Pi_{k-1}\mathrm{div}\boldsymbol{v}_h)_K \quad \forall K\in\mathcal{T}_h, \end{aligned} \tag{3.2}$$

where the bilinear forms $a_{\mu,h}^K(\boldsymbol{u}_h,\boldsymbol{v}_h)$ are constructed with the strategy that we followed to construct the bilinear forms a_h^K in section 2.5. (Think, for instance, of what you would get in the previous case if λ was equal to zero.)

Following the obvious notation

$$a_\mu(\boldsymbol{u},\boldsymbol{v}) = \sum_{K\in\mathcal{T}_h} a_\mu^K(\boldsymbol{u},\boldsymbol{v}) \quad \forall \boldsymbol{u},\boldsymbol{v}\in\mathbf{V}, \tag{3.3}$$

the local bilinear forms $a_{\mu,h}^K(\boldsymbol{u}_h,\boldsymbol{v}_h)$ satisfy an analogous version of (2.11)–(2.12):

H2. For all h and for all K in $\mathcal{T}_h$,

- $\forall \boldsymbol{p}\in(\mathbb{P}_k)^2$, $\forall \boldsymbol{v}_h\in\mathbf{V}_h$;

$$a_{\mu,h}^K(\boldsymbol{p},\boldsymbol{v}_h) = a_\mu^K(\boldsymbol{p},\boldsymbol{v}_h); \tag{3.4}$$

- $\exists$ two positive constants α_* and α^*, independent of h,μ and of K, such that

$$\forall \boldsymbol{v}_h\in\mathbf{V}_h \qquad \alpha_*\, a_\mu^K(\boldsymbol{v}_h,\boldsymbol{v}_h) \le a_{\mu,h}^K(\boldsymbol{v}_h,\boldsymbol{v}_h) \le \alpha^*\, a_\mu^K(\boldsymbol{v}_h,\boldsymbol{v}_h). \tag{3.5}$$

We note now that for every $\boldsymbol{v}_h\in\mathbf{V}_h$ and for every $q\in Q_h$ the integral

$$\int_K \mathrm{div}\,\boldsymbol{v}_h\, q\mathrm{d}\mathbf{x} = \int_{\partial K} \boldsymbol{v}_h\cdot\mathbf{n}\, q\mathrm{d}s - \int_K \boldsymbol{v}_h\cdot\nabla q\mathrm{d}\mathbf{x} \tag{3.6}$$

is computable once we know $\boldsymbol{v}_h$ at the boundary and the projection of $\boldsymbol{v}_h$ on $(\mathbb{P}_{k-2})^2$ (that is, $\boldsymbol{\Pi}_{k-2}\boldsymbol{v}_h$). Therefore the projection operator Π_{k-1} appearing in (3.2) is computable for any function in $\mathbf{V}_h$ by using the available degrees of freedom.

The discrete problem reads as (2.15) with the bilinear form given now by (3.2). In order to study the convergence properties of the proposed method we consider $k \geq 2$ and prove a discrete inf-sup condition:

$$\exists\, \beta^* > 0 \text{ such that } \forall\, h \inf_{q \in Q_h} \sup_{\boldsymbol{v} \in \mathbf{V}_h} \frac{(\mathrm{div}\boldsymbol{v}, q)}{\|q\|_Q \, \|\boldsymbol{v}\|_{\mathbf{V}}} \geq \beta^* > 0. \tag{3.7}$$

We have the following theorem.

THEOREM 3.1. *Let $k \geq 2$. In the above assumptions,* (3.7) *holds true.*

Proof. We recall the continuous inf-sup condition (see, for instance, [12]),

$$\exists\, \beta > 0 \text{ such that } \inf_{q \in Q} \sup_{\boldsymbol{v} \in \mathbf{V}} \frac{(\mathrm{div}\boldsymbol{v}, q)}{\|q\|_Q \, \|\boldsymbol{v}\|_{\mathbf{V}}} \geq \beta > 0\ , \quad Q = L^2(\Omega). \tag{3.8}$$

For every $q^* \in Q_h \subset Q$, using (3.8) we have that there exists a $\boldsymbol{w} \in \mathbf{V}$ such that

$$2 \frac{(\mathrm{div}\boldsymbol{w}, q^*)}{\|q^*\|_Q \, \|\boldsymbol{w}\|_{\mathbf{V}}} \geq \beta > 0. \tag{3.9}$$

