DOMAIN DECOMPOSITION METHODS

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1. Introduction

The aim of this paper is to present a three-field formulation for linear elliptic problems which is particularly well suited for domain decomposition methods. The formulation is inspired by the hybrid formulation of Tong [4] for elasticity problems, the main difference being that we work here at the macro-element (≡subdomain) level instead of working at the element level. The effect of this is that we obtain a new formulation of the continuous problem which can then be discretized in many different ways, including the possibility of using different methods (or the same method with different meshes) from one subdomain to another. Hence, we get a fairly general framework in which most of the domain decomposition methods using non overlapping subdomains can be reinterpreted. At the end of the paper we also propose a "continuous pre-conditioner" that seems particularly appealing for dealing with non symmetric problems and unstructured decompositions.

2. The three-field formulation

Let us consider, for the sake of simplicity, a polygonal domain $\Omega \subset \mathbf{R}^2$ split into a finite number of polygonal subdomains Ω_k $(k = 1, .., K)$:

$$\Omega = \overset{\circ}{\overline{\bigcup_k \Omega_k}}, \tag{2.1}$$

and define

$$\Gamma_k = \partial\Omega_k \quad ; \quad \Sigma = \bigcup_k \Gamma_k. \tag{2.2}$$

Let A be a linear elliptic operator of the form

$$Au = \sum_i \{\sum_j (-\frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u}{\partial x_i} + b_j(x)u) + c_i(x)\frac{\partial u}{\partial x_i}\} + d(x)u. \tag{2.3}$$

in each Ω_k, and we consider the bilinear forms associated with A in each Ω_k, that is,

$$\begin{aligned} & for\ u,\, v \in\ H^1(\Omega_k) : \\ & a_k(u,\, v) := \int_{\Omega_k} \{ \sum_i (\sum_j (a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + b_j u \frac{\partial v}{\partial x_j}) + c_i \frac{\partial u}{\partial x_i} v) + duv \} \, dx. \end{aligned} \tag{2.4}$$

We also set, for $u,\, v \in \prod_k H^1(\Omega_k)$

$$a(u,\, v) := \sum_k a_k(u,\, v); \tag{2.5}$$

for the sake of simplicity we also assume that there exists a constant $\alpha > 0$ such that

$$a(v,\, v) \geq \alpha ||v||^2_{H^1(\Omega)} \qquad \forall v \in H^1_0(\Omega). \tag{2.6}$$

From now on we are going to use the following notation: $(.,.)$ will be the usual inner product in $L^2(\Omega)$; for $k = 1, ...K$, $(.,.)_k$ will be the inner product in $L^2(\Omega_k)$ and $< .,. >_k$ will be the inner product in $L^2(\Gamma_k)$ (or, when necessary, the duality pairing between $H^{-\frac{1}{2}}(\Gamma_k)$ and $H^{\frac{1}{2}}(\Gamma_k)$). Similarly, we will use $||.||_s$ for the $H^s(\Omega)$ norm, and $||.||_{s,k}$, $|||.|||_{s,k}$ for the $H^s(\Omega_k)$ and $H^s(\Gamma_k)$ norms respectively ($k = 1, ..., K$). Let us now introduce the spaces that will be used in our macro-hybrid formulation. For $k = 1, ..., K$ we set

$$V_k := H^1(\Omega_k) \qquad ; \qquad \overset{\circ}{V}_k := V_k \bigcap H^1_0(\Omega_k); \tag{2.7}$$

$$M_k := H^{-\frac{1}{2}}(\Gamma_k). \tag{2.8}$$

We then define

$$V := \prod_k V_k, \tag{2.9}$$

$$M := \prod_k M_k, \tag{2.10}$$

and

$$\Phi := \{ \phi \in L^2(\Sigma) \ :\ \exists v \in H^1_0(\Omega) \text{ with } \phi = v_{|\Sigma} \} \equiv H^1_0(\Omega)_{|\Sigma}, \tag{2.11}$$

with the obvious norms

$$||v||^2_V = \sum_k ||v^k||^2_{1,k} \qquad (v \in V;\ v = (v^1, ..., v^K)); \tag{2.12}$$

