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$H(\mathrm{div})$ and $H(\mathbf{curl})$-conforming Virtual Element Methods

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Abstract In the present paper we construct Virtual Element Spaces that are $H(\mathrm{div})$-conforming and $H(\mathbf{curl})$-conforming on general polygonal and polyhedral elements; these spaces can be interpreted as a generalization of well known Finite Elements. We moreover present the basic tools needed to make use of these spaces in the approximation of partial differential equations. Finally, we discuss the construction of exact sequences of VEM spaces.

1 Introduction

The Virtual Element Methods (in short, VEM) were initially introduced in [10], as a variant of classical Lagrange Finite Element Methods to accommodate the use of polygonal and polyhedral elements. Needless to say, they could be seen as an evolution of nodal Mimetic Finite Differences (see [23, 14]) as well as a variant of other Galerkin methods for polygonal and polyhedral elements (see e.g. [4, 7, 8, 9, 18, 21, 30, 31, 33, 34, 35, 36, 40, 42, 44, 45, 46, 47, 48, 49] and the references therein). Even more recently, in [24] we started the extension

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to polygonal elements of Raviart-Thomas or BDM elements for mixed formulations (see e.g. [20] and the references therein). These, in a sense, constitute the most natural and direct evolution of the original "flux based" Mimetic Finite Differences, as for instance in [39]. See also, for the more mathematical aspects, [25, 28, 27], as well as [16, 37, 13], the review papers [19, 32, 41], and the book [15]. In addition to [10], see for instance [29, 11, 1, 17, 24] and references therein for applications of the Virtual Element Method to various types of problems.

On the other hand, to deal with a sufficiently wide range of mixed formulations (see again [20] and the references therein), one needs to use a big variety of $H(\mathrm{div})$ and $H(\mathbf{curl})$-conforming spaces (to be used together with the more classical H^1-conforming and L^2-conforming ones). See for instance [43] or [3]. See also the recent overview on Finite Element spaces presented in [2].

The purpose of this paper is to indicate a possible strategy to construct the extensions of all these types of spaces to more general elemental geometries, and typically to polygonal and polyhedral elements. The use of curved edges or curved faces (that so far, in this context, was tackled only in [26]) will be the object of future research.

As a general matter, the (vector valued) functions to be used, in each element, in the Virtual Element Methods are **not** polynomials (although they contain suitable polynomial spaces within each element), and are presented as solutions of (systems of) partial differential equations. However "the name of the game", in the VEM context, is to avoid solving these PDE systems, even in a roughly approximate way. Hence, in order to be able to construct, element by element, the necessary local matrices, we have to be able to construct suitable *projectors* from the local VEM spaces to some polynomial spaces (whose degree will determine the final accuracy of the method).

In presenting our $H(\mathrm{div})$-conforming and $H(\mathbf{curl})$-conforming spaces we will therefore take care to show how, for them, one can construct suitable L^2-projection operators on the corresponding polynomial spaces. This of course will not always solve all the problems, but (as pointed out for instance in [1] for some particular cases) will surely be a precious instrument.

As the range of possible variants (required by different applications) is overwhelming, we decided to limit ourselves, here, to the presentation of a few typical cases (that in our opinion could be sufficient to give the general idea), leaving to the very last (and short) section the task to give hints on some of the possible variants. In the same spirit, we decided not to present direct applications. We believe that, for the readers with some experience in the approximation of mixed formulations, the general ideas outlined in this paper should be enough to understand the possible use of our spaces for most of the applications discussed in [20]. Clearly, a lot of additional work, and a lot of numerical experiments, will be needed for the tune-up of these methods in each particular type of application. To have an idea on the implementation of Virtual Element Methods we refer to the guidelines given in [12] for nodal virtual elements.

Here is an outline of the paper: in the next section we will introduce a suitable notation and recall a few classical results of Calculus in several variables. Then we will present, each in a separate section, the $H(\mathrm{div})$-conforming and the $H(\mathrm{curl})$-conforming spaces for *polygonal* elements, and the corresponding ones for *polyhedral* elements. Next, we will briefly recall the H^1-conforming and L^2-conforming spaces (as introduced for instance in [10]) and discuss the possibility of having *exact sequences* of VEM spaces, in the spirit of [3].

In the last section, as announced already, we will give a short hint of the huge range of possible variants.

2 Notation, Assumptions, and Known Results

In the present section we introduce some preliminaries.

2.1 Basic notation on mesh and operators

In what follows, we will detail the spaces and their degrees of freedom mainly **at the element level**. One of the best features of Virtual Element Methods is the possibility to use elements having a very general geometry, and actually, in order to give the definition of the space we could use, in 2D, arbitrary *simply connected polygons*, and in 3D arbitrary *simply connected polyhedra with simply connected faces*. In order to have optimal *interpolation errors*, as well as suitable stability properties in the applications to different problems, we would however need some additional assumptions (see, for instance, [10]). For every geometrical object $\mathcal{O}$ that we are going to use in what follows (segment, polygon, polyhedron) and for a generic space of functions $\mathcal{F}(\mathcal{O})$ defined on $\mathcal{O}$, we denote by $\mathcal{F}(\mathcal{O})/\mathbb{R}$ (or simply by $\mathcal{F}/\mathbb{R}$ when the context is clear) the subset of functions having zero mean value on $\mathcal{O}$.

In two dimensions E denotes a polygon, and ℓ_e^E (or simply ℓ_e) the number of edges. Moreover,

- For a polygon E, $\boldsymbol{n}_E$ or simply $\boldsymbol{n}$ will be the outward normal unit vector, and $\boldsymbol{t}_E$, or simply $\boldsymbol{t}$, will be the tangent counterclockwise unit vector.
- For a scalar field q and a vector field $\boldsymbol{v} = (v_1, v_2)$, we will set (with a usual notation)

$$\mathbf{rot}\, q := \Big(\frac{\partial q}{\partial y}, -\frac{\partial q}{\partial x}\Big) \qquad \mathrm{rot}\, \boldsymbol{v} := \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

In three dimensions P denotes a polyhedron having ℓ_e^P (or simply ℓ_e) edges, and ℓ_f faces. Moreover,

- For a face f of a polyhedron P, the **tangential** differential operators will be denoted by a subscript 2, as in: div_2, rot_2, $\mathbf{rot}_2$, $\mathbf{grad}_2$, Δ_2, and so on.
- When dealing with a single polyhedron, we will always assume that all its faces are oriented with the *outward normal*, while, when necessary, we will have to choose an orientation for every edge. Obviously when dealing

with a decomposition in several polyhedra we will also have to decide an orientation for every face.

- On a polyhedron P, on each face f we will have to distinguish between the unit outward *normal to the plane of the face* (that we denote by $\boldsymbol{n}_P^f$), and the unit vector *in the plane of the face* that is normal to the boundary ∂f (that will be denoted, on each edge e, by $\boldsymbol{n}_f^e$). On each face, $\boldsymbol{t}_f$ or simply $\boldsymbol{t}$ will again be the unit counterclockwise tangent vector on ∂f.
- For a (smooth enough) three dimensional vector-valued function $\boldsymbol{\varphi}$ on P, and for a face f with normal $\boldsymbol{n}_P^f$, we define the tangential component of $\boldsymbol{\varphi}$ as
$$\boldsymbol{\varphi}_f := \boldsymbol{\varphi} - (\boldsymbol{\varphi}\cdot\boldsymbol{n}_P^f)\boldsymbol{n}_P^f, \tag{2.1}$$
while $\boldsymbol{\varphi}_t$ denotes the vector field defined on ∂P such that, on each face $f \in \partial P$, its restriction to the face f satisfies:
$$\boldsymbol{\varphi}_{t|f} = \boldsymbol{\varphi}_f. \tag{2.2}$$
- Note that $\boldsymbol{\varphi}_f$ as defined in (2.1) is different from
$$\boldsymbol{\varphi}\wedge\boldsymbol{n}_P^f; \tag{2.3}$$
indeed, for instance, if $\boldsymbol{n}_P^f = (0,0,1)$ and $\boldsymbol{\varphi} = (\phi_1,\phi_2,\phi_3)$, then
$$\boldsymbol{\varphi}_f = (\phi_1,\phi_2,0) \qquad \boldsymbol{\varphi}\wedge\boldsymbol{n}_P^f = (\phi_2,-\phi_1,0).$$
- With an abuse of language, sometimes we will treat both $\boldsymbol{\varphi}_f$ and $\boldsymbol{\varphi}\wedge\boldsymbol{n}_P^f$ as 2D vectors in the plane of the face. In the previous case, then, we would often take $\boldsymbol{\varphi}_f = (\phi_1,\phi_2)$ and $\boldsymbol{\varphi}\wedge\boldsymbol{n}_P^f = (\phi_2,-\phi_1)$.

2.2 Polynomial spaces and exact sequences

We will now recall some basic properties of Calculus of several variables, applied in particular to polynomial spaces. For a generic non negative number k and for a generic geometrical object $\mathcal{O}$ in 1,2, or 3 dimensions $\mathbb{P}_k(\mathcal{O})$ denotes the space of by polynomials of degree $\leq k$ on $\mathcal{O}$, with the additional (common) convention that $\mathbb{P}_{-1}(\mathcal{O}) = \{0\}$. Moreover, with a common abuse of language, we will often say "polynomial of degree k" meaning actually "polynomial of degree $\leq k$". Often the geometrical object $\mathcal{O}$ will be omitted when no confusion arises.

In all the following diagrams (2.4), (2.5), and (2.6), as well as in those at the end, i.e. (8.1), (8.2), and (8.10) we will denote by i the mapping that to every real *number* c associates the constant *function* identically equal to c, and by o the mapping that to every *function* associates the *number* 0. Then we recall that, in 2 and in 3 dimensions, we have the exactness of the following sequences. In 2 dimensions
$$\mathbb{R} \xrightarrow{\ i\ } \mathbb{P}_r \xrightarrow{\ \mathbf{grad}\ } (\mathbb{P}_{r-1})^2 \xrightarrow{\ \mathrm{rot}\ } \mathbb{P}_{r-2} \xrightarrow{\ o\ } 0 \tag{2.4}$$

or, equivalently,

$$\mathbb{R} \xrightarrow{\;i\;} \mathbb{P}_r \xrightarrow{\;\mathbf{rot}\;} (\mathbb{P}_{r-1})^2 \xrightarrow{\;\mathrm{div}\;} \mathbb{P}_{r-2} \xrightarrow{\;o\;} 0 \tag{2.5}$$

are exact sequences. In three dimensions we have that

$$\mathbb{R} \xrightarrow{\;i\;} \mathbb{P}_r \xrightarrow{\;\mathbf{grad}\;} (\mathbb{P}_{r-1})^3 \xrightarrow{\;\mathbf{curl}\;} (\mathbb{P}_{r-2})^3 \xrightarrow{\;\mathrm{div}\;} \mathbb{P}_{r-3} \xrightarrow{\;o\;} 0 \tag{2.6}$$

is also an exact sequence. We recall that the *exactness* means that *the image of every operator coincides with the kernel of the following one.* To better explain the consequences of these statements we introduce an additional notation. For s integer ≥ 1, in two dimensions we denote by

$$\mathcal{G}_{s-1} \text{ the set } \mathbf{grad}(\mathbb{P}_s), \qquad \mathcal{R}_{s-1} \text{ the set } \mathbf{rot}(\mathbb{P}_s), \tag{2.7}$$

and in three dimensions

$$\mathcal{G}_{s-1} \text{ the set } \mathbf{grad}(\mathbb{P}_s), \qquad \mathcal{R}_{s-1} \text{ the set } \mathbf{curl}\Big((\mathbb{P}_s)^3\Big). \tag{2.8}$$

When considering polynomials on a domain $\mathcal{O}$ (not too irregular) we might use the $L^2(\mathcal{O})$ or (in d dimensions) the $(L^2(\mathcal{O}))^d$ inner product, and introduce

$$\mathcal{G}_s^\perp := \text{ orthogonal of } \mathcal{G}_s \text{ in } (\mathbb{P}_s)^d, \quad \mathcal{R}_s^\perp := \text{ orthogonal of } \mathcal{R}_s \text{ in } (\mathbb{P}_s)^d. \tag{2.9}$$

Obviously, $(\mathbb{P}_s)^d = \mathcal{G}_s \oplus \mathcal{G}_s^\perp = \mathcal{R}_s \oplus \mathcal{R}_s^\perp$. In a similar way, the space $\mathbb{P}_s$ could be seen as decomposed in the subspace of constants (the image of $i : \mathbb{R} \longrightarrow \mathbb{P}_s$) and the polynomials in $\mathbb{P}_s$ having zero mean value on $\mathcal{O}$ (and hence orthogonal to the constants), that is $(\mathbb{P}_s(\mathcal{O}))/\mathbb{R}$.