Now, in order to use the so-called Fortin's trick, we want to construct a $\boldsymbol{w}_h \in \mathbf{V}_h$ such that

$$(\mathrm{div}(\boldsymbol{w} - \boldsymbol{w}_h), q) = 0 \quad \forall\, q \in Q_h \tag{3.10}$$

with

$$\|\boldsymbol{w}_h\|_{\mathbf{V}} \leq C_F \, \|\boldsymbol{w}\|_{\mathbf{V}} \tag{3.11}$$

for some constant C_F independent of h. For this, we proceed as in the classical finite element theory (see, e.g., [12]). We first construct a $\overline{\boldsymbol{w}}_h \in \mathbf{V}_h$ such that

$$(\mathrm{div}(\boldsymbol{w} - \overline{\boldsymbol{w}}_h), \overline{q}) = 0 \quad \forall\, \overline{q} \text{ piecewise constant in } \Omega \tag{3.12}$$

with

$$\|\boldsymbol{w} - \overline{\boldsymbol{w}}_h\|_{r,K} \leq C\, h_K^{1-r} \|\boldsymbol{w}\|_{1,K}\, \forall\, K \in \mathcal{T}_h, \qquad r = 0, 1, \tag{3.13}$$

with a constant C that does not depend on h. Note that for doing this, as is well known, we must use in an essential way the degree of freedom "average of the normal component of $\overline{\boldsymbol{w}}$ on each edge of K," which allows us to enforce

$$\int_e (\boldsymbol{w} - \overline{\boldsymbol{w}}_h) \cdot \mathbf{n}\, \mathrm{d}s = 0\, \forall \text{ edge } e \text{ in } \mathcal{T}_h, \tag{3.14}$$

and this is why we require $k \geq 2$. Once $\overline{\boldsymbol{w}}_h$ is constructed, always following the finite element track, we choose a "bubble" $\widetilde{\boldsymbol{w}}_h$ having all the degrees of freedom on each ∂K equal to zero and relying only, in each K, on the $k(k-1)$ internal degrees of freedom, such that on each K

$$(\mathrm{div}\, \widetilde{\boldsymbol{w}}_h, q)_{0,K} \equiv -(\widetilde{\boldsymbol{w}}_h, \nabla q)_{0,K} = (\mathrm{div}(\boldsymbol{w} - \overline{\boldsymbol{w}}_h), q)_{0,K} \quad \forall\, q \in (Q_h)_{|K}. \tag{3.15}$$

We note that (3.15) amounts to $k(k+1)/2-1$ conditions (as the dimension of the space of gradients of polynomials of degree $\le k-1$). We note that this is $\le k(k-1)$ (the number of internal degrees of freedom for the space $\mathbf{V}_h^K$) for $k \ge 2$. An additional scaling argument, together with (3.13), shows that we also have

$$\|\widetilde{\boldsymbol{w}}_h\|_{r,K} \le C\, h_K^{1-r} \|\boldsymbol{w}\|_{1,K}\ \forall\, K \in \mathcal{T}_h, \qquad r = 0, 1. \tag{3.16}$$

Finally we set

$$\boldsymbol{w}_h := \overline{\boldsymbol{w}}_h + \widetilde{\boldsymbol{w}}_h$$

and we note that (3.15) implies (3.10), while (3.13) and (3.16) imply (3.11). Finally from (3.9), (3.10), and (3.11) we have, as usual,

$$2\frac{(\mathrm{div}\boldsymbol{w}_h, q^*)}{\|q^*\|_Q\ \|\boldsymbol{w}_h\|_{\mathbf{V}}} \ge \frac{2}{C_F}\frac{(\mathrm{div}\boldsymbol{w}_h, q^*)}{\|q^*\|_Q\ \|\boldsymbol{w}\|_{\mathbf{V}}} = \frac{2}{C_F}\frac{(\mathrm{div}\boldsymbol{w}, q^*)}{\|q^*\|_Q\ \|\boldsymbol{w}\|_{\mathbf{V}}} \ge \beta^*, \tag{3.17}$$

which gives (3.7) with $\beta^* = 2\beta/C_F$. ∎

As a consequence of the inf-sup condition (3.7) we have with classical arguments (see, for instance, Proposition 2.5 in [12]) the following property: *For all smooth enough vector valued function* $\boldsymbol{u}$, $\boldsymbol{u}_I \in V_h$ *exists such that*

$$\Pi_{k-1}\mathrm{div}\,\boldsymbol{u}_I = \Pi_{k-1}\mathrm{div}\,\boldsymbol{u}, \tag{3.18}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_I\|_{\mathbf{V}} \le C \inf_{\boldsymbol{v}_h \in \mathbf{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathbf{V}}. \tag{3.19}$$

We are now able to prove the following result.