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$$\|\phi\|_\Phi = \inf\{\|v\|_1 \mid v \in H^1_0(\Omega),\ v_{|\Sigma} = \phi\}. \tag{2.14}$$

For every f, say, in $L^2(\Omega)$, we can now consider the following two problems:

$$\begin{cases} \text{find } w \in H^1_0(\Omega) \text{ such that} \\ a(w,v) = (f,v) \qquad \forall v \in H^1_0(\Omega) \end{cases} \tag{2.15}$$

and

$$\begin{cases} \text{find } u \in V,\ \lambda \in M \text{ and } \phi \in \Phi \text{ such that} \\ \text{i) } a(u,v) - \sum_k < \lambda^k, v^k >_k \; = \; (f,v) \qquad \forall v \in V \\ \text{ii) } \sum_k < \mu^k, \psi - u^k >_k = \; 0 \qquad \forall \mu \in M \\ \text{iii) } \sum_k < \lambda^k, \phi >_k = 0 \qquad \forall \phi \in \Phi. \end{cases} \tag{2.16}$$

Theorem 1 For every $f \in L^2(\Omega)$, both problems (2.15) and (2.16) have a unique solution. Moreover we have

$$u^k = w \qquad \text{in } \Omega_k \quad (k = 1, ..., K), \tag{2.17}$$

$$\lambda^k = \frac{\partial w}{\partial n^k_A} \qquad \text{on } \Gamma_k \quad (k = 1, ..., K), \tag{2.18}$$

$$\psi = w \qquad \text{on } \Sigma \tag{2.19}$$

where $\partial w / \partial n^k_A$ is the outward conormal derivative (of the restriction of w to Ω_k) with respect to the operator A.

Proof It follows from (2.6) that (2.15) has a unique solution w. Setting u, λ, ψ as in (2.17) – (2.19) it is easy to verify that this is a solution of (2.16). Hence, we only need to show that (2.16) cannot have two different solutions or, in other words, that $f = 0$ in (2.16) implies $u = 0,\ \lambda = 0,\ \psi = 0$. Let then $f = 0$; from (2.16;ii) we get $u^k = \psi$ on Γ_k for every k, and therefore the existence of a function $w \in H^1_0(\Omega)$ such that $\psi = w_{|\Sigma}$ and $u^k = w_{|\Omega_k}$. From (2.16;i) with $v = w$, and (2.16;iii) with $\phi = w$ we have

$$a(w,w) = 0 \tag{2.20}$$

yielding $u = 0$ and $\psi = 0$. From (2.16;i) we have now

$$< \lambda^k, v >_k = 0 \qquad \forall v \in V_k, \tag{2.21}$$

It is very important, for applications to domain decomposition methods, to remark explicitly that the first two equations of (2.16) can be written as

$$\begin{cases} a_k(u^k, v^k) - < \lambda^k, v^k >_k = (f, v^k) & \forall v^k \in V_k, \quad \forall k \\ < \mu^k, u^k >_k = < \psi, \mu^k >_k & \mu^k \in M_k, \quad \forall k. \end{cases} \tag{2.22}$$

In particular, for all fixed k, assuming f ans ψ as data, (2.22) is the variational formulation of the Dirichlet problem

$$\begin{cases} Au^k = f & \text{in } \Omega_k, \\ u^k = \psi & \text{on } \Gamma_k, \end{cases} \tag{2.23}$$

where the boundary condition is imposed by means of a Lagrange multiplier (that finally comes out to be $\lambda^k \equiv \partial u^k / \partial n_A^k$) as in Babuška [1]. Hence, for f and ψ given, the resolution of the first two equations of (2.16) amounts to the resolution of K independent Dirichlet problems. In operator form (2.16) can be written as