We recall now some of the properties following from the exactness of the above sequences. The exactness of the sequence (2.4) implies in particular that for all integer s:

$$\begin{aligned}
&\text{i) } \mathbf{grad} \text{ is an isomorphism from } (\mathbb{P}_s)/\mathbb{R} \text{ to } \mathcal{G}_{s-1},\\
&\text{ii) } \{\boldsymbol{v} \in (\mathbb{P}_s)^2\} \Rightarrow \{\mathrm{rot}\,\boldsymbol{v} = 0 \text{ iff } \boldsymbol{v} \in \mathcal{G}_s\},\\
&\text{iii) rot is an isomorphism from } \mathcal{G}_s^\perp \text{ to the whole } \mathbb{P}_{s-1},
\end{aligned} \tag{2.10}$$

and equivalently (2.5) implies that

$$\begin{aligned}
&\text{i) } \mathbf{rot} \text{ is an isomorphism from } (\mathbb{P}_s)/\mathbb{R} \text{ to } \mathcal{R}_{s-1},\\
&\text{ii)} \{\boldsymbol{v} \in (\mathbb{P}_s)^2\} \Rightarrow \{\mathrm{div}\,\boldsymbol{v} = 0 \text{ iff } \boldsymbol{v} \in \mathcal{R}_s\},\\
&\text{iii) div is an isomorphism from } \mathcal{R}_s^\perp \text{ to the whole } \mathbb{P}_{s-1}.
\end{aligned} \tag{2.11}$$

Finally, the exactness of the sequence (2.6) implies in particular that, for all integer s:

$$\begin{aligned}
&\text{i) } \{\boldsymbol{v} \in (\mathbb{P}_s)^3\} \Rightarrow \{\mathbf{curl}\,\boldsymbol{v} = 0 \text{ iff } \boldsymbol{v} \in \mathcal{G}_s\},\\
&\text{ii)} \{\boldsymbol{v} \in (\mathbb{P}_s)^3\} \Rightarrow \{\mathrm{div}\,\boldsymbol{v} = 0 \text{ iff } \boldsymbol{v} \in \mathcal{R}_s\},\\
&\text{iii) } \mathbf{grad} \text{ is an isomorphism from } (\mathbb{P}_s)/\mathbb{R} \text{ to } \mathcal{G}_{s-1},\\
&\text{iv) } \mathbf{curl} \text{ is an isomorphism from } \mathcal{G}_s^\perp \text{ to } \mathcal{R}_{s-1},\\
&\text{v) div is an isomorphism from } \mathcal{R}_s^\perp \text{ to the whole } \mathbb{P}_{s-1}.
\end{aligned} \tag{2.12}$$

Remark 1 Properties [2.10;ii)], [2.11;ii)], and [2.12;i) and ii)] are just particular cases of well known results in Calculus. Indeed, on a simply connected domain, we know that a (smooth enough) vector field $\boldsymbol{v}$ having $\mathrm{rot}\,\boldsymbol{v} = 0$ (in 2 dimensions) or $\mathbf{curl}\,\boldsymbol{v} = 0$ (in 3 dimensions) is necessarily a gradient, and a (smooth enough) vector $\boldsymbol{v}$ field having $\mathrm{div}\,\boldsymbol{v} = 0$ is necessarily a **rot** (in 2 dimensions) or a **curl** (in 3 dimensions).

To all these spaces we can attach their dimensions. To start with, we denote by $\pi_{k,d}$ the dimension of the space $\mathbb{P}_k(\mathbb{R}^d)$, that is,

$$\pi_{k,1} = k+1;\ \pi_{k,2} = \frac{(k+1)(k+2)}{2};\ \pi_{k,3} = \frac{(k+1)(k+2)(k+3)}{6}. \tag{2.13}$$

The dimension of vector-valued polynomials $(\mathbb{P}_k)^d$ is then

$$\dim\{(\mathbb{P}_k)^d\} = d\pi_{k,d}. \tag{2.14}$$

We denote by $\gamma_{k,d}$ the dimension of $\mathcal{G}_k \subseteq (\mathbb{P}_k)^d$ defined in (2.7)-(2.8):

$$\dim\{\mathcal{G}_k\} \text{ in } d \text{ dimensions } \equiv \gamma_{k,d} = \pi_{k+1,d} - 1. \tag{2.15}$$

Clearly, $\gamma_{k,2}$ also equals the dimension $\rho_{k,2}$ of $\mathbf{rot}(\mathbb{P}_{k+1})$ (that is, $\mathcal{R}_k$):

$$\dim\{\mathcal{R}_k\} \text{ in 2 dimensions } = \rho_{k,2} = \gamma_{k,2} = \pi_{k+1,2} - 1. \tag{2.16}$$

We also have (obviously), in d dimensions,

$$\dim\{\mathcal{G}_k^\perp\} \text{ in } d \text{ dimensions } = d\pi_{k,d} - \gamma_{k,d} = d\pi_{k,d} - \pi_{k+1,d} + 1. \tag{2.17}$$

In 2 dimensions, looking at [2.10;iii)] and at [2.11;iii)] we see that the dimension of $\mathcal{G}_k^\perp$ as well as that of $\mathcal{R}_k^\perp$ equal that of $\mathbb{P}_{k-1}$, that is

$$\dim\{\mathcal{G}_k^\perp\} = \dim\{\mathcal{R}_k^\perp\} = \pi_{k-1,2} \qquad \text{in two dimensions.} \tag{2.18}$$

On the other hand, for $d = 3$, we can use [2.12;iv)] and see that the dimension $\rho_{k-1,3}$ of $\mathcal{R}_{k-1} = \mathbf{curl}((P_k)^3)$ is given by

$$\rho_{k-1,3} = \dim\{\mathcal{R}_{k-1}\} = \dim\{\mathcal{G}_k^\perp\} = 3\pi_{k,3} - \pi_{k+1,3} + 1, \tag{2.19}$$

while, following [2.12;v], we have

$$\dim\{\mathcal{R}_k^\perp\} = \pi_{k-1,3} \qquad \text{in three dimensions.} \tag{2.20}$$

We summarize all the above results on the dimensions of polynomial spaces in the following equations. In **two dimensions**:

$$\dim\{\mathcal{G}_k\} = \dim\{\mathcal{R}_k\} = \pi_{k+1,2} - 1 \qquad \dim\{\mathcal{R}_k^\perp\} = \dim\{\mathcal{G}_k^\perp\} = \pi_{k-1,2} \tag{2.21}$$

and in **three dimensions**:

$$\begin{aligned} &\dim\{\mathcal{G}_k\} = \pi_{k+1,3} - 1, \qquad \dim\{\mathcal{G}_k^\perp\} = 3\pi_{k,3} - \pi_{k+1,3} + 1 \\ &\qquad \dim\{\mathcal{R}_k\} = 3\pi_{k+1,3} - \pi_{k+2,3} + 1 \qquad\qquad \dim\{\mathcal{R}_k^\perp\} = \pi_{k-1,3}. \end{aligned} \tag{2.22}$$

2.3 Spaces H_{div}, H_{rot}, $H_{\mathbf{curl}}$

As announced, the definition of our local Virtual Element spaces will be done as the solution, within each element, of a suitable *div-curl* system. In view of that, it will be convenient to recall the *compatibility conditions* (between the data inside the element and the ones at the boundary) that are required in order to have a solution. To start with, for a polygon E we define

$$H(\mathrm{div};E) := \{\boldsymbol{v} \in (L^2(E))^2 \text{ such that } \mathrm{div}\,\boldsymbol{v} \in L^2(E)\}, \tag{2.23}$$

$$H(\mathrm{rot};E) := \{\boldsymbol{v} \in (L^2(E))^2 \text{ such that } \mathrm{rot}\,\boldsymbol{v} \in L^2(E)\}, \tag{2.24}$$

and for a polyhedron P

$$H(\mathrm{div};P) := \{\boldsymbol{v} \in (L^2(P))^3 \text{ such that } \mathrm{div}\,\boldsymbol{v} \in L^2(P)\}, \tag{2.25}$$

$$H(\mathbf{curl};P) := \{\boldsymbol{v} \in (L^2(P))^3 \text{ such that } \mathbf{curl}\,\boldsymbol{v} \in (L^2(P))^3\}. \tag{2.26}$$

We now assume that we are given, on a simply connected polygon E, two smooth functions f_d and f_r, and, on the boundary ∂E, two edge-wise smooth functions g_n and g_t. We recall that the problem: *find* $\boldsymbol{v} \in H(\mathrm{div};E) \cap H(\mathrm{rot};E)$ *such that:*

$$\mathrm{div}\,\boldsymbol{v} = f_d \text{ and } \mathrm{rot}\,\boldsymbol{v} = f_r \text{ in } E \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{n} = g_n \text{ on } \partial E \tag{2.27}$$

has a unique solution if and only if

$$\int_E \mathrm{div}\,\boldsymbol{v}\,\mathrm{d}E = \int_{\partial E} g_n\,\mathrm{d}s. \tag{2.28}$$

Similarly the problem: *find* $\boldsymbol{v} \in H(\mathrm{div};E) \cap H(\mathrm{rot};E)$ *such that:*

$$\mathrm{div}\,\boldsymbol{v} = f_d \text{ and } \mathrm{rot}\,\boldsymbol{v} = f_r \text{ in } E \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{t} = g_t \text{ on } \partial E \tag{2.29}$$

has a unique solution if and only if

$$\int_E \mathrm{rot}\,\boldsymbol{v}\,\mathrm{d}E = \int_{\partial E} g_t\,\mathrm{d}s. \tag{2.30}$$

In three dimensions, on a simply connected polyhedron P we assume that we are given a smooth scalar function f_d and a smooth vector valued function $\boldsymbol{f}_r$ with $\mathrm{div}\,\boldsymbol{f}_r = 0$. On the boundary ∂P we assume that we are given a face-wise smooth scalar function g_n and a face-wise smooth tangent vector field $\boldsymbol{g}_t$ whose tangential components are continuous (with a natural meaning) at the edges of ∂P. Then we recall that the problem: *find* $\boldsymbol{v} \in H(\mathrm{div};P) \cap H(\mathbf{curl};P)$ *such that:*

$$\mathrm{div}\,\boldsymbol{v} = f_d \text{ and } \mathbf{curl}\,\boldsymbol{v} = \boldsymbol{f}_r \text{ in } P \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{n} = g_n \text{ on } \partial P \tag{2.31}$$

has a unique solution if and only if

$$\int_P \mathrm{div}\,\boldsymbol{v}\,\mathrm{d}P = \int_{\partial P} g_n\,\mathrm{d}s, \tag{2.32}$$

and similarly the problem: *find* $\boldsymbol{v} \in H(\mathrm{div}; P) \cap H(\mathbf{curl}; P)$ *such that:*

$$\mathrm{div}\, \boldsymbol{v} = f_d \text{ and } \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{f}_r \text{ in } P \quad \text{and} \quad \boldsymbol{v}_t = \boldsymbol{g}_t \text{ on } \partial P \tag{2.33}$$

has a unique solution if and only if

$$\boldsymbol{f}_r \cdot \boldsymbol{n} = \mathrm{rot}_2\, \boldsymbol{g}_t \text{ on } \partial P. \tag{2.34}$$

For more details concerning the solutions of the *div-curl system* we refer, for instance, to [5], [6] and the references therein.

Finally, in order to clarify what we consider as *feasible* (in a code), we recall that we assume to be able to integrate any polynomial on any polygon or polyhedron, for instance through formulae of the type

$$\int_E x_1^k = \frac{1}{k+1} \int_{\partial E} x_1^{k+1}\, n_1 \,\mathrm{d}s. \tag{2.35}$$

Remark 2 The virtual spaces presented in this work will be indexed by an integer number k, to underline that, in all cases, the scalar or vector-valued polynomial space $\mathbb{P}_k$ is contained in the local virtual space, and k will be the expected order of accuracy.

3 2D Face Elements

These spaces are the same of Brezzi-Falk-Marini [24], although here we propose a different set of degrees of freedom.

