Mixed finite element approximation of a degenerate elliptic problem

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Abstract. We present a mixed finite element approximation of an elliptic problem with degenerate coefficients, arising in the study of the electromagnetic field in a resonant structure with cylindrical symmetry. Optimal error bounds are derived.

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1. Introduction

We shall present a mixed finite element approximation of the following elliptic problem with degenerate coefficients

$$\begin{cases} -\mathrm{div}\,(\frac{1}{x_1}\underline{\nabla} u) = \frac{f}{x_1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set of R^2 defined by

$$\Omega = \{(x_1, x_2) \in R^2 \ : \ 0 < x_1 < g(x_2),\ x_2 \in (a,b)\}, \tag{1.2}$$

with g smooth and positive. We shall assume Ω to be a convex polygon defined as in (1.2) with $g : [a,b] \to R$ piecewise linear continuous and resonant structure with cylindrical symmetry. Physical considerations and symmetry properties allow to transform the original 3-D problem, governed by Maxwell's equations in the vacuum, into an eigenvalue problem for the operator in (1.1), see Fernandez, Parodi (1985). A conforming finite element discretization for the eigenvalue problem has been introduced and analyzed in Marini, Pietra (1993). In this paper a mixed finite element scheme for (1.1) is presented and optimal error bounds are derived. As in the conforming case, special care for the presence of the singular weight x_1^{-1} must be taken, since classical elements cannot be used and standard techniques do not directly apply. Introducing the spaces

$$W = \left\{ v : x_1^{-1/2} v \in L^2(\Omega) \right\},$$

and

$$H = \{ v : x_1^{-1/2} v \in L^2(\Omega), x_1^{-1/2} \underline{\nabla} v \in (L^2(\Omega))^2, v = 0 \text{ on } \partial\Omega \backslash \{x_1 = 0\}\},$$

if $f \in W$, problem (1.1) has a unique weak solution in H, and the following regularity result holds

$$f \in W \implies x_1^{-\alpha} D^2 u \in L^2(\Omega) \quad \forall \alpha < \frac{1}{2}. \tag{1.3}$$

Moreover, (1.3) implies

$$f \in W \implies x_1^{-\alpha} D u \in L^2(\Omega) \quad \forall \alpha < \frac{3}{2}, \tag{1.4a}$$

$$\left\| x_1^{-\alpha} D u \right\|_{0,\Omega} \leq C(\alpha, \Omega) \, \|f\|_W \quad \forall \alpha < \frac{3}{2}. \tag{1.4b}$$

In order to introduce the mixed formulation of (1.1), let us define the space

$$V = \left\{ \underline{\tau} : x_1^{1/2} \underline{\tau} \in (L^2(\Omega))^2, \; x_1^{1/2} \mathrm{div} \underline{\tau} \in L^2(\Omega) \right\},$$

with the usual graph norm $\|\underline{\tau}\|_V^2 = \|x_1^{1/2}\underline{\tau}\|_{0,\Omega}^2 + \|x_1^{1/2}\mathrm{div}\underline{\tau}\|_{0,\Omega}^2$ (here and in the following $\|\cdot\|_{0,D}$ denotes the norm in $L^2(D)$ or in $(L^2(D))^k, k = 2, 4$). Define

$$a(\underline{\sigma}, \underline{\tau}) = \int_\Omega x_1 \underline{\sigma} \cdot \underline{\tau} dx \quad \underline{\sigma}, \underline{\tau} \in V,$$

$$b(\underline{\tau}, v) = \int_\Omega \mathrm{div} \underline{\tau} \, v dx \quad \underline{\tau} \in V, v \in W, \quad L(v) = \int_\Omega \frac{1}{x_1} f v dx \quad v \in W.$$

The mixed formulation of (1.1) is then the following

$$\begin{cases} \text{find } (\underline{\sigma}, u) \in V \times W \text{ such that} \\ a(\underline{\sigma}, \underline{\tau}) - b(\underline{\tau}, u) = 0 & \forall \underline{\tau} \in V, \\ b(\underline{\sigma}, v) = L(v) & \forall v \in W. \end{cases} \tag{1.5}$$

$$a(\underline{\tau},\underline{\tau}) = ||x_1 \underline{\tau}||_{0,\Omega}, \quad \forall \underline{\tau} \in V. \tag{1.6}$$

Moreover, the *Inf-Sup* condition (see Brezzi, Fortin (1991), e.g.) holds:

$$\exists \beta > 0 \ : \ \forall v \in W \backslash \{0\}, \ \exists \underline{\tau} \in V \backslash \{0\} \ : \ \frac{b(\underline{\tau}, v)}{||\underline{\tau}||_V ||v||_W} \geq \beta. \tag{1.7}$$