$$\begin{cases} Au - B\lambda = f \\ -B^T u + C\psi = 0 \\ C^T u = 0 \end{cases} \tag{2.24}$$

where, as already noticed, the operator

$$\mathcal{A} := \begin{pmatrix} A & -B \\ -B^T & 0 \end{pmatrix} \tag{2.25}$$

is "block diagonal" in $V \times M$ and invertible. Setting now

$$\mathcal{C}^T(u, \lambda) := C^T \lambda, \tag{2.26}$$

$$(u_f, \lambda_f) := \mathcal{A}^{-1}(f, 0), \tag{2.27}$$

$$(u_\psi, \lambda_\psi) := -\mathcal{A}^{-1}(0, C\psi) = -\mathcal{A}^{-1}\mathcal{C}\psi, \tag{2.28}$$

$$g := \mathcal{C}^T(u_f, \lambda_f) = C^T \lambda_f, \tag{2.29}$$

problem (2.24) can now be written as

$$\mathcal{C}^T \mathcal{A}^{-1} \mathcal{C} \psi = g. \tag{2.30}$$

Setting

$$\mathcal{S} := \mathcal{C}^T \mathcal{A}^{-1} \mathcal{C}, \tag{2.31}$$

Theorem 2 $\mathcal{S}$ is an isomorphism from Φ onto Φ'. Moreover, for every $\phi \in \Phi$ we have

$$\underset{\Phi'}{<} \mathcal{S}\phi, \phi \underset{\Phi}{>} \geq \alpha ||\phi||^2_{\Phi}. \tag{2.32}$$

Proof We first prove that, for every $\psi \in \Phi$, $\mathcal{S}\psi$ is a linear continuous functional on Φ. For this, we associate with any $\phi \in \Phi$ a function $\tilde{\phi} \in H^1_0(\Omega)$ defined by

$$\begin{cases} A\tilde{\phi} &= 0 \qquad \text{in } \Omega_k, \\ \tilde{\phi} &= \phi \qquad \text{on } \Gamma_k, \end{cases} \tag{2.33}$$

for all k, and we remark that

$$\sum_k ||\tilde{\phi}||^2_{1,k} \leq \sum_k c_k |||\phi|||^2_{\frac{1}{2},k} \leq c||\phi||^2_{\Phi}. \tag{2.34}$$

Let now $\psi \in \Phi$ be given. We set $(u, \lambda) = \mathcal{A}^{-1}\mathcal{C}\psi$, so that, in particular, $u = \tilde{\psi}$. For every $\phi \in \Phi$ we have now

$$< \mathcal{S}\psi, \phi >= \sum_k < \lambda^k, \phi >_k = \sum_k a_k(u^k, \tilde{\phi}) = a(\tilde{\psi}, \tilde{\phi}) \tag{2.35}$$

From (2.34) – (2.35) we easily get that

$$< \mathcal{S}\psi, \phi > \leq \gamma ||\psi||_{\Phi} \, ||\phi||_{\Phi} \tag{2.36}$$

for some $\gamma > 0$, independent of ψ and ϕ. Taking $\phi = \psi$ in (2.35) and using (2.6) and (2.14) we have (2.32). This and (2.36) tell us that $\mathcal{S}$ is an isomorphism and the proof is complete. ∎

Remark 1 In the usual language of domain decomposition methods, $\mathcal{S}$ is the Poincaré-Steklov operator on Σ, associated with the elliptic operator A. ∎

Remark 2 It is easy to check that, since $A\tilde{\psi} = 0$ in each Ω_k we have

$$a(\tilde{\psi}, \chi) = 0 \qquad \forall \chi \in \prod \overset{\circ}{V}_k. \tag{2.37}$$