3.1 The local space

On a polygon E, for k integer ≥ 1, we set:

$$V_{2,k}^{\mathrm{face}}(E) := \{\boldsymbol{v} \in H(\mathrm{div}; E) \cap H(\mathrm{rot}; E) : \boldsymbol{v} \cdot \boldsymbol{n}_{|e} \in \mathbb{P}_k(e)\ \forall \text{ edge } e \text{ of } E, \\ \mathbf{grad}\,\mathrm{div}\, \boldsymbol{v} \in \mathcal{G}_{k-2}(E), \text{ and } \mathrm{rot}\, \boldsymbol{v} \in \mathbb{P}_{k-1}(E)\}. \tag{3.1}$$

3.2 Dimension of the space $V_{2,k}^{\mathrm{face}}(E)$

We recall from Subsection 2.3 that, given

- a function g defined on ∂E such that $g_{|e} \in \mathbb{P}_k(e)$ for all $e \in \partial E$,
- a polynomial $f_d \in \mathbb{P}_{k-1}(E)$ such that

$$\int_E f_d \,\mathrm{d}E = \int_{\partial E} g \,\mathrm{d}s, \tag{3.2}$$

- a polynomial $f_r \in \mathbb{P}_{k-1}(E)$, we can find a unique vector $\boldsymbol{v} \in V_{2,k}^{\mathrm{face}}(E)$ such that

$$\boldsymbol{v} \cdot \boldsymbol{n} = g \text{ on } \partial E, \quad \operatorname{div} \boldsymbol{v} = f_d \text{ in } E, \quad \operatorname{rot} \boldsymbol{v} = f_r \text{ in } E. \tag{3.3}$$

This easily implies that the dimension of $V_{2,k}^{\mathrm{face}}(E)$ is given by:

$$\begin{aligned}\dim V_{2,k}^{\mathrm{face}}(E) &= \ell_e \dim \mathbb{P}_k(e) + \{\dim \mathbb{P}_{k-1}(E) - 1\} + \dim \mathbb{P}_{k-1}(E) \\ &= \ell_e \pi_{k,1} + \pi_{k-1,2} - 1 + \pi_{k-1,2}\end{aligned} \tag{3.4}$$

3.3 The local Degrees of Freedom

A convenient set of degrees of freedom for functions $\boldsymbol{v}$ in $V_{2,k}^{\mathrm{face}}(E)$ will be:

$$\int_e \boldsymbol{v} \cdot \boldsymbol{n}\, p_k \,\mathrm{d}e \qquad \text{for all edge } e, \text{ for all } p_k \in \mathbb{P}_k(e), \tag{3.5}$$

$$\int_E \boldsymbol{v} \cdot \boldsymbol{g}_{k-2} \,\mathrm{d}E \qquad \text{for all } \boldsymbol{g}_{k-2} \in \mathcal{G}_{k-2}, \tag{3.6}$$

$$\int_E \boldsymbol{v} \cdot \boldsymbol{g}_k^{\perp} \,\mathrm{d}E \qquad \text{for all } \boldsymbol{g}_k^{\perp} \in \mathcal{G}_k^{\perp}. \tag{3.7}$$

A depiction of the degrees of freedom above, for the cases $k = 1, 2, 3$, can be found in Figure 1. Here and in the rest of the paper we adopt a common abuse of notation, calling "degrees of freedom" (3.5)–(3.7). To be more precise, in order to get true d.o.f. out of (3.5)–(3.7) one should first choose a basis in the involved polynomial spaces. Remembering (2.21) we easily see that the number of degrees of freedom in (3.5)–(3.7) equals the dimension of $V_{2,k}^{\mathrm{face}}(E)$ as given in (3.4).

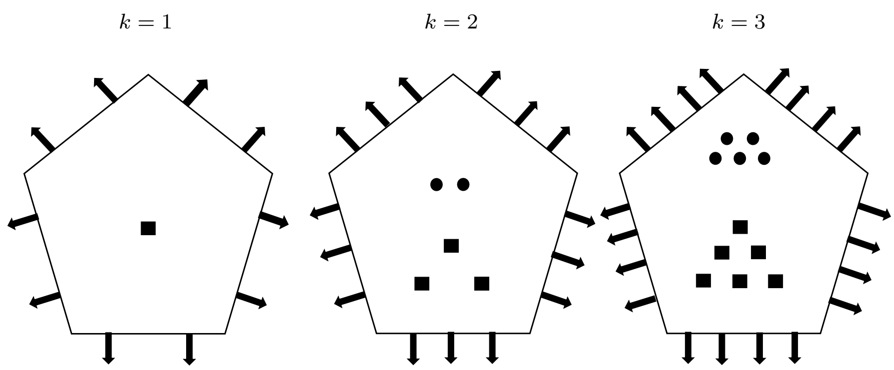

Fig. 1 Representation of the degrees of freedom for 2D face elements, $k = 1, 2, 3$. The arrows represent (3.5), while the circles and squares stand for (3.6) and (3.7), respectively.

3.4 Unisolvence

Since the number of degrees of freedom (3.5)-(3.7) equals the dimension of $V_{2,k}^{\rm face}(E)$, to prove unisolvence we just need to show that if for a given $\boldsymbol{v}$ in $V_{2,k}^{\rm face}(E)$ all the degrees of freedom (3.5)-(3.7) are zero, that is if

$$\begin{aligned} &\textstyle\int_e \boldsymbol{v}\cdot\boldsymbol{n}\, p_k \,{\rm d}e = 0 \qquad \text{for all edge } e, \text{ for all } \ p_k\in\mathbb{P}_k(e), && (3.8)\\ &\textstyle\int_E \boldsymbol{v}\cdot\boldsymbol{g}_{k-2}\,{\rm d}E = 0 \qquad \text{for all } \boldsymbol{g}_{k-2}\in\mathcal{G}_{k-2}, && (3.9)\\ &\textstyle\int_E \boldsymbol{v}\cdot\boldsymbol{g}_k^{\perp}\,{\rm d}E = 0 \qquad \text{for all } \boldsymbol{g}_k^{\perp}\in\mathcal{G}_k^{\perp}, && (3.10) \end{aligned}$$

then we must have $\boldsymbol{v} = 0$. For this we introduce a couple of preliminary observations.

Lemma 1 *If $\boldsymbol{v}\in V_{2,k}^{\rm face}(E)$ and if* (3.8) *and* (3.9) *hold, then*

$$\int_E \boldsymbol{v}\cdot\mathbf{grad}\,\varphi\,{\rm d}E = 0 \quad \forall\varphi\in H^1(E). \tag{3.11}$$

Proof Using the fact that ${\rm div}\,\boldsymbol{v}\in\mathbb{P}_{k-1}$ and setting $q_{k-1} := {\rm div}\,\boldsymbol{v}$ we have

$$\begin{aligned}\int_E |\,{\rm div}\,\boldsymbol{v}|^2\,{\rm d}E &= \int_E {\rm div}\,\boldsymbol{v}\, q_{k-1}\,{\rm d}E\\ &= \int_{\partial E}\boldsymbol{v}\cdot\boldsymbol{n}q_{k-1}\,{\rm d}s - \int_E \boldsymbol{v}\cdot\mathbf{grad}\,q_{k-1}\,{\rm d}E = 0, \end{aligned}\tag{3.12}$$

where the last step follows from (3.8) and (3.9). Hence we have that ${\rm div}\,\boldsymbol{v}=0$ and since (using again (3.8)) $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on ∂E, the result (3.11) follows by using a simple integration by parts.

Lemma 2 *If $\boldsymbol{v}\in V_{2,k}^{\rm face}(E)$ then there exist a $\boldsymbol{q}_k^{\perp}$ in $\mathcal{G}_k^{\perp}$ and a $\varphi\in H^1(E)$ such that*

$$\boldsymbol{v} = \boldsymbol{q}_k^{\perp} + \mathbf{grad}\,\varphi. \tag{3.13}$$

Proof We first note that according to (3.1) if $\boldsymbol{v}\in V_{2,k}^{\rm face}(E)$ then ${\rm rot}\,\boldsymbol{v}\in\mathbb{P}_{k-1}$. Looking at [2.10;iii)] we have then that ${\rm rot}\,\boldsymbol{v} = {\rm rot}\,\boldsymbol{q}_k^{\perp}$ for some $\boldsymbol{q}_k^{\perp}\in\mathcal{G}_k^{\perp}$. Now the difference $\boldsymbol{v}-\boldsymbol{q}_k^{\perp}$ satisfies ${\rm rot}(\boldsymbol{v}-\boldsymbol{q}_k^{\perp}) = 0$, and as E is simply connected the result follows from Remark 1.

We can now easily prove the following theorem.

Theorem 1 *The degrees of freedom* (3.5)-(3.7) *are unisolvent in $V_{2,k}^{\rm face}(E)$.*

Proof Assume that for a certain $\boldsymbol{v}\in V_{2,k}^{\rm face}(E)$ we have (3.8)-(3.10). From Lemma 2 we have $\boldsymbol{v} = \boldsymbol{q}_k^{\perp} + \mathbf{grad}\,\varphi$ for some $\boldsymbol{q}\in\mathcal{G}_k^{\perp}$ and some $\varphi\in H^1(E)$. Then

$$\int_E |\boldsymbol{v}|^2\,{\rm d}E = \int_E \boldsymbol{v}\cdot(\boldsymbol{q}_k^{\perp} + \mathbf{grad}\,\varphi)\,{\rm d}E = 0 \tag{3.14}$$

since the first term is zero by (3.10) and the second term is zero by (3.8)-(3.9) and Lemma 1.

Remark 3 The degrees of freedom (3.5) are pretty obvious. A natural variant would be to use, on each edge e, the value of $\boldsymbol{v} \cdot \boldsymbol{n}$ at $k+1$ suitable points on e. On the other hand, for the degrees of freedom (3.6) we could integrate by parts, and replace them with

$$\int_E \mathrm{div}\, \boldsymbol{v}\, q_{k-1}\, \mathrm{d}E \qquad \text{for all } q_{k-1} \in \mathbb{P}_{k-1}/\mathbb{R}. \tag{3.15}$$

Moreover, the degrees of freedom (3.7) could be replaced by

$$\int_E \mathrm{rot}\, \boldsymbol{v}\, q_{k-1}\, \mathrm{d}E \qquad \text{for all } q_{k-1} \in \mathbb{P}_{k-1} \tag{3.16}$$

as we had in the original work [24]. Finally, in (3.7) $\mathcal{G}_k^\perp$ could be replaced by any subspace $\mathcal{G}_k^\oplus$ of $(\mathbb{P}_k)^d$ such that $(\mathbb{P}_k)^d = \mathcal{G}_k \oplus \mathcal{G}_k^\oplus$.

3.5 Computing the L^2 projection

Since the VEM spaces contain functions which are not polynomials, and their reconstruction can be too hard, for the practical use of a virtual element method it is often important to be able to compute different types of projections onto spaces of polynomials. Here we show how to construct the one that is possibly the most convenient, and surely the most commonly used: the L^2 projection onto $(\mathbb{P}_k(E))^2$.

For this, we begin by recalling that to assign $\mathbf{grad}\, \mathrm{div}\, \boldsymbol{v} \in \mathcal{G}_{k-2}(E)$ (as we do with our degrees of freedom (3.6) for $\boldsymbol{v} \in V_{2,k}^{\mathrm{face}}(E)$), is equivalent to assign $\mathrm{div}\, \boldsymbol{v} \in \mathbb{P}_{k-1}(E)$ up to an additive constant. This constant will be assigned by the integral of $\boldsymbol{v} \cdot \boldsymbol{n}$ over ∂E, that can be deduced from the degrees of freedom (3.5). Indeed, using the same integration by parts applied in (3.12), the degrees of freedom (3.5) and (3.6) allow us to compute $\int_E \mathrm{div}\, \boldsymbol{v}\, q_{k-1}\, \mathrm{d}E$ for all $q_{k-1} \in \mathbb{P}_{k-1}(E)$, and since $\mathrm{div}\, \boldsymbol{v} \in \mathbb{P}_{k-1}(E)$, we can compute exactly the divergence of any $\boldsymbol{v} \in V_{2,k}^{\mathrm{face}}(E)$. In turn this implies, again by using an integration by parts and (3.5), that we are able to compute also

$$\int_E \boldsymbol{v} \cdot \boldsymbol{g}_k\, \mathrm{d}E \quad \forall \boldsymbol{g}_k \in \mathcal{G}_k,$$

and actually

$$\int_E \boldsymbol{v} \cdot \mathbf{grad}\, \varphi\, \mathrm{d}E \quad \forall \varphi \text{ polynomial on } E.$$

The above property, combined with (3.7), allows to compute the integrals against any $\boldsymbol{q}_k \in (\mathbb{P}_k(E))^2$ and thus yields the following important result.

Theorem 2 *The $L^2(E)$ projection operator*

$$\Pi_k^0 : V_{2,k}^{\mathrm{face}}(E) \longrightarrow (\mathbb{P}_k(E))^2$$

is computable using the degrees of freedom (3.5)–(3.7).