To prove (1.7), let us consider the following auxiliary problem: for $v \in W$, let w be the solution of $-\text{div}(x_1^{-1} \underline{\nabla} w) = x_1^{-1} v$ in Ω, $w = 0$ on $\partial\Omega$. Take then $\underline{\tau} = -x_1^{-1} \underline{\nabla} w$. Clearly, $\underline{\tau} \in V$ and $||\underline{\tau}||_V \leq C ||v||_W$. Hence we deduce

$$\frac{b(\underline{\tau}, v)}{||\underline{\tau}||_V ||v||_W} = \frac{||v||_W}{||\underline{\tau}||_V} \geq \frac{1}{C}, \tag{1.8}$$

and (1.7) holds with $\beta = 1/C$. According to the general theory (Brezzi, Fortin (1991)), (1.6) and (1.7) imply that problem (1.5) has a unique solution $(\underline{\sigma}, u)$, with

$$\underline{\sigma} = -x_1^{-1} \underline{\nabla} u. \tag{1.9}$$

For $\alpha < 1/2$, define the space

$$\widetilde{V}_{\alpha,\Omega} = \{\underline{\tau} : x_1^{-\alpha} \underline{\tau} \in (L^2(\Omega))^2, x_1^{1-\alpha} D\underline{\tau} \in (L^2(\Omega))^4\} \cap V \ , \tag{1.10}$$

with the graph norm $||\underline{\tau}||^2_{\widetilde{V}_{\alpha,\Omega}} = ||x_1^{-\alpha} \underline{\tau}||^2_{0,\Omega} + ||x_1^{1-\alpha} D\underline{\tau}||^2_{0,\Omega} + ||x_1^{1/2} \text{div} \underline{\tau}||^2_{0,\Omega}$. Note that, due to (1.9) and the regularity (1.3)-(1.4) of the solution u of (1.1), one has $\underline{\sigma} \in \widetilde{V}_{\alpha,\Omega}$, $\forall \alpha < 1/2$, and

$$||\underline{\sigma}||_{\widetilde{V}_{\alpha,\Omega}} \leq C ||f||_W \ . \tag{1.11}$$

Moreover, the *Inf-Sup* condition (1.7) holds with $\widetilde{V}_{\alpha,\Omega}$ instead of V:

$$\exists \beta > 0 \ : \ \forall v \in W \backslash \{0\}, \ \exists \underline{\tau} \in \widetilde{V}_{\alpha,\Omega} \backslash \{0\} \ : \ \frac{b(\underline{\tau}, v)}{||\underline{\tau}||_{\widetilde{V}_{\alpha,\Omega}} ||v||_W} \geq \beta \ . \tag{1.12}$$

The outline of the paper is the following. In Section 2 the mixed finite element discretization is presented and the interpolant operators are defined. Section 3 contains the error estimates.

2. The discrete formulation

Let $\{T_h\}_h$ be a regular family of decompositions (see Ciarlet (1978), e.g.) of Ω into rectangles and triangles as in fig.1. For each T_h, denote by T (resp. K) the generic triangle (resp. rectangle) of T_h; h_{T} will denote the element mesh size of T, h_1, h_2 the edges of K, and h the global mesh size.

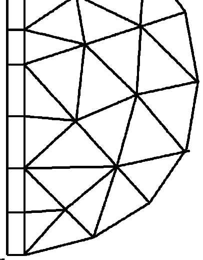

Figure 1. Example of mesh

Note that the regularity assumption on the family $\{T_h\}_h$ implies the existence of three positive constants c_1, c_2, and c_3, independent of h, such that, for every element of T_h

$$c_1 \leq \frac{\rho_{\rm T}}{h_{\rm T}} \leq 1 \qquad \text{or} \qquad c_2 \leq \frac{h_2}{h_1} \leq c_3, \tag{2.1}$$

where, as usual, $\rho_{\rm T}$ is the diameter of the inscribed circle. Immediate consequence of the regularity assumption on T_h is the following property that will be used throughout the paper.

Proposition 2.1 *Let* T *be a triangle of* T_h. *Set* $\tilde{a} = \min_{\rm T} x_1$. *Then, there exists a constant* C *independent of* h *such that*

$$h_{\rm T}/\tilde{a} \leq C. \tag{2.2}$$

∎

Next, define our finite element spaces as

$$V_h = \{\underline{\tau} \in V : \underline{\tau}_{|\rm K} \in RT({\rm K}) \ \ \forall {\rm K} \in T_h, \underline{\tau}_{|\rm T} \in RT({\rm T}) \ \ \forall {\rm T} \in T_h\}, \tag{2.3}$$

$$W_h = \left\{v \in W : v_{|\rm K} = ax_1 \ \forall {\rm K} \in T_h, \ (a \in R); \ v_{|\rm T} \in P_0({\rm T}) \ \forall {\rm T} \in T_h\right\}, \tag{2.4}$$

where RT (K) and RT (T) denote the lowest order Raviart-Thomas elements on rectangles and triangles, resp. (see Raviart, Thomas (1977)):

$$\begin{aligned} RT({\rm K}) &= \{\underline{\tau} = (ax_1 + b, cx_2 + d) \ , \ a, b, c, d \in \ R\}, \\ RT({\rm T}) &= \{\underline{\tau} = (ax_1 + b, ax_2 + c) \ , \ a, b, c \in \ R\}. \end{aligned} \tag{2.5}$$