In particular, let $\tilde{\phi}^* \in H^1_0(\Omega)$ be defined by

$$\begin{cases} A^*\tilde{\phi}^* &= 0 \qquad \text{in } \Omega_k, \\ \tilde{\phi}^* &= \phi \qquad \text{on } \Gamma_k, \end{cases} \tag{2.38}$$

for $k = 1, ..., K$, where A^* is the formal adjoint of A. From (2.35), (2.37) and (2.38) we have

$$< \mathcal{S}\psi, \phi >= a(\tilde{\psi}, \tilde{\phi}^*) \tag{2.39}$$

$$< \mathcal{S}^T\phi,\psi > = < \mathcal{S}\psi,\phi > = a(\tilde{\psi},\tilde{\phi}^*) = a^T(\tilde{\phi}^*,\tilde{\psi}) \tag{2.40}$$

so that the dual operator of $\mathcal{S}$ is the Poincaré-Steklov operator associated with A^*. ∎

Problem (2.16) can now be approximated in many different ways. Choosing $V_h,\ M_h$, and Φ_h finite dimensional subspaces of $V,\ M,\ \Phi$, we can consider the discretized problem

$$\begin{cases} \text{find } u_h \in V_h,\ \lambda_h \in M_h \text{ and } \phi_h \in \Phi_h \text{ such that} \\ \text{i) } a(u_h,v) - \sum_k < \lambda_h^k, v^k >_k = (f,v) \qquad \forall v \in V_h \\ \text{ii) } \sum_k < \mu^k, \psi_h - u_h^k >_k = 0 \qquad \forall \mu \in M_h \\ \text{iii) } \sum_k < \lambda_h^k, \phi >_k = 0 \qquad \forall \phi \in \Phi_h. \end{cases} \tag{2.41}$$

It is clear that suitable inf-sup conditions have to be assumed for $V_h,\ M_h$ and Φ_h in order to ensure stability and optimal error bounds for the discrete problems (2.41). We shall not address these questions here. We note however that one can always stabilize (2.41), for general discretizations, by adding proper stabilizing terms “à la Hughes”. In this respect, see Barbosa- Hughes [2] for a way of stabilizing (2.25).

We finally point out that, if one takes a finite element approximation $\tilde{V}_h$ of $H^1(\Omega)$, on a mesh compatible with the decomposition (2.1), one can set

$$\begin{cases} V_h^k := \tilde{V}_{h|\Omega_k} & ; \quad V_h = \prod V_h^k; \\ \tilde{V}_h^\circ = \tilde{V}_h \bigcap H_0^1(\Omega_k) & ; \quad \Phi_h = \tilde{V}^\circ_{h|\Sigma}; \\ M_h^k := (V^k_{h\ |\Gamma_k})' & ; \quad M_h := \prod M_h^k. \end{cases} \tag{2.42}$$

It is easy to check that with these choices the solution of (2.41) is nothing else but the standard finite element approximation of the solution of (2.15) by means of the subspace $\tilde{V}_h^\circ$. Moreover, the discrete analogue of $\mathcal{S}$ is the classical Schur complement, and most of the recent literature on non overlapping domain decomposition methods has been devoted to find optimal preconditioners for it.(See, e.g., [3] and the references therein).

Out of our analysis of the continuous problem (2.30) we can suggest a “continuous preconditioner” that can then be applied to a big variety of discretized problems. Its simplicity makes it appealing for nasty situations as: non dimensional problems etc.

Let $\Phi_1 = \Phi \bigcap H^1(\Sigma)$, and let $T_2 \; : \; \Phi_1 \longmapsto \Phi_1'$ be the operator defined by

$$< T_2\psi, \phi >= \int_\Sigma \psi_{/t}\phi_{/t}\, ds \tag{2.43}$$

where $\phi_{/t}$ is the tangential derivative of ϕ (in a three-dimensional problem, Σ will be two-dimensional and the tangential derivative has to be replaced by the tangential gradient). Then, one can see that the operator

$$\mathcal{S}^T T_2^{-1} \mathcal{S} \tag{2.44}$$

is symmetric and uniformly bounded, for instance, from $L^2(\Sigma)$ into itself. Hence, it is a good starting point for designing a discrete preconditioner.

References

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