Remark 4 We point out that, for instance, the $(L^2(E))^2$ projection would be much more difficult to compute if we used the original degrees of freedom of [24] discussed in Remark 3.

3.6 The global 2D-face space

Given a polygon Ω and a decomposition $\mathcal{T}_h$ of Ω into a finite number of polygonal elements E, we can now consider the *global* space

$$V_{2,k}^{\text{face}}(\Omega) := \{\boldsymbol{v} \in H(\text{div};\Omega) \text{ s. t. } \boldsymbol{v}_{|E} \in V_{2,k}^{\text{face}}(E)\ \forall \text{ element } E \in \mathcal{T}_h\}. \quad (3.17)$$

Note that in (3.17) we assumed that the elements $\boldsymbol{v}$ of $V_{2,k}^{\text{face}}(\Omega)$ have a divergence that is *globally* (and not just element-wise) in $L^2(\Omega)$. Hence the normal component of vectors $\boldsymbol{v} \in V_{2,k}^{\text{face}}(\Omega)$ will have to be "continuous" (with obvious meaning) at the inter-element edges. From the local degrees of freedom (3.5)-(3.7) we deduce the global degrees of freedom:

$$\int_e \boldsymbol{v}\cdot\boldsymbol{n}\, p_k \,\mathrm{d}e \qquad \text{for all edge } e, \text{ for all } p_k \in \mathbb{P}_k(e), \quad (3.18)$$

$$\int_E \boldsymbol{v}\cdot\boldsymbol{g}_{k-2}\,\mathrm{d}E \qquad \text{for all element } E, \text{ for all } \boldsymbol{g}_{k-2} \in \mathcal{G}_{k-2}(E), \quad (3.19)$$

$$\int_E \boldsymbol{v}\cdot\boldsymbol{g}_k^{\perp}\,\mathrm{d}E \qquad \text{for all element } E, \text{ for all } \boldsymbol{g}_k^{\perp} \in \mathcal{G}_k^{\perp}(E). \quad (3.20)$$

From the above discussion it follows immediately that the degrees of freedom (3.18)-(3.20) are unisolvent, and that the dimension of $V_{2,k}^{\text{face}}(\Omega)$ is given by

$$\dim(V_{2,k}^{\text{face}}(\Omega)) = \pi_{k,1} \times \{\text{number of edges in } \mathcal{T}_h\} + (2\pi_{k-1,2} - 1) \times \{\text{number of elements in } \mathcal{T}_h\}.$$

4 2D Edge Elements

The edge elements in 2D exactly correspond to the face elements, just rotating everything by $\pi/2$. For the sake of completeness we just recall the definition of the spaces and the corresponding degrees of freedom.

4.1 The local space

On a polygon E we set

$$V_{2,k}^{\text{edge}}(E) := \{\boldsymbol{v} \in H(\text{div};E) \cap H(\text{rot};E) : \boldsymbol{v}\cdot\boldsymbol{t}_{|e} \in \mathbb{P}_k(e)\ \forall \text{ edge } e \text{ of } E, \ \mathbf{rot}\,\text{rot}\,\boldsymbol{v} \in \mathbb{P}_{k-2}(E), \text{ and } \text{div}\,\boldsymbol{v} \in \mathbb{P}_{k-1}(E)\}. \quad (4.1)$$

4.2 The local Degrees of Freedom

A convenient set of degrees of freedom for elements $\boldsymbol{v}$ in $V_{2,k}^{\text{edge}}(E)$ will be:

$$\int_e \boldsymbol{v}\cdot\boldsymbol{t}\, p_k \,\mathrm{d}e \qquad \text{for all edge } e, \text{ for all } p_k \in \mathbb{P}_k(e), \quad (4.2)$$

$$\int_E \boldsymbol{v}\cdot\boldsymbol{r}_{k-2}\,\mathrm{d}E \qquad \text{for all } \boldsymbol{r}_{k-2} \in \mathcal{R}_{k-2}, \quad (4.3)$$

$$\int_E \boldsymbol{v}\cdot\boldsymbol{r}_k^{\perp}\,\mathrm{d}E \qquad \text{for all } \boldsymbol{r}_k^{\perp} \in \mathcal{R}_k^{\perp}. \quad (4.4)$$

Remark 5 Here too we could use alternative degrees of freedom, in analogy with the ones discussed in Remarks 3. In particular we point out that we can identify uniquely an element $\boldsymbol{v}$ of $V_{2,k}^{\mathrm{edge}}(E)$ by prescribing its tangential component $\boldsymbol{v}\cdot\boldsymbol{t}$ (in $\mathbb{P}_k(e)$) on every edge, its rotation $\mathrm{rot}\,\boldsymbol{v}$ (in $(\mathbb{P}_{k-1}(E))/\mathbb{R}$), and its divergence $\mathrm{div}\,\boldsymbol{v}$ (in $\mathbb{P}_{k-1}(E)$).

Remark 6 Obviously, here too we can define the L^2-projection onto $\mathbb{P}_k$, exactly as we did in subsection 3.5, with $\mathcal{R}_k^{\perp}$ taking the role of $\mathcal{G}_k^{\perp}$.

4.3 The global 2D-edge space

Given a polygon Ω and a decomposition $\mathcal{T}_h$ of Ω into a finite number of polygonal elements E, we can now consider the *global* space

$$V_{2,k}^{\mathrm{edge}}(\Omega) := \{\boldsymbol{v}\in H(\mathrm{rot};\Omega) \text{ s. t. } \boldsymbol{v}_{|E}\in V_{2,k}^{\mathrm{edge}}(E)\ \forall \text{ element } E\in\mathcal{T}_h\}. \quad (4.5)$$

Note that the tangential component of vectors $\boldsymbol{v}\in V_{2,k}^{\mathrm{edge}}(\Omega)$ will have to be "continuous" (with obvious meaning) at the inter-element edges. Mimicking what we did for the 2D-face elements, the degrees of freedom for $\boldsymbol{v}\in V_{2,k}^{\mathrm{edge}}(\Omega)$ are the obvious extension of the local d.o.f. (4.2)-(4.4), and the dimension of $V_{2,k}^{\mathrm{edge}}(\Omega)$ is

$$\begin{aligned}\dim(V_{2,k}^{\mathrm{edge}}(\Omega)) = \pi_{k,1}\times\{&\text{number of edges in } \mathcal{T}_h\}+\\ &(2\pi_{k-1,2}-1)\times\{\text{number of elements in } \mathcal{T}_h\}.\end{aligned}$$

5 3D Face Elements

The three-dimensional $H(\mathrm{div})$-conforming spaces follow in a very natural way the path of their two-dimensional companions.

5.1 The local space

On a polyhedron P we set

$$\begin{aligned}V_{3,k}^{\mathrm{face}}(P) := \{\boldsymbol{v}\in H(\mathrm{div};P)\cap H(\mathbf{curl};P) \text{ s. t. } \boldsymbol{v}\cdot\boldsymbol{n}_P^f\in\mathbb{P}_k(f)\,\forall \text{ face } f \text{ of } P,\\ \mathbf{grad}\,\mathrm{div}\,\boldsymbol{v}\in\mathcal{G}_{k-2}(P),\ \ \mathbf{curl}\,\boldsymbol{v}\in\mathcal{R}_{k-1}(P)\}. \quad (5.1)\end{aligned}$$

The dimension of $V_{3,k}^{\mathrm{face}}(P)$ is given by:

$$\dim(V_{3,k}^{\mathrm{face}}(P)) = \ell_f\pi_{k,2}+\gamma_{k-2,3}+\rho_{k-1,3}. \quad (5.2)$$

5.2 The local Degrees of Freedom

The degrees of freedom will be:

$$\int_f \boldsymbol{v} \cdot \boldsymbol{n}_P^f \, p_k \, \mathrm{d}f \qquad \text{for all face } f, \text{ for all } \; p_k \in \mathbb{P}_k(f), \tag{5.3}$$

$$\int_P \boldsymbol{v} \cdot \boldsymbol{g}_{k-2} \, \mathrm{d}P \qquad \text{for all } \boldsymbol{g}_{k-2} \in \mathcal{G}_{k-2}, \tag{5.4}$$

$$\int_P \boldsymbol{v} \cdot \boldsymbol{g}_k^{\perp} \, \mathrm{d}P \qquad \text{for all } \boldsymbol{g}_k^{\perp} \in \mathcal{G}_k^{\perp}. \tag{5.5}$$

A depiction of the degrees of freedom for the case $k = 1$ can be found in Figure 2. It is not difficult to check, using (2.15) and (2.19), that the number of the above degrees of freedom is given by

$$\ell_f \pi_{k,2} + \dim\{\mathcal{G}_{k-2}\} + \dim\{\mathcal{G}_k^{\perp}\} = \ell_f \pi_{k,2} + \gamma_{k-2,3} + \rho_{k-1,3}, \tag{5.6}$$

which equals the dimension of $V_{3,k}^{\text{face}}(P)$ as given in (5.2).

The proof that the above operators are degrees of freedom for the space $V_{3,k}^{\text{face}}(P)$ follows the same steps as in the two dimensional case and is therefore omitted.

Remark 7 We note that also in the three dimensional case there are alternative choices of degrees of freedom, similarly as in Remark 3.

Remark 8 Obviously, here too we can compute the L^2-projection onto $\mathbb{P}_k$, exactly as we did in subsection 3.5.

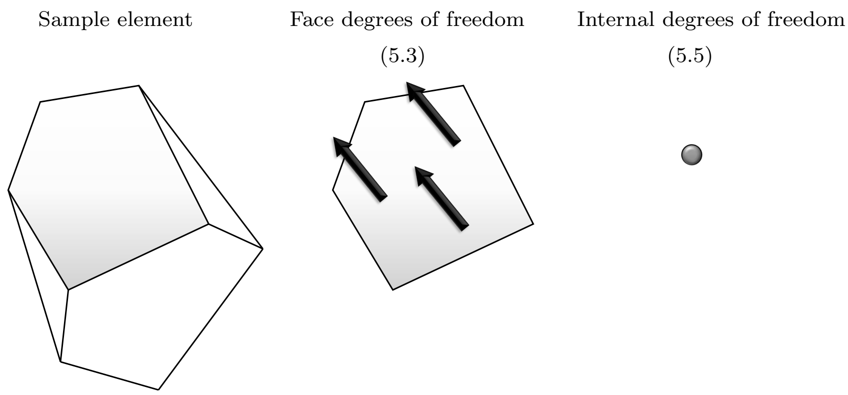

Fig. 2 Representation of the degrees of freedom for 3D face elements, case $k = 1$. Note that the set (5.4) is empty in this case.

5.3 The global 3D-face space

Having now a polyhedron Ω and a decomposition $\mathcal{T}_h$ of Ω into a finite number of polyhedral elements P, we can consider the global space:

$$V_{3,k}^{\mathrm{face}}(\Omega) := \{\boldsymbol{v} \in H(\mathrm{div};\Omega) \text{ s. t. } \boldsymbol{v}_{|P} \in V_{3,k}^{\mathrm{face}}(P) \ \forall \text{ element } P \in \mathcal{T}_h\}. \quad (5.7)$$

As we did for the 2D case, we note that the normal component of the elements of $V_{3,k}^{\mathrm{face}}(\Omega)$ will be "continuous" at the inter-element face. The degrees of freedom for the global space $V_{3,k}^{\mathrm{face}}$ are the obvious extension of the local ones already described, and the dimension of $V_{3,k}^{\mathrm{face}}(\Omega)$ is

$$\dim(V_{3,k}^{\mathrm{face}}(\Omega)) = \pi_{k,2} \times \{\text{number of faces in } \mathcal{T}_h\} + (\pi_{k-1,3} - 1 + \rho_{k-1,3}) \times \{\text{number of elements in } \mathcal{T}_h\}.$$

6 3D Edge Elements

This time we cannot just rotate the $3D$-face case. However we can get some inspiration. We recall, from the very beginning, the Green formula:

$$\int_P \mathbf{curl}\,\boldsymbol{\psi} \cdot \boldsymbol{\varphi} \,\mathrm{d}P = \int_P \boldsymbol{\psi} \cdot \mathbf{curl}\,\boldsymbol{\varphi} \,\mathrm{d}P + \int_{\partial P} \boldsymbol{\psi} \cdot (\boldsymbol{\varphi} \wedge \boldsymbol{n}) \,\mathrm{d}S, \quad (6.1)$$

as well as

$$\int_P \mathbf{curl}\,\boldsymbol{\psi} \cdot \mathbf{curl}\,\boldsymbol{\varphi} \,\mathrm{d}P = \int_P \boldsymbol{\psi} \cdot \Big[-\Delta \boldsymbol{\varphi} + \mathbf{grad}\,\mathrm{div}\,\boldsymbol{\varphi} \Big] \,\mathrm{d}P + \int_{\partial P} \boldsymbol{\psi} \cdot (\mathbf{curl}\,\boldsymbol{\varphi} \wedge \boldsymbol{n}) \,\mathrm{d}S. \quad (6.2)$$

We also recall the observation that we made in Section 2 concerning the difference between $\boldsymbol{\varphi} \wedge \boldsymbol{n}_f$ and $\boldsymbol{\varphi}_f$. We introduce moreover the following space.