THEOREM 3.2. *Under the assumptions* H0*,* H1*, and* H2 *the discrete problem* (2.15) *with bilinear form* (3.2) *has a unique solution* $\boldsymbol{u}_h$*. Moreover, let* $\boldsymbol{u}_I$ *be the interpolant of* $\boldsymbol{u}$ *defined in* (3.18)–(3.19)*. Then for every approximation* $\boldsymbol{u}_\pi$ *of* $\boldsymbol{u}$ *that is piecewise in* $(\mathbb{P}_k)^2$*, we have*

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathbf{V}} \le C\big(\|\boldsymbol{u} - \boldsymbol{u}_I\|_{\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \|p - \Pi_{k-1}p\|_Q + \mathfrak{F}\big), \tag{3.20}$$

where C *is a constant depending only on* μ, α_*, α^**, where* $\mathfrak{F}$ *is still defined by* (2.16) *and where*

$$p := \lambda\,\mathrm{div}\,\boldsymbol{u} \tag{3.21}$$

is the "pressure."

Proof. The existence of a unique $\boldsymbol{u}_h$ follows immediately from the definite positive property of the discrete bilinear form (see (3.5)) on the space with boundary conditions $\mathbf{V}_h$. Setting $\boldsymbol{\delta}_h := \boldsymbol{u}_h - \boldsymbol{u}_I$ we have

$$\begin{aligned}
\alpha_*\, 2\mu\|\boldsymbol{\delta}_h\|_{\mathbf{V}}^2 &\le \alpha_* 2\mu\, a_\mu(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \le a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) \\
&= a_h(\boldsymbol{u}_h, \boldsymbol{\delta}_h) - a_h(\boldsymbol{u}_I, \boldsymbol{\delta}_h) \text{ (use (2.15) and (3.2))} \\
&= < \boldsymbol{f}_h, \boldsymbol{\delta}_h > - \sum_K \big(2\mu a_{\mu,h}^K(\boldsymbol{u}_I, \boldsymbol{\delta}_h) \\
&\quad + \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}_I, \Pi_{k-1}\mathrm{div}\boldsymbol{\delta}_h)_K\big) \text{ (use } \pm\, \boldsymbol{u}_\pi \text{ and (3.18))} \\
&= < \boldsymbol{f}_h, \boldsymbol{\delta}_h > -2\mu \sum_K \big(a_{\mu,h}^K(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a_{\mu,h}^K(\boldsymbol{u}_\pi, \boldsymbol{\delta}_h)\big)
\end{aligned} \tag{3.22}$$

$$
\begin{aligned}
&- \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{\delta}_h)_\Omega \text{ (use (3.4))}\\
=&< \boldsymbol{f}_h, \boldsymbol{\delta}_h > -2\mu \sum_K \left(a^K_{\mu,h}(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K_\mu(\boldsymbol{u}_\pi, \boldsymbol{\delta}_h)\right)\\
&- \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{\delta}_h)_\Omega \text{ (use } \pm u \text{ and (3.3))}\\
=&< \boldsymbol{f}_h, \boldsymbol{\delta}_h > -2\mu \sum_K \left(a^K_{\mu,h}(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h) + a^K_\mu(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{\delta}_h)\right)\\
&- 2\mu a_\mu(\boldsymbol{u}, \boldsymbol{\delta}_h) - \lambda(\Pi_{k-1}\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{\delta}_h)_\Omega \text{ (use (2.2), (2.6), and (3.21))}\\
=&< \boldsymbol{f}_h, \boldsymbol{\delta}_h > -2\mu \sum_K \Big(a^K_{\mu,h}(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h)\\
&+ a^K_\mu(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{\delta}_h)\Big) - < \boldsymbol{f}, \boldsymbol{\delta}_h > -(\Pi_{k-1}p - p, \mathrm{div}\boldsymbol{\delta}_h)_{0,\Omega}\\
=&< \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{\delta}_h > -2\mu \sum_K \Big(a^K_{\mu,h}(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{\delta}_h)\\
&+ a^K_\mu(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{\delta}_h)\Big) - (\Pi_{k-1}p - p, \mathrm{div}\boldsymbol{\delta}_h)_{0,\Omega}.
\end{aligned}
$$