The discrete problem is then

$$\begin{cases} \text{find } \ (\underline{\sigma}_h, u_h) \in V_h \times W_h \text{ such that} & \\ a(\underline{\sigma}_h, \underline{\tau}) \ - \ b(\underline{\tau}, u_h) \ = \ 0 & \forall \underline{\tau} \in V_h, \\ b(\underline{\sigma}_h, v) \qquad\qquad\quad = \ L(v) & \forall v \in W_h \ . \end{cases} \tag{2.6}$$

here. In order to satisfy the requirement $W_h \subset W$, $v \in W_h$ has to behave (at least) as x_1 in a strip close to $\{x_1 = 0\}$. The choice of subdividing the strip into rectangles, although not crucial, is the simplest one for the error analysis.

Note that, although the finite element spaces (2.3)-(2.5) are very similar to the Raviart-Thomas spaces, the analysis is not straightforward, and properties such as the commuting diagram property (see Douglas, Roberts (1985)) fail here ($\mathrm{div}\, V_h \neq W_h$). As usual in mixed finite elements, the analysis will rely on a proper definition of the interpolant operators and on the study of their properties. First, for any $0 \leq \alpha < 1/2$, define $\Pi_h : \widetilde{V}_{\alpha,\Omega} \longrightarrow V_h$ locally on K by

$$\int_{\mathrm{K}} (\tau_1 - (\Pi_h \underline{\tau})_1) dx = 0 \tag{2.7}$$

$$\int_{e} x_1 (\underline{\tau} - \Pi_h \underline{\tau}) \cdot \underline{n} ds = 0 \qquad \forall \; edge \; e \; of \; \mathrm{K} \backslash \{x_1 = 0\} \; , \tag{2.8}$$

and locally on T by

$$\int_{e} (\underline{\tau} - \Pi_h \underline{\tau}) \cdot \underline{n} ds = 0 \qquad \forall \; edge \; e \; of \; \mathrm{T} \; . \tag{2.9}$$

It is easy to check that Π_h is well defined. In particular, note that (2.8) and (2.9) are compatible, since the only possible common edge e of a rectangle and a triangle is a vertical edge (see fig.1). Moreover, it follows immediately from Gauss theorem and the definition of Π_h that

$$b(\underline{\tau} - \Pi_h \underline{\tau}, v) = 0 \qquad \forall \; v \in W_h \; . \tag{2.10}$$

Proposition 2.2 *Let* $\Pi_h : \widetilde{V}_{\alpha,\Omega} \longrightarrow V_h$ *be the interpolant operator defined by* (2.7)-(2.9). *Then, there exists a constant* γ *independent of* h, *such that*

$$||\Pi_h \underline{\tau}||_V \leq \gamma ||\underline{\tau}||_{\widetilde{V}_{\alpha,\Omega}} \qquad \forall \underline{\tau} \in \widetilde{V}_{\alpha,\Omega}, \; with \; 0 \leq \alpha < 1/2 \; . \tag{2.11}$$

In order to prove Proposition 2.2 we shall use the following Lemma.

Lemma 2.1 *For* $\underline{\tau} \in \widetilde{V}_{\alpha,\Omega}$, *with* $0 \leq \alpha < 1/2$, *on any rectangle* K *we have*

$$\int_{\mathrm{K}} x_1 |\underline{\tau}|^2 dx \leq C h^{1+2\alpha} ||\underline{\tau}||^2_{\widetilde{V}_{\alpha,\mathrm{K}}}, \tag{2.12}$$

$$\int_{\mathrm{K}} x_1 |\Pi_h \underline{\tau}|^2 dx \leq C h^{1+2\alpha} ||\underline{\tau}||^2_{\widetilde{V}_{\alpha,\mathrm{K}}}. \tag{2.13}$$

$$\int_K x_1|\underline{\tau}|^2dx = \int_K |x_1^{-\alpha}\underline{\tau}|^2 x_1^{1+2\alpha}dx \le h_1^{1+2\alpha}\int_K |x_1^{-\alpha}\underline{\tau}|^2dx\ . \tag{2.14}$$