Definition 1 We define the boundary space $\mathcal{B}(\partial P)$ as the space of $\boldsymbol{v}$ in $(L^2(\partial P))^3$ such that $\boldsymbol{v}_f \in H(\mathrm{div}; f) \cap H(\mathrm{rot}; f)$ on each face $f \in \partial P$, and such that on each edge e (common to the faces f_1 and f_2), $\boldsymbol{v}_{f_1} \cdot \mathbf{t}_e$ and $\boldsymbol{v}_{f_2} \cdot \mathbf{t}_e$ (where $\mathbf{t}_e$ is a unit tangential vector to e) coincide . Then we define $\mathcal{B}_t(\partial P)$ as the space of the tangential components of the elements of $\mathcal{B}(\partial P)$.

Definition 2 We now define the boundary VEM space $B_k^{\mathrm{edge}}(\partial P)$ as

$$B_k^{\mathrm{edge}}(\partial P) := \big\{ \boldsymbol{v} \in \mathcal{B}_t(\partial P) \text{ such that } \boldsymbol{v}_f \in V_{2,k}^{\mathrm{edge}}(f) \text{ on each face } f \in \partial P \big\}.$$

Recalling the previous discussion on the two-dimensional virtual elements $V_{2,k}^{\mathrm{edge}}(f)$, we can easily see that for a polyhedron with ℓ_e edges and ℓ_f faces the dimension β_k of $B_k^{\mathrm{edge}}(\partial P)$ is given by

$$\beta_k = \ell_e \pi_{k,1} + \ell_f \ (2\pi_{k-1,2} - 1). \quad (6.3)$$

6.1 The local space

On a polyhedron P we set

$$V_{3,k}^{\text{edge}}(P) := \{\boldsymbol{v}|\ \boldsymbol{v}_t \in B_k^{\text{edge}}(\partial P),\\ \operatorname{div}\boldsymbol{v} \in \mathbb{P}_{k-1}(P),\ \text{and}\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \in \mathcal{R}_{k-2}(P)\}. \quad (6.4)$$

6.2 Dimension of the space $V_{3,k}^{\text{edge}}(P)$

We start by observing that, given a vector $\boldsymbol{g}$ in $B_k^{\text{edge}}(\partial P)$, a function f_d in $\mathbb{P}_{k-1}$, and a vector $\boldsymbol{f}_r \in \mathcal{R}_{k-2}(P)$ we can find a unique $\boldsymbol{v}$ in $V_{3,k}^{\text{edge}}(P)$ such that

$$\boldsymbol{v}_t = \boldsymbol{g} \text{ on } \partial P,\ \operatorname{div}\boldsymbol{v} = f_d \text{ in } P,\ \text{and } \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} = \boldsymbol{f}_r \text{ in } P. \quad (6.5)$$

To prove it we consider the following auxiliary problems. The first is: find $\boldsymbol{H}$ in $H(\text{div}; P) \cap H(\mathbf{curl}; P)$ such that

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{f}_r \text{ in } P,\ \operatorname{div}\boldsymbol{H} = 0 \text{ in } P,\ \text{and } \boldsymbol{H}\cdot\boldsymbol{n} = \operatorname{rot}_2\boldsymbol{g} \text{ on } \partial P, \quad (6.6)$$

that is uniquely solvable since

$$\int_{\partial P} \operatorname{rot}_2\boldsymbol{g}\,\mathrm{d}S = 0. \quad (6.7)$$

The second is: find $\boldsymbol{\psi}$ in $H(\text{div}; P) \cap H(\mathbf{curl}; P)$ such that

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{H} \text{ in } P,\ \operatorname{div}\boldsymbol{\psi} = 0 \text{ in } P,\ \text{and } \boldsymbol{\psi}_t = \boldsymbol{g} \text{ on } \partial P, \quad (6.8)$$

that is also uniquely solvable since

$$\boldsymbol{H}\cdot\boldsymbol{n} = \operatorname{rot}_2\boldsymbol{g}. \quad (6.9)$$

The third problem is: find $\varphi \in H_0^1(P)$ such that:

$$\Delta\varphi = f_d \text{ in } P, \quad (6.10)$$

that also has a unique solution. Then it is not difficult to see that the choice

$$\boldsymbol{v} := \boldsymbol{\psi} + \mathbf{grad}\,\varphi \quad (6.11)$$

solves our problem. Indeed, it is clear that $(\mathbf{grad}\,\boldsymbol{\varphi})_t = 0$, that $\operatorname{div}(\mathbf{grad}\,\boldsymbol{\varphi}) = f_d$ and that $\mathbf{curl}\,\mathbf{curl}(\mathbf{grad}\,\boldsymbol{\varphi}) = 0$; all these, added to (6.6) and (6.8), produce the right conditions. It is also clear that the solution $\boldsymbol{v}$ of (6.5) is unique.

Hence we can conclude that the dimension of $V_{3,k}^{\text{edge}}(P)$ is given by

$$\dim(V_{3,k}^{\text{edge}}(P)) = \beta_k + \pi_{k-1,3} + \rho_{k-2,3}. \quad (6.12)$$

6.3 The local Degrees of Freedom.

A possible set of degrees of freedom will be:

– for every edge e:
$$\int_e \boldsymbol{v}\cdot\boldsymbol{t}\,p_k\;\mathrm{d}e \quad \text{for all } \; p_k\in\mathbb{P}_k(e), \tag{6.13}$$

– for every face f:
$$\int_f \boldsymbol{v}\cdot\boldsymbol{r}_k^{\perp}\,\mathrm{d}f \quad \text{for all } \boldsymbol{r}_k^{\perp}\in\mathcal{R}_k^{\perp}(f), \tag{6.14}$$
$$\int_f \boldsymbol{v}\cdot\boldsymbol{r}_{k-2}\,\mathrm{d}f \quad \text{for all } \boldsymbol{r}_{k-2}\in\mathcal{R}_{k-2}(f), \tag{6.15}$$

– and inside P
$$\int_P \boldsymbol{v}\cdot\boldsymbol{r}_k^{\perp}\,\mathrm{d}P \quad \text{for all } \boldsymbol{r}_k^{\perp}\in\mathcal{R}_k^{\perp}, \tag{6.16}$$
$$\int_P \boldsymbol{v}\cdot\boldsymbol{r}_{k-2}\,\mathrm{d}P \quad \text{for all } \boldsymbol{r}_{k-2}\in\mathcal{R}_{k-2}. \tag{6.17}$$

A depiction of the degrees of freedom for the case $k = 1$ can be found in Figure 3. The total number of degrees of freedom (6.13)-(6.15) is clearly equal to β_k as given in (6.3), and the number of degrees of freedom (6.17) is equal to $\rho_{k-2,3}$. On the other hand, using [2.12;v)] we see that the number of degrees of freedom (6.16) is equal to $\pi_{k-1,3}$, so that the total number of degrees of freedom (6.13)-(6.17) equals the dimension of $V_{3,k}^{\mathrm{edge}}(P)$ as computed in (6.12).

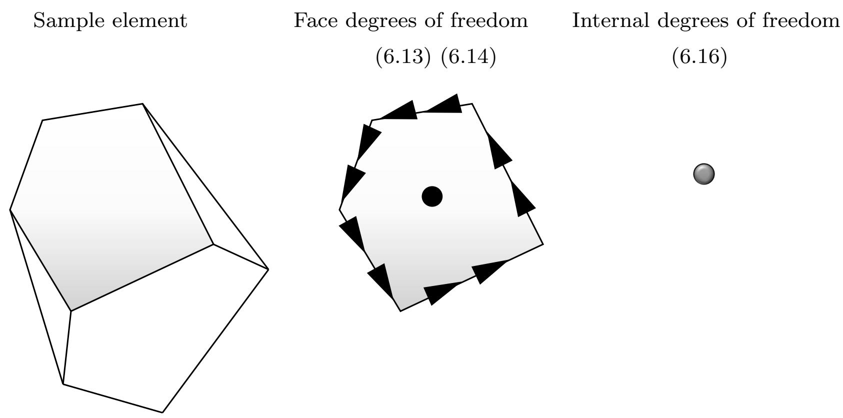

Fig. 3 Representation of the degrees of freedom for 3D edge elements, case $k = 1$. Note that the sets (6.15), (6.17) are empty in this case.

6.4 Unisolvence.

Having seen that the number of degrees of freedom (6.13)-(6.17) equals the dimension of $V_{3,k}^{\mathrm{edge}}(P)$, in order to see their unisolvence we only need to check that a vector $\boldsymbol{v} \in V^{\text{edge}}_{3,k}(P)$ that satisfies

$$\int_e \boldsymbol{v} \cdot \boldsymbol{t}\, p_k \,\mathrm{d}e = 0 \qquad \forall \text{ edge } e \text{ of } P \text{ and } \forall\, p_k \in \mathbb{P}_k(e), \tag{6.18}$$

$$\int_f \boldsymbol{v} \cdot \boldsymbol{r}_k^{\perp} \,\mathrm{d}f = 0 \qquad \forall \text{ face } f \text{ of } P \text{ and } \forall\, \boldsymbol{r}_k^{\perp} \in \mathcal{R}_k^{\perp}(f), \tag{6.19}$$

$$\int_f \boldsymbol{v} \cdot \boldsymbol{r}_{k-2} \,\mathrm{d}f = 0 \qquad \forall \text{ face } f \text{ of } P \text{ and } \forall\, \boldsymbol{r}_{k-2} \in \mathcal{R}_{k-2}(f), \tag{6.20}$$

$$\int_P \boldsymbol{v} \cdot \boldsymbol{r}_k^{\perp} \,\mathrm{d}P = 0 \qquad \forall\, \boldsymbol{r}_k^{\perp} \in \mathcal{R}_k^{\perp}(P), \tag{6.21}$$

$$\int_P \boldsymbol{v} \cdot \boldsymbol{r}_{k-2} \,\mathrm{d}P = 0 \qquad \forall\, \boldsymbol{r}_{k-2} \in \mathcal{R}_{k-2}(P), \tag{6.22}$$

is necessarily equal to zero.

Actually, recalling the results of Section 4, it is pretty obvious that (6.18)-(6.20) imply that $\boldsymbol{v}_t = 0$ on ∂P. Moreover, since $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \in \mathcal{R}_{k-2}(P)$, we are allowed to take $\boldsymbol{r}_{k-2} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}$ as a test function in (6.22). An integration by parts (using $\boldsymbol{v}_t = 0$) gives

$$0 = \int_P \boldsymbol{v} \cdot \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \,\mathrm{d}P = \int_P (\mathbf{curl}\,\boldsymbol{v}) \cdot (\mathbf{curl}\,\boldsymbol{v}) \,\mathrm{d}P \tag{6.23}$$

and therefore we get $\mathbf{curl}\,\boldsymbol{v} = 0$. Using this, and again $\boldsymbol{v}_t = 0$, we easily check, integrating by parts, that

$$\int_P \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} \,\mathrm{d}P = 0 \quad \forall \boldsymbol{\varphi} \in H(\mathbf{curl}; P). \tag{6.24}$$

Now we recall that from the definition (6.4) of $V^{\text{edge}}_{3,k}(P)$ we have that $\operatorname{div} \boldsymbol{v}$ is in $\mathbb{P}_{k-1}$. From [(2.12);v] we then deduce that there exists a $\boldsymbol{q}_k^{\perp} \in \mathcal{R}_k^{\perp}$ with $\operatorname{div} \boldsymbol{q}_k^{\perp} = \operatorname{div} \boldsymbol{v}$, so that the divergence of $\boldsymbol{v} - \boldsymbol{q}_k^{\perp}$ is zero, and then (since P is simply connected)

$$\boldsymbol{v} - \boldsymbol{q}_k^{\perp} = \mathbf{curl}\,\boldsymbol{\varphi} \tag{6.25}$$

for some $\boldsymbol{\varphi} \in H(\mathbf{curl}; P)$. At this point we can use (6.24) and (6.25) to conclude as in (3.14)

$$\int_P |\boldsymbol{v}|^2 \,\mathrm{d}P = \int_P \boldsymbol{v} \cdot (\boldsymbol{q}_k^{\perp} + \mathbf{curl}\,\boldsymbol{\varphi}) \,\mathrm{d}P = \int_P \boldsymbol{v} \cdot \boldsymbol{q}_k^{\perp} \,\mathrm{d}P + \int_P \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} \,\mathrm{d}P = 0.$$

6.5 Alternative degrees of freedom

As we did in the previous cases, we observe that the degrees of freedom (6.13)-(6.17) are not (by far) the only possible choice. To start with, we can change the degrees of freedom in each face, according to Remark 3. Moreover, in the spirit of (6.5) we could assign, instead of (6.16) and/or (6.17), $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}$ in $\mathcal{R}_{k-2}(P)$ and/or $\operatorname{div} \boldsymbol{v}$ in $\mathbb{P}_{k-1}(P)$, respectively.