Recalling (2.16), (3.5), and the continuity of each a^K_μ in (3.22) easily yields

$$
\|\boldsymbol{\delta}_h\|^2_{\mathbf{V}} \le C\big(\mathfrak{F} + \|\boldsymbol{u}_I - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \|\boldsymbol{u} - \boldsymbol{u}_\pi\|_{h,\mathbf{V}} + \|\Pi_{k-1}p - p\|_{0,\Omega}\big)\, \|\boldsymbol{\delta}_h\|_{\mathbf{V}} \tag{3.23}
$$

for some constant C depending only on μ, α_*, α^*. Then the result follows easily by the triangle inequality. ∎

Combining the above error bound with (3.19), (2.29), (2.39), and standard approximation results on star-shaped domains gives the following uniform convergence estimate.

Remark 3.1. We point out that using the inf-sup (3.8) one deduces that the "pressure" $p = \lambda\boldsymbol{u}$ remains uniformly bounded in λ when $\lambda \to +\infty$. Indeed, from (3.9) and then (2.2) one has, for some $\boldsymbol{w} \in \mathbf{V}$ different from zero, that

$$
\begin{aligned}
\frac{\beta}{2}\|p\|\,\|\boldsymbol{w}\|_{\mathbf{V}} &\le (\mathrm{div}\boldsymbol{w}, p) = \lambda(\mathrm{div}\boldsymbol{w}, \mathrm{div}\boldsymbol{u})\\
&=< \boldsymbol{f}, \boldsymbol{w} > -2\mu a_\mu(\boldsymbol{u}, \boldsymbol{w}) \le (\|\boldsymbol{f}\|_{0,\Omega} + 2\mu\|\boldsymbol{u}\|_{\mathbf{V}})\|\boldsymbol{w}\|_{\mathbf{V}},
\end{aligned}
$$

as classical in nearly incompressible elasticity (see, for instance, [12]).

Remark 3.2. Under the same assumptions of Theorem 3.2 it holds that

$$
\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathbf{V}} \le Ch^{s-1}\big(|\boldsymbol{u}|_{s,\Omega} + |p|_{s-1,\Omega}\big), \quad 1 \le s \le k+1,
$$

with C independent of the material constants and on the mesh $\mathcal{T}_h$.

4. Numerical tests. In this section we present some simple numerical tests for the method proposed in this paper, restricted to the lower order case $k = 1$ for compressible elasticity. In all the tests we take the material constant values $\mu = \lambda = 1$.

The local stiffness matrices $\mathsf{M} \in \mathbb{R}^{N^K \times N^K}$ corresponding to the local bilinear forms $a^K_h(\cdot,\cdot)$, $K \in \mathcal{T}_h$, are implemented following the guidelines in [14, 7]. We here describe such construction very briefly. Any symmetric bilinear form satisfying the conditions of consistency (2.11) and stability (2.12) can be written as

$$
a^K_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = a^K(\Pi^E\boldsymbol{u}_h, \Pi^E\boldsymbol{v}_h) + b_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \qquad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in \mathbf{V}_h, \tag{4.1}
$$

where Π^E is the (energy) projection on the space of polynomials $(\mathbb{P}_k)^2$ and $b_h(\cdot,\cdot)$ is a symmetric and positive semidefinite bilinear form with kernel given by $(\mathbb{P}_k)^2$. It turns out that the first bilinear form in (4.1) is explicitly computable as a matrix M_0, so that the matrix M can be constructed as

$$\mathsf{M} = \mathsf{M}_0 + \mathsf{M}_1,$$

where M_1 corresponds to $b_h(\cdot,\cdot)$ and can be built in various ways. Here we follow the simple choice of taking

$$\mathsf{M}_1 = \alpha(\mathsf{I} - \mathsf{P}) \tag{4.2}$$

with I the identity matrix, P a projection matrix on the polynomial space $(\mathbb{P}_k)^2$, and $\alpha \in \mathbb{R}^+$ an arbitrary constant, typically chosen as

$$\alpha = \frac{1}{2}\mathrm{trace}(\mathsf{M}_0). \tag{4.3}$$

Test case 1. We consider the square domain $\Omega = [0,1]^2$ and problem (2.6) with homogeneous Dirichlet boundary conditions with the load $\boldsymbol{f}$ taken accordingly to the solution

$$u(x,y) = \big(\sin(\pi x)\sin(\pi y)\,,\ \sin(\pi x)\sin(\pi y)\big).$$

We implement the VEM with α as in (4.3) for two different families of meshes. The first family is composed of square meshes of 4×4, 8×8, 16×16, and 32×32 elements, respectively. The second family of meshes is composed of four regular hexagonal grids, where each successive mesh is roughly the half the previous one. (An example is shown in Figure 4.1, left.)