6.6 The global 3D-edge space

Here too we can assume that we have a polyhedral domain Ω and its decomposition $\mathcal{T}_h$ in a finite number of polyhedra P. In this case we can define the global space

$$V_{3,k}^{\mathrm{edge}}(\Omega) := \{\boldsymbol{v} \in H(\mathbf{curl};\Omega) \text{ s. t. } \boldsymbol{v}_{|P} \in V_{3,k}^{\mathrm{edge}}(P) \forall \, \text{element } P \in \mathcal{T}_h\}. \quad (6.26)$$

Accordingly, we could take, as degrees of freedom:

– for every edge e in $\mathcal{T}_h$:

$$\int_e \boldsymbol{v}\cdot\boldsymbol{t}\, p_k \,\mathrm{d}e \quad \text{ for all } \; p_k \in \mathbb{P}_k(e), \quad (6.27)$$

– for every face f in $\mathcal{T}_h$:

$$\int_f \boldsymbol{v}_f \cdot \boldsymbol{r}_k^{\perp} \,\mathrm{d}f \quad \text{ for all } \boldsymbol{r}_k^{\perp} \in \mathcal{R}_k^{\perp}(f), \quad (6.28)$$

$$\int_f \boldsymbol{v}_f \cdot \boldsymbol{r}_{k-2} \,\mathrm{d}f \quad \text{ for all } \boldsymbol{r}_{k-2} \in \mathcal{R}_{k-2}(f) \quad (6.29)$$

– and for every element P in $\mathcal{T}_h$

$$\int_P \boldsymbol{v} \cdot \boldsymbol{r}_k^{\perp} \,\mathrm{d}P \quad \text{ for all } \boldsymbol{r}_k^{\perp} \in \mathcal{R}_k^{\perp}, \quad (6.30)$$

$$\int_P \boldsymbol{v} \cdot \boldsymbol{r}_{k-2} \,\mathrm{d}P \quad \text{ for all } \boldsymbol{r}_{k-2} \in \mathcal{R}_{k-2}. \quad (6.31)$$

From the above discussion it follows immediately that the degrees of freedom (6.27)-(6.31) are unisolvent, and that the dimension of $V_{3,k}^{\mathrm{edge}}(\Omega)$ is

$$\begin{aligned}\dim(V_{3,k}^{\mathrm{edge}}(\Omega)) = \pi_{k,1} &\times \{\text{number of edges in } \mathcal{T}_h\} \\ &+ (2\pi_{k-1,2} - 1) \times \{\text{number of faces in } \mathcal{T}_h\} \\ &\quad + (\pi_{k-1,3} + \rho_{k-1,3}) \times \{\text{number of elements in } \mathcal{T}_h\}.\end{aligned}$$

6.7 An enhanced edge space

It is immediate to check that the degrees of freedom (6.16)-(6.17) allow to compute the moments of $\boldsymbol{v} \in V_{3,k}^{\mathrm{edge}}(P)$ up to order $k-2$. Nevertheless, in order to be able to compute the $L^2(P)$ projection operator on the space $(\mathbb{P}_k(P))^3$ we need to be able to compute the moments up to order k. In the present section, in the spirit of [1], we will introduce an enhanced space $W_{3,k}^{\mathrm{edge}}(P)$ with the additional property that the L^2 projector on $(\mathbb{P}_k(P))^3$ is computable.

We consider the larger virtual space

$$\begin{aligned}\widetilde{V}_{3,k}^{\mathrm{edge}}(P) := \{\boldsymbol{v} | \; \boldsymbol{v}_{|\partial P} \in B_k^{\mathrm{edge}}(\partial P), \operatorname{div}\boldsymbol{v} \in \mathbb{P}_{k-1}(P), \\ \text{and } \; \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \in \mathcal{R}_k(P)\}. \quad (6.32)\end{aligned}$$

Following the same identical arguments used in the previous section and introducing the space

$$\mathcal{R}_k/\mathcal{R}_{k-2}(P) := \big\{\boldsymbol{q}_k \in \mathcal{R}_k \ : \ \int_P \boldsymbol{q}_k \cdot \boldsymbol{r}_{k-2} \,\mathrm{d}P = 0 \ \forall \boldsymbol{r}_{k-2} \in \mathcal{R}_{k-2}\big\},$$

it is immediate to check that (6.13)-(6.17), with the addition of

$$\int_P \boldsymbol{v} \cdot \boldsymbol{q}_k \,\mathrm{d}P \quad \text{ for all } \boldsymbol{q}_k \in \mathcal{R}_k/\mathcal{R}_{k-2}(P), \tag{6.33}$$

constitute a set of degrees of freedom for $\widetilde{V}^{\text{edge}}_{3,k}(P)$. Note moreover that $V^{\text{edge}}_{3,k}(P)$ is a subset of $\widetilde{V}^{\text{edge}}_{3,k}(P)$ and that the combination of (6.16), (6.17) and (6.33) allows, for any function in $\widetilde{V}^{\text{edge}}_{3,k}(P)$, to compute all the integrals against polynomials in $\mathbb{P}_k(P)$. Therefore the L^2 projection operator

$$\Pi^0_k \ : \ \widetilde{V}^{\text{edge}}_{3,k}(P) \to \Big(\mathbb{P}_k(P)\Big)^3$$

is computable.

For the time being we *assume* the existence of a projection operator

$$\widetilde{\Pi}_k \ : \ \widetilde{V}^{\text{edge}}_{3,k}(P) \to \Big(\mathbb{P}_k(P)\Big)^3, \tag{6.34}$$

with the fundamental property of depending *only* on the degrees of freedom (6.13)-(6.17) (meaning that if $\boldsymbol{v}$ satisfies (6.18)-(6.22) then $\widetilde{\Pi}_k \boldsymbol{v} = 0$). We now introduce the space

$$W^{\text{edge}}_{3,k}(P) := \{\boldsymbol{v} \in \widetilde{V}^{\text{edge}}_{3,k}(P) \text{ such that:} \\ \int_P (\widetilde{\Pi}_k \boldsymbol{v}) \cdot \boldsymbol{q}_k \,\mathrm{d}P = \int_P (\Pi^0_k \boldsymbol{v}) \cdot \boldsymbol{q}_k \,\mathrm{d}P \quad \forall \boldsymbol{q}_k \in \mathcal{R}_k/\mathcal{R}_{k-2}(P)\}. \tag{6.35}$$

We then have the following lemma.

Lemma 3 *The dimension of the space $W^{\text{edge}}_{3,k}(P)$ is equal to the dimension of the original edge space $V^{\text{edge}}_{3,k}(P)$. Moreover, the operators* (6.13)-(6.17) *constitute a set of degrees of freedom for $W^{\text{edge}}_{3,k}(P)$.*

Proof By definition of $W^{\text{edge}}_{3,k}(P)$ we have

$$\dim\Big(W^{\text{edge}}_{3,k}(P)\Big) \geq \dim\Big(\widetilde{V}^{\text{edge}}_{3,k}(P)\Big) - \dim\Big(\mathcal{R}_k/\mathcal{R}_{k-2}(P)\Big) = \dim\Big(V^{\text{edge}}_{3,k}(P)\Big).$$

Therefore, in order to conclude the lemma, it is sufficient to show the unisolvence of (6.13)-(6.17). For this, let $\boldsymbol{v} \in W^{\text{edge}}_{3,k}(P)$ satisfying (6.18)-(6.22). Note that, by the previously mentioned property of the (linear) projection operator

$\widetilde{\Pi}_k$, we immediately have that $\widetilde{\Pi}_k(\boldsymbol{v})$ is equal to 0. Therefore, by definition of $W^{\mathrm{edge}}_{3,k}(P)$, for all $\boldsymbol{q}_k \in \mathcal{R}_k/\mathcal{R}_{k-2}(P)$ it holds

$$\int_P \boldsymbol{v}\cdot\boldsymbol{q}_k\,\mathrm{d}P = \int_P \Big(\Pi^0_k\boldsymbol{v}\Big)\cdot\boldsymbol{q}_k\,\mathrm{d}P = \int_P \Big(\widetilde{\Pi}_k\boldsymbol{v}\Big)\cdot\boldsymbol{q}_k\,\mathrm{d}P = 0. \tag{6.36}$$

Since $W^{\mathrm{edge}}_{3,k}(P) \subseteq \widetilde{V}^{\mathrm{edge}}_{3,k}(P)$ and the set of degrees of freedom (6.13)-(6.17) plus (6.33) is unisolvent for $\widetilde{V}^{\mathrm{edge}}_{3,k}(P)$, we conclude that (6.18)-(6.22) plus (6.36) imply $\boldsymbol{v} = 0$.

Note that, due to the above lemma, the enhanced space $W^{\mathrm{edge}}_{3,k}(P)$ has the same degrees of freedom as $V^{\mathrm{edge}}_{3,k}(P)$. Moreover, since the condition in (6.35) is satisfied by polynomials of degree k, we still have $(\mathbb{P}_k(P))^3 \subseteq W^{\mathrm{edge}}_{3,k}(P)$. The advantage of the space $W^{\mathrm{edge}}_{3,k}(P)$ with respect to $V^{\mathrm{edge}}_{3,k}(P)$ is that in $W^{\mathrm{edge}}_{3,k}(P)$ we can compute all the moments of order up to k. Indeed, the moments

$$\int_P \boldsymbol{v}\cdot\boldsymbol{q}_{k-2}\,\mathrm{d}P \quad \text{for all } \boldsymbol{q}_{k-2}\in\mathcal{R}_{k-2}(P),$$
$$\int_P \boldsymbol{v}\cdot\boldsymbol{q}_k^{\perp}\,\mathrm{d}P \quad \text{for all } \boldsymbol{q}_k\in\mathcal{R}^{\perp}_k(P)$$

can be computed using the degrees of freedom (6.16) and (6.17), while

$$\int_P \boldsymbol{v}\cdot\boldsymbol{q}_k\,\mathrm{d}P = \int_P \Big(\Pi^0_k\boldsymbol{v}\Big)\cdot\boldsymbol{q}_k\,\mathrm{d}P = \int_P \Big(\widetilde{\Pi}_k\boldsymbol{v}\Big)\cdot\boldsymbol{q}_k\,\mathrm{d}P \tag{6.37}$$

for all $\boldsymbol{q}_k \in \mathcal{R}_k/\mathcal{R}_{k-2}(P)$.

We are therefore left with the duty to build a projection operator $\widetilde{\Pi}_k$ as in (6.34). Let N denote the dimension of the space $V^{\mathrm{edge}}_{3,k}(P)$, i.e. the number of degrees of freedom (6.13)-(6.17). Let us introduce the operator

$$\mathcal{D} : \widetilde{V}^{\mathrm{edge}}_{3,k}(P) \longrightarrow \mathbb{R}^N$$

that associates, to any $\boldsymbol{v} \in \widetilde{V}^{\mathrm{edge}}_{3,k}(P)$, a vector with components given by the evaluation of all the (ordered) operators (6.13)-(6.17) on $\boldsymbol{v}$ (in other words, $\mathcal{D}$ associates to every element of $\widetilde{V}^{\mathrm{edge}}_{3,k}(P)$ its "first N" degrees of freedom). Note that the operator $\mathcal{D}$ is not injective (as the dimension of $\widetilde{V}^{\mathrm{edge}}_{3,k}(P)$ is *bigger* than that of $V^{\mathrm{edge}}_{3,k}$, that in turn is equal to N). On the other hand, since $(\mathbb{P}_k(P))^3 \subseteq V^{\mathrm{edge}}_{3,k}(P)$ and the above N operators are a set of degrees of freedom for $V^{\mathrm{edge}}_{3,k}(P)$, the operator $\mathcal{D}$ *restricted to* $(\mathbb{P}_k(P))^3$ *is injective.* Given now *any* symmetric and positive definite bilinear form $\mathcal{S}$ defined on $\mathbb{R}^N\times\mathbb{R}^N$ we define the projection operator $\widetilde{\Pi}^{\mathcal{S}}_k$ as follows. For all $\boldsymbol{v}\in\widetilde{V}^{\mathrm{edge}}_{3,k}(P)$:

$$\begin{cases} \widetilde{\Pi}^{\mathcal{S}}_k\boldsymbol{v} \in \big(\mathbb{P}_k(P)\big)^3 \\ \mathcal{S}\Big(\mathcal{D}\,\widetilde{\Pi}^{\mathcal{S}}_k\boldsymbol{v} - \mathcal{D}\boldsymbol{v}, \mathcal{D}\mathbf{q}_k\Big) = 0 \qquad \forall \mathbf{q}_k \in \big(\mathbb{P}_k(P)\big)^3. \end{cases} \tag{6.38}$$

By recalling that $\mathcal{D}$ is injective on $(\mathbb{P}_k(P))^3$, it is immediate to check that the above operator is well defined. Moreover, by definition it depends only on the degrees of freedom (6.13)-(6.17).

Remark 9 Our construction is pretty general. Actually it is not difficult to prove that for every projector $\mathcal{P}$ onto $(\mathbb{P}_k(P))^3$ depending only on the degrees of freedom (6.13)-(6.17) we can find a bilinear symmetric positive definite form $\mathcal{S}$ such that $\mathcal{P} = \widetilde{\Pi}_k^{\mathcal{S}}$.

Remark 10 The construction of the enhanced space $W_{3,k}^{\text{edge}}(P)$ has basically a theoretical interest. In practice (meaning, in writing the code) one does not even need to know what this space is. If one needs to use the L^2 projection of the elements of $V_{3,k}^{\text{edge}}$, one can just use the construction (6.38) (typically, with $\mathcal{S}$ equal to the Euclidean scalar product in $\mathbb{R}^N$) in order to define $\widetilde{\Pi}_k$, and then (6.37) to get the L^2 projection. For stability reasons in practical problems, what is really important is that the degrees of freedom $\mathcal{D}$ scale all in the same way (see. for instance, [10] and [12]), and that the eigenvalues of $\mathcal{S}$ are uniformly bounded from above and from below.

Remark 11 The construction (6.38) could be used with any operator $\mathcal{D}$ that, restricted to $(\mathbb{P}_k(P))^d$, is injective. This could be used in order to eliminate some internal degrees of freedom without loosing the inclusion of $(\mathbb{P}_k(P))^d$ in the VEM-space.

7 Scalar VEM spaces

In the present section we restrict ourselves to the three dimensional case, the two dimensional one being simpler and analogous. We denote as usual with P a generic polyhedron.

7.1 VEM vertex elements

Let us brieflyy recall the H^1-conforming scalar space introduced in [10], here generalized to three dimensions. For computing the L^2-projection in this case we refer to [1]. Let k be, as usual, an integer ≥ 1.

Definition 3 We define $B_k^{\text{vert}}(\partial P)$ as the set of functions $v \in C^0(\partial P)$ such that $v_{|e} \in \mathbb{P}_k(e)$ on each edge $e \in \partial P$, and $\Delta_2 v_{|f} \in \mathbb{P}_{k-2}(f)$ on each face $f \in \partial P$, where Δ_2 is the planar Laplace operator on f.

We introduce the family of local vertex spaces $V_{3,k}^{\text{vert}}(P) \subset H^1(P)$ as

$$V_{3,k}^{\text{vert}}(P) := \{v |\ v_{|\partial P} \in B_k^{\text{vert}}(\partial P) \text{ and } \Delta v \in \mathbb{P}_{k-2}(P)\}, \tag{7.1}$$

with the associated set of degrees of freedom:

- the pointwise value $v(\nu)$ for all vertex ν, (7.2)
- $\int_e v\, p_{k-2}\, \mathrm{d}e$ for all edge e, for all $p_{k-2} \in \mathbb{P}_{k-2}(e)$, (7.3)
- $\int_f v\, p_{k-2}\, \mathrm{d}f$ for all face f, for all $p_{k-2} \in \mathbb{P}_{k-2}(f)$, (7.4)
- $\int_P v\, p_{k-2}\, \mathrm{d}P \quad$ for all $p_{k-2} \in \mathbb{P}_{k-2}(P)$. (7.5)

The dimension of the space is thus given by

$$\dim\big(V^{\mathrm{vert}}_{3,k}(P)\big) = \ell_v + \ell_e \pi_{k-2,1} + \ell_f \pi_{k-2,2} + \pi_{k-2,3}\,. \tag{7.6}$$

Remark 12 The following observation follows immediately applying the results for the two dimensional space in [10]. Given any face $f \in \partial P$ the operators (7.4) combined with the operators (7.2)-(7.3) restricted to the vertices and edges of ∂f, form a set of degrees of freedom for the restriction of $V^{\mathrm{vert}}_{3,k}(P)$ to the face f.

As in the above section we can also consider the global spaces. Assuming that we have a polyhedral domain Ω and a decomposition $\mathcal{T}_h$ in a finite number of polyhedra P, we can define the global space

$$V^{\mathrm{vert}}_{3,k}(\Omega) := \{v \in H^1(\Omega) \text{ s. t. } v_{|P} \in V^{\mathrm{vert}}_{3,k}(P) \text{ for all elements } P \in \mathcal{T}_h\}, \tag{7.7}$$

with the associated set of degrees of freedom:

- the pointwise value $v(\nu)$ for all vertex ν, (7.8)
- $\int_e v\, p_{k-2}\, \mathrm{d}e$ for all edge e, for all $p_{k-2} \in \mathbb{P}_{k-2}(e)$, (7.9)
- $\int_f v\, p_{k-2}\mathrm{d}f$ for all face f, for all $p_{k-2} \in \mathbb{P}_{k-2}(f)$, (7.10)
- $\int_P v\, p_{k-2}\, \mathrm{d}P \quad$ for all element P, for all $p_{k-2} \in \mathbb{P}_{k-2}(P)$. (7.11)

Note that, thanks to the observation in Remark 12, the above degrees of freedom indeed guarantee the continuous gluing of the local spaces. In other words, the condition $V^{\mathrm{vert}}_{3,k}(\Omega) \subseteq H^1(\Omega)$ is compatible with the choice of degrees of freedom. The dimension of the global space is given by

$$\dim\big(V^{\mathrm{vert}}_{3,k}(\Omega)\big) = \{\text{number of vertices } \in \mathcal{T}_h\} + \pi_{k-2,1} \times \{\text{number of edges } \in \mathcal{T}_h\} + \pi_{k-2,2} \times \{\text{number of faces } \in \mathcal{T}_h\} + \pi_{k-2,3} \times \{\text{number of elements } \in \mathcal{T}_h\}.$$

7.2 VEM volume elements

We finally introduce, for all integer $k \geq 0$, the family of volume spaces $V_{3,K}^{\text{elem}}(P) := \mathbb{P}_k(P) \subset L^2(P)$, with the associated degrees of freedom

$$\int_P v\, p_k \,\mathrm{d}P \qquad \text{for all } p_k \in \mathbb{P}_k(P).$$

This is a actually a space of polynomials (like the ones used, for instance, in Discontinuous Galerkin methods), and to deal with it does not require any particular care. The corresponding global space will be

$$V_{3,k}^{\text{elem}}(\Omega) = \{v \in L^2(\Omega) \text{ such that } v_{|P} \in \mathbb{P}_k(P)\, \forall \text{ element } P \in \mathcal{T}_h\}. \qquad (7.12)$$

8 Virtual exact sequences

We show now that, for the obvious choices of the polynomial degrees, the set of virtual spaces introduced in this paper constitutes an exact sequence. We start with the (simpler) two-dimensional case. Let $V_{2,k}^{\text{vert}}(\Omega)$ denote the same H^1 conforming space introduced in [10] and $V_{2,k}^{\text{elem}}(\Omega) = \mathbb{P}_{k-2}(\Omega)$ the obvious two dimensional counterpart of (7.12).

Theorem 3 *Let $k \geq 2$, and assume that Ω is a simply connected polygon, decomposed in a finite number of polygons E. Then the sequences*

$$\mathbb{R} \xrightarrow{\;i\;} V_{2,k}^{\text{vert}}(\Omega) \xrightarrow{\;\mathbf{grad}\;} V_{2,k-1}^{\text{edge}}(\Omega) \xrightarrow{\;\text{rot}\;} \mathbb{P}_{k-2}(\Omega) \xrightarrow{\;o\;} 0 \qquad (8.1)$$

and

$$\mathbb{R} \xrightarrow{\;i\;} V_{2,k}^{\text{vert}}(\Omega) \xrightarrow{\;\mathbf{rot}\;} V_{3,k-1}^{\text{edge}}(\Omega) \xrightarrow{\;\text{div}\;} \mathbb{P}_{k-2}(\Omega) \xrightarrow{\;o\;} 0 \qquad (8.2)$$

are both exact sequences.

Proof We note first that the two sequences are practically the same, up to a rotation of $\pi/2$. Hence we will just show the exactness of the sequence (8.1). Essentially, the only non-trivial part will be to show that

- **a.1** for every $\boldsymbol{v} \in V_{2,k-1}^{\text{edge}}(\Omega)$ with $\text{rot}\,\boldsymbol{v} = 0$ there exists a $\varphi \in V_{2,k}^{\text{vert}}(\Omega)$ such that $\mathbf{grad}\,\varphi = \boldsymbol{v}$.
- **a.2** for every $q \in V_{2,k-2}^{\text{elem}}(\Omega)$ there exists a $\boldsymbol{v} \in V_{2,k-1}^{\text{edge}}(\Omega)$ such that $\text{rot}\,\boldsymbol{v} = q$.

We start with **a.1**. As Ω is simply connected, we have that the condition $\text{rot}\,\boldsymbol{v} = 0$ implies that there exists a function $\varphi \in H^1(\Omega)$ such that $\mathbf{grad}\,\varphi = \boldsymbol{v}$ in Ω. On every edge e of $\mathcal{T}_h$ such φ will obviously satisfy, as well:

$$\frac{\partial \varphi}{\partial \boldsymbol{t}_e} = \boldsymbol{v} \cdot \boldsymbol{t}_e \in \mathbb{P}_{k-1}(e). \qquad (8.3)$$

Then the restriction of φ to each $E \in \mathcal{T}_h$ verifies:

$$\varphi_{|e} \in \mathbb{P}_k(e)\ \forall e \in \partial E; \qquad \Delta\varphi \equiv \operatorname{div} \boldsymbol{v} \in \mathbb{P}_{k-2}(E) \tag{8.4}$$

so that clearly $\varphi \in V_{2,k}^{\mathrm{vert}}(\Omega)$.

To deal with **a.2**, we first construct a $\boldsymbol{\varphi}$ in $(H^1(\Omega))^2$ such that $\operatorname{rot}\boldsymbol{\varphi} = q$ and

$$\boldsymbol{\varphi} \cdot \boldsymbol{t} = \frac{\int_\Omega q\,\mathrm{d}x}{|\partial\Omega|} \quad \text{ on } \partial\Omega, \tag{8.5}$$

where $\boldsymbol{t}$ is the unit counterclockwise tangent vector to $\partial\Omega$ and $|\partial\Omega|$ is the length of $\partial\Omega$. Then we consider the element $\boldsymbol{v} \in V_{2,k-1}^{\mathrm{edge}}(\Omega)$ such that

$$\boldsymbol{v} \cdot \boldsymbol{t}_e := \Pi_{k-1}^0(\boldsymbol{\varphi} \cdot \boldsymbol{t}_e)\ \forall \text{ edge } e \text{ in } \mathcal{T}_h \tag{8.6}$$

and, within each element E:

$$\operatorname{rot}\boldsymbol{v} = \operatorname{rot}\boldsymbol{\varphi} = q, \qquad \operatorname{div}\boldsymbol{v} = 0. \tag{8.7}$$

Clearly such a $\boldsymbol{v}$ solves the problem.