In Table 4.1 we present the relative errors in the maximum and energy discrete norms

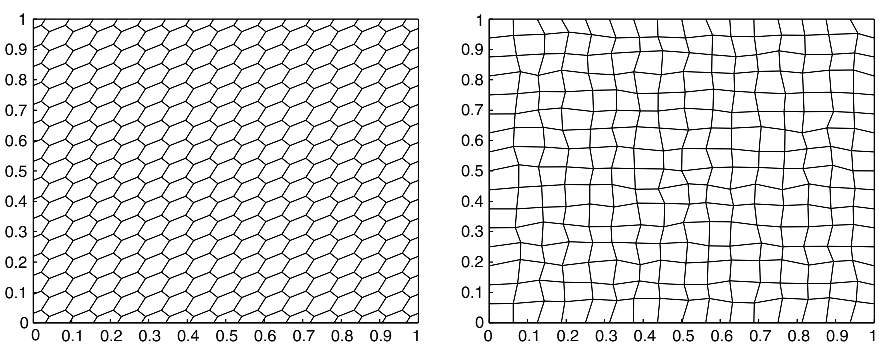

FIG. 4.1. *Sample of hexagonal mesh (left) and randomly distorted quadrilateral mesh (right).*

TABLE 4.1
Test case 1. *Relative error in discrete maximum and energy norms.*

	Square meshes			Hexagonal meshes		
Mesh #	h	E_∞	E_{ene}	h	E_∞	E_{ene}
1	1/4	4.84e-2	5.70e-2	1/5	3.30e-2	6.44e-2
2	1/8	1.08e-2	1.48e-2	1/9	1.24e-2	2.08e-2
3	1/16	2.64e-3	3.76e-3	1/17	3.54e-3	6.88e-3
4	1/32	6.56e-4	9.45e-4	1/33	9.43e-4	2.37e-3

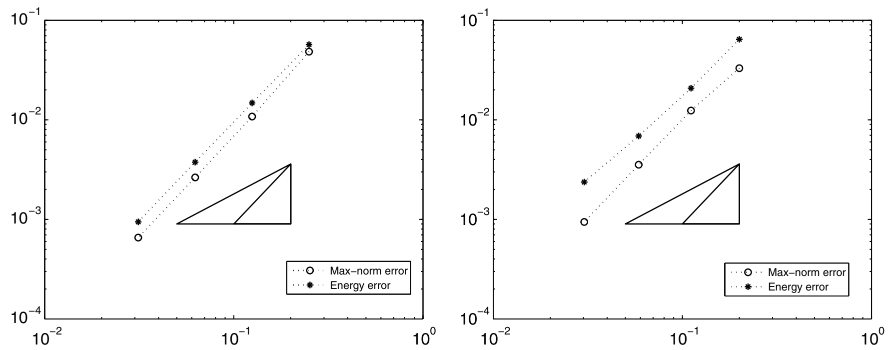

FIG. 4.2. *Test case* 1*. Logarithmic plot of the relative error in the discrete energy (stars) and maximum (circles) norms for square (left) and hexagonal (right) meshes. We plot also the $O(h)$ and $O(h^2)$ slopes for comparison.*

$$E_\infty = \frac{\max_{\nu\in\mathcal{V}_h} ||\boldsymbol{u}_h(\nu) - \boldsymbol{u}(\nu)||}{\max_{\nu\in\mathcal{V}_h} ||\boldsymbol{u}(\nu)||}\ , \qquad E_{ene} = \frac{a^h(\boldsymbol{u}_h - \boldsymbol{u}_I, \boldsymbol{u}_h - \boldsymbol{u}_I)^{1/2}}{a^h(\boldsymbol{u}_I, \boldsymbol{u}_I)^{1/2}},$$

where $\boldsymbol{u}_I \in \mathbf{V}_h$ is the standard interpolant of $\boldsymbol{u}$ and $\mathcal{V}_h$ represents the set of all vertices of the mesh. In the table we indicate with h the mesh size given by $|\Omega|^{1/2} = 1$ divided by the square root of the number of elements. We plot the results of Table 4.1 in Figure 4.2.