Remark 13 The construction in the proof of **a.2** could also be done if the two-dimensional domain Ω is a *closed surface*, obtained as union of polygons. To fix the ideas, assume that we deal with the boundary ∂P of a polyhedron P, and that we are given on every face f of P a polynomial q_f of degree $k-2$, in such a way that

$$\sum_{f\in\partial P} \int_f q_f \mathrm{d}f = 0. \tag{8.8}$$

Then there exists an element $\boldsymbol{v} \in B_{k-1}^{\mathrm{edge}}(\partial P)$ such that on each face f we have $\operatorname{rot}_2(\boldsymbol{v}_{|f}) = q_f$. To see that this is true, we define first, for each face f, the number

$$\tau_f := \int_f q_f \mathrm{d}f.$$

Then we fix, on each edge e, an orientation $\boldsymbol{t}_e$, we orient each face f with the outward normal, and we define, *for each edge e of f*, the counterclockwise tangent unit vector $\boldsymbol{t}_c^f$. Then we consider the *combinatorial problem* (defined on the topological decomposition $\mathcal{T}_h$) of finding for each edge e a real number σ_e such that for each face f

$$\sum_{e\in\partial f} \sigma_e \boldsymbol{t}_e \cdot \boldsymbol{t}_c^f = \tau_f. \tag{8.9}$$

This could be solved using the same approach used in the above proof, applied on a flat polygonal decomposition that is topologically equivalent to the decomposition of ∂P without one face. The last face will fit automatically, due to (8.8). Then we take $\boldsymbol{v}$ such that on each edge $\boldsymbol{v}\cdot\boldsymbol{t} \in \mathbb{P}_{k-1}$ with $\int_e \boldsymbol{v}\cdot\boldsymbol{t}_e\,\mathrm{d}e = \sigma_e$, and for each face, $\operatorname{div}\boldsymbol{v}_f = 0$, $\operatorname{rot}\boldsymbol{v}_f = q_f$.

We are now ready to consider the three-dimensional case.

Theorem 4 *Let $k \geq 3$, and assume that Ω is a simply connected polyhedron, decomposed in a finite number of polyhedra P. Then the sequence*

$$\mathbb{R} \xrightarrow{i} V^{\text{vert}}_{3,k}(\Omega) \xrightarrow{\textbf{grad}} V^{\text{edge}}_{3,k-1}(\Omega) \xrightarrow{\textbf{curl}} V^{\text{face}}_{3,k-2}(\Omega) \xrightarrow{\text{div}} \mathbb{P}_{k-3}(\Omega) \xrightarrow{o} 0 \quad (8.10)$$

is exact.

Proof It is pretty much obvious, looking at the definitions of the spaces, that

- a constant function is in $V^{\text{vert}}_{3,k}(\Omega)$ and has zero gradient,
- the gradient of a function of $V^{\text{vert}}_{3,k}(\Omega)$ is in $V^{\text{edge}}_{3,k-1}(\Omega)$ and has zero **curl**,
- the **curl** of a vector in $V^{\text{edge}}_{3,k-1}(\Omega)$ is in $V^{\text{face}}_{3,k-2}(\Omega)$ and has zero divergence,
- the divergence of a vector of $V^{\text{face}}_{3,k-2}(\Omega)$ is in $V^{\text{elem}}_{3,k-3}(\Omega)$.

Hence, essentially, we have to prove that:

- **b.1** for every $\boldsymbol{v} \in V^{\text{edge}}_{3,k-1}(\Omega)$ with $\textbf{curl}\,\boldsymbol{v} = 0$ there exists a $\varphi \in V^{\text{vert}}_{3,k}$ such that $\textbf{grad}\,\varphi = \boldsymbol{v}$.
- **b.2** for every $\boldsymbol{\tau} \in V^{\text{face}}_{3,k-2}(\Omega)$ with $\text{div}\,\boldsymbol{\tau} = 0$ there exists a $\boldsymbol{\varphi} \in V^{\text{edge}}_{3,k-1}(\Omega)$ such that $\textbf{curl}\,\boldsymbol{\varphi} = \boldsymbol{\tau}$
- **b.3** for every $q \in V^{\text{elem}}_{3,k-3}(\Omega)$ there exists a $\boldsymbol{\sigma} \in V^{\text{face}}_{3,k-2}$ such that $\text{div}\,\boldsymbol{\sigma} = q$.

The proof of **b.1** is immediate, as in the two-dimensional case [2.1]: the function (unique up to a constant) φ such that $\textbf{grad}\,\varphi = \boldsymbol{v}$ will verify (8.3) on each edge. Moreover, its restriction φ_f to each face f will satisfy $\textbf{grad}_2\,\varphi = \boldsymbol{v}_f$, and so on.

Let us therefore look at **b.2**. Given $\boldsymbol{\tau} \in V^{\text{face}}_{3,k-2}(\Omega)$ with $\text{div}\,\boldsymbol{\tau} = 0$ we first consider (as in Remark 13) the element $\boldsymbol{g} \in B^{edge}_{k-1}(\partial\Omega)$ such that, on each face $f \in \partial\Omega$

$$\text{rot}_2(\boldsymbol{g}_{|f}) = \boldsymbol{\tau}\cdot\boldsymbol{n}\ (\in \mathbb{P}_{k-2}(f)). \quad (8.11)$$

Note that

$$\sum_{f\in\partial\Omega}\int_f \boldsymbol{\tau}\cdot\boldsymbol{n}^f_\Omega\,\mathrm{d}f = \int_\Omega \text{div}\,\boldsymbol{\tau}\,\mathrm{d}\Omega\ =\ 0, \quad (8.12)$$

so that the compatibility condition (8.8) is satisfied. Then we solve in Ω the $Div - Curl$ problem

$$\text{div}\,\boldsymbol{\psi} = 0 \quad\text{and}\quad \textbf{curl}\,\boldsymbol{\psi} = \boldsymbol{\tau} \quad\text{in } \Omega, \qquad\text{with}\quad \boldsymbol{\psi}_t = \boldsymbol{g} \quad\text{on } \partial\Omega. \quad (8.13)$$

The (unique) solution of (8.13) has enough regularity to take the trace of its tangential component on each edge e, and therefore, after deciding an orientation $\boldsymbol{t}_e$ for every edge e in $\mathcal{T}_h$, we can take

$$\eta_e := \Pi^0_{k-1}(\boldsymbol{\psi}\cdot\boldsymbol{t}_e) \qquad \text{on each edge } e \text{ in } \mathcal{T}_h. \quad (8.14)$$

At this point, for each element P we construct $\boldsymbol{\varphi} \in B^{\mathrm{edge}}_{k-1}(\partial P)$ by requiring that

$$\boldsymbol{\varphi}\cdot\boldsymbol{t}_e = \eta_e \text{ on each edge,}$$
$$\mathrm{rot}_2\,\boldsymbol{\varphi}_f = \boldsymbol{\tau}\cdot\boldsymbol{n}^f_P \text{ and } \mathrm{div}\,\boldsymbol{\varphi}_f = 0 \text{ in each face } f \in \partial P. \quad (8.15)$$

Then we can define $\boldsymbol{\varphi}$ inside each element by choosing, together with (8.15),

$$\mathbf{curl}\,\boldsymbol{\varphi} = \boldsymbol{\tau} \text{ and } \mathrm{div}\,\boldsymbol{\varphi} = 0 \quad \text{in each element } P. \quad (8.16)$$

It is easy to see that the boundary conditions given in (8.15) are *compatible* with the requirement $\mathbf{curl}\,\boldsymbol{\varphi} = \boldsymbol{\tau}$, so that the solution of (8.16) exists. Moreover it is easy to see that all the necessary orientations fit, in such a way that $\mathbf{curl}\,\boldsymbol{\varphi}$ is globally in $(L^2(\Omega))^3$, so that actually $\boldsymbol{\varphi} \in V^{\mathrm{edge}}_{3,k-1}(\Omega)$.

Finally, we have to prove **b.3**. The proof follows very closely the two dimensional case: given $q \in V^{\mathrm{elem}}_{3,k-3}(\Omega)$, we first choose $\boldsymbol{\eta} \in (H^1(\Omega))^3$ such that

$$\mathrm{div}\,\boldsymbol{\eta} = q \text{ in } \Omega \quad \text{and } \boldsymbol{\eta}\cdot\boldsymbol{n}_\Omega = \frac{\int_\Omega q\,\mathrm{d}\Omega}{|\partial\Omega|} \quad (8.17)$$

where, now, $|\partial\Omega|$ is obviously the *area* of $\partial\Omega$. Then on each face f of $\mathcal{T}_h$ we take

$$\boldsymbol{\sigma}\cdot\boldsymbol{n}^f = \Pi^0_{k-2}(\boldsymbol{\eta}\cdot\boldsymbol{n}^f) \quad (8.18)$$

and inside each element P we take $\mathrm{div}\,\boldsymbol{\sigma} = q$ and $\mathbf{curl}\,\boldsymbol{\sigma} = 0$. Note again that condition $\mathrm{div}\,\boldsymbol{\sigma} = q$ is *compatible* with the boundary conditions (8.18) and the orientations will fit in such a way that actually $\mathrm{div}\,\boldsymbol{\sigma} \in L^2(\Omega)$, so that $\boldsymbol{\sigma} \in V^{\mathrm{face}}_{3,k-2}(\Omega)$.

Remark 14 Although here we are not dealing with applications, we point out that, as is well known (see e.g. [22], [43], [38], [3]), the exactness of the above sequences are of paramount importance in proving several properties (as the various forms of *inf-sup*, the ellipticity in the kernel, etc.) that are crucial in the study of convergence of mixed formulations (see e.g. [20]).

9 A hint on more general cases

As already pointed out in the final part of [24] for the particular case of 2D face elements, we observe here that actually in all four cases considered in this paper (face elements and edge elements in 2D and in 3D), we have at least three parameters to play with in order to create variants of our elements.

For instance, considering the case of 3D face elements, we could choose three different integers k_b, k_r and k_d (all ≥ -1) and consider, instead of (5.1) the spaces

$$V^{\mathrm{face}}_{3,\mathbf{k}}(P) := \{\boldsymbol{v} \in H(\mathrm{div};P)\cap H(\mathbf{curl};P) \text{ s. t. } \boldsymbol{v}\cdot\boldsymbol{n}^f_P \in \mathbb{P}_{k_b}(f)\,\forall \text{ face } f \text{ of } P,$$
$$\mathbf{grad}\,\mathrm{div}\,\boldsymbol{v} \in \mathcal{G}_{k_d-1}(P),\ \ \mathbf{curl}\,\boldsymbol{v} \in \mathcal{R}_{k_r}(P)\}, \quad (9.1)$$

where obviously $\mathbf{k}$ is given by $\mathbf{k} := (k_b, k_d, k_r)$. Taking, for a given integer k, the three indices as $k_b = k,\ k_d = k-1,\ k_r = k-1$ we re-obtain the elements in (5.1), that in turn are the natural extension of the BDM $H(\text{div})$-conforming elements. We notice that, like for the BDM elements, the case $k = 0$ is useless in practice (even the construction of a local basis would be cumbersome). On simplices, for $k = 1$ the VEM spaces (3.1) and (5.1) have one d.o.f. more than BDM_1, which could however be eliminated in the spirit of Remark 11. On the other hand, taking instead $k_b = k,\ k_d = k,\ k_r = k-1$, for $k \geq 0$ we would mimic more the Raviart-Thomas elements. In particular, on simplices and for $k = 0$ we recover exactly the RT_0 element.

We also point out that if we know a priori that (say, in a mixed formulation) the *vector part* of the solution of our problem will be a gradient, we could consider the choice $k_b = k,\ k_d = k-1,\ k_r = -1$ obtaining a space that contains all polynomial vectors in $\mathcal{G}_k$ (that is: vectors that are gradients of some scalar polynomial of degree $\leq k+1$), a space that is rich enough to provide an optimal approximation of our unknown.

Similarly, for the spaces in (6.4) one can consider the variants

$$V^{\text{edge}}_{3,\mathbf{k}}(P) := \{\boldsymbol{v} |\ \boldsymbol{v}_t \in B^{\text{edge}}_{k_b}(\partial P), \\ \operatorname{div} \boldsymbol{v} \in \mathbb{P}_{k_d}(P),\ \text{and}\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \in \mathcal{R}_{k_r-1}(P)\}. \quad (9.2)$$

Taking $k_b = k_r = k$ and $k_d = k-1$ (inside the element and, in three dimensions, also on every face), we would mimick instead the Nédélec elements of the first kind; in particular, for $k = 0$ on simplices we would get the lowest order Nédélec elements. On the other hand, for nodal VEMs we can play with two indices, say k_b and k_Δ, to have

$$V^{\text{vert}}_{3,\mathbf{k}}(P) := \{v |\ v_{|\partial P} \in B^{\text{vert}}_{k_b}(\partial P) \text{ and } \Delta v \in \mathbb{P}_{k_\Delta - 2}(P)\}, \quad (9.3)$$

and, needless to say, in the definition of $B_{k_b}(\partial P)$, the degree of Δ_2 in each face could be different from k_b.

Actually, to be sincere, the amount of possible variants looks overwhelming, and the need of numerical experiments (for different applications of practical interest) is enormous.

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