From the results shown one can appreciate the good behavior of the method and in particular the convergence of order $O(h^2)$ in the discrete maximum norm. In the discrete energy norm the convergence rate is $O(h^2)$ for the square grids and almost that same rate for the hexagonal meshes; this is clearly a superconvergence phenomenon due to the regularity of the meshes, the smoothness of the solution, and the type of error considered.

Test case 2. We consider the same domain $\Omega = [0,1]^2$ and apply a constant load $\boldsymbol{f} = (1,0)$, still with homogeneous Dirichlet boundary conditions. We build a family of four distorted quadrilateral meshes obtained by the square grids introduced previously and randomly perturbing all internal vertices, except the central one $C = (1/2, 1/2)$. We therefore obtain four meshes with 4×4, 8×8, 16×16 and 32×32 quadrilateral elements, respectively. (See, as an example, Figure 4.1, right.) Here too α is chosen as in (4.3).

In Table 4.2 we write the relative error in the center point

$$E_c = \frac{||u_h(C) - u(C)||}{||u(C)||},$$

where we take as $u(C)$ the value obtained with a 96×96 square mesh. From the results we can see that the convergence rate of the scheme in terms of pointwise error is close to $O(h^2)$.

TABLE 4.2
Test case 2*. Relative error in the center point.*

Mesh	4×4	8×8	16×16	32×32
E_c	3.98e-2	1.62e-2	3.76e-3	1.13e-3

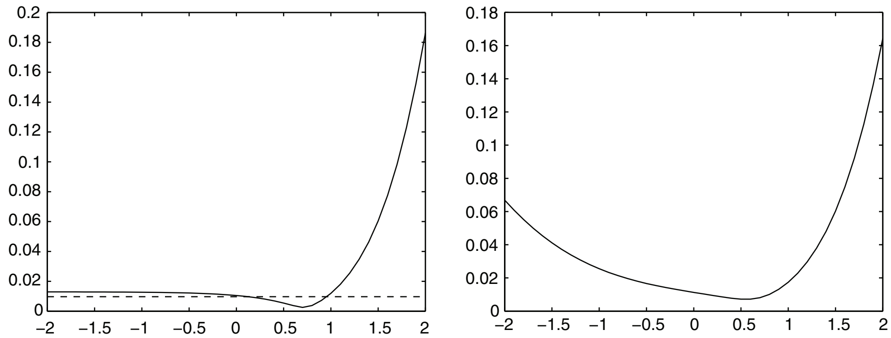

FIG. 4.3. *Test case* 3. *Behavior of the discrete relative energy error versus* $Log_{10}\,\alpha$ *for a square mesh (left) and a hexagonal mesh (right).*

Test case 3. We consider the same problem as in test case 1 but for a fixed mesh. Namely, we consider the mesh number 3 of both families, i.e., the 16×16 square grid and the hexagonal mesh depicted in Figure 4.1, left. We then study the behavior of the discrete relative energy error E_{ene} in dependence of the parameter α in (4.2). We plot the result in Figure 4.3 for the square mesh on the left and for the hexagonal one on the right. The parameter α varies from 10^{-2} to 10^2. In the left figure we also plot a horizontal dashed line, representing the error obtained on the same grid for the Q_1 (bilinear) finite element method. Note that there is no value of α that reproduces exactly Q_1 finite element method; in order to reproduce the stiffness matrix of the Q_1 element one would need to use a choice for M_1 different from (4.2). We moreover observe that the value of α that corresponds to the choice (4.3) is $\alpha = 4$ for the square case and $\alpha \simeq 4.38$ for the hexagonal one (when considering internal regular hexagons; for the boundary ones with a different shape it is slightly different). In the semilogarithmic plot shown in the figure such numbers correspond roughly to 0.602 and 0.641, respectively. We note that these values of the parameter are very near to the point of minimum for the error. This indicates, at least on the basis of the present test, that the above choice based on the trace of M_0 is a good one. Moreover, observe that the error is in general not too sensitive with respect to α. Indeed, varying α all the way from 0.1 to 10 the results are almost the same. Finally, for the square grid the error does not explode when $\alpha \to 0$ since in this case the global stiffness matrix turns out to be positive definite also for $\alpha = 0$. This is not the case for the hexagonal mesh.

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