In order to prove (2.13), consider the affine mapping $F\ :\ \mathrm{K}\longrightarrow\widehat{\mathrm{K}} = (0,1)\times(0,1)$, and set $\widehat{\Pi\underline{\tau}}(\widehat{x}) = \Pi_h\underline{\tau}(F^{-1}(\widehat{x}))$. It is immediate to check that $\widehat{\Pi\underline{\tau}}(\widehat{x}) = \widehat{\Pi}\widehat{\underline{\tau}}(\widehat{x})$, so that

$$\int_K x_1|\Pi_h\underline{\tau}|^2dx = |\mathrm{K}|h_1\int_{\widehat{K}}\widehat{x}_1|\widehat{\Pi}\widehat{\underline{\tau}}|^2d\widehat{x}. \tag{2.15}$$

From the explicit expression for $\widehat{\Pi}\widehat{\underline{\tau}}(\widehat{x})$, which can be deduced from (2.7)-(2.8), we get

$$\int_{\widehat{K}}\widehat{x}_1|\widehat{\Pi}\widehat{\underline{\tau}}|^2d\widehat{x}\ \le\ C(||\widehat{\underline{\tau}}||^2_{0,\widehat{K}} + ||\widehat{x}_1\widehat{\underline{\tau}}\cdot\widehat{\underline{n}}||^2_{0,\partial\widehat{K}}). \tag{2.16}$$

Since $\underline{\tau}\in\widetilde{V}_{\alpha,\Omega}$, with $0\le\alpha<1/2$ we have

$$||\widehat{\underline{\tau}}||^2_{0,\widehat{K}} = \int_{\widehat{K}}|\widehat{x}_1^{-\alpha}\widehat{\underline{\tau}}|^2\widehat{x}_1^{2\alpha}d\widehat{x} \le \int_{\widehat{K}}|\widehat{x}_1^{-\alpha}\widehat{\underline{\tau}}|^2d\widehat{x} = |\mathrm{K}|^{-1}h_1^{2\alpha}\int_K|x_1^{-\alpha}\underline{\tau}|^2dx. \tag{2.17}$$

On the other hand we can write

$$||\widehat{x}_1\widehat{\underline{\tau}}\cdot\widehat{\underline{n}}||^2_{0,\partial\widehat{K}} \le C||\widehat{x}_1\widehat{\underline{\tau}}||^2_{1,\widehat{K}} \le C\,(||\widehat{x}_1\widehat{\underline{\tau}}||^2_{0,\widehat{K}} + ||\widehat{\underline{\tau}}||^2_{0,\widehat{K}} + ||\widehat{x}_1D\widehat{\underline{\tau}}||^2_{0,\widehat{K}}), \tag{2.18}$$

where

$$||\widehat{x}_1\widehat{\underline{\tau}}||^2_{0,\widehat{K}} \le ||\widehat{\underline{\tau}}||^2_{0,\widehat{K}}\ , \tag{2.19}$$

$$\begin{aligned}||\widehat{x}_1D\widehat{\underline{\tau}}||^2_{0,\widehat{K}} &= \int_{\widehat{K}}|\widehat{x}_1^{1-\alpha}D\widehat{\underline{\tau}}|^2\widehat{x}_1^{2\alpha}d\widehat{x} \le \int_{\widehat{K}}|\widehat{x}_1^{1-\alpha}D\widehat{\underline{\tau}}|^2d\widehat{x}\\ &\le C\,h_1^{2\alpha-2}\int_K|x_1^{1-\alpha}D\underline{\tau}|^2dx\ .\end{aligned} \tag{2.20}$$

Finally, from (2.15)-(2.20) we obtain (2.13).

■

Proof of Proposition 2.2 Consider first the contribution of a generic rectangle K. We have from (2.10), with $v = x_1$ on K, and $v=0$ elsewhere,

$$\mathrm{div}\Pi_h\underline{\tau}_{|\mathrm{K}} = \int_K x_1\mathrm{div}\underline{\tau}dx\Big/\int_K x_1dx\ . \tag{2.21}$$

$$\int_K x_1(\mathrm{div}\Pi_h \underline{\tau})^2 dx \leq \int_K x_1(\mathrm{div}\underline{\tau})^2 dx \ . \tag{2.22}$$

Finally, (2.13) and (2.22) imply

$$\forall \mathrm{K} \qquad ||\Pi_h\underline{\tau}||^2_{V_\mathrm{K}} \leq C||\underline{\tau}||^2_{\widetilde{V}_{\alpha,\mathrm{K}}} \ , \tag{2.23}$$

where V_K denotes the restriction of V to the generic rectangle K. Let us consider now a triangle T, and recall that on T $\underline{\tau} \in (H^1(\mathrm{T}))^2$ and Π_h is the usual Raviart-Thomas-interpolant. Let $\tilde{a} = \min_\mathrm{T} x_1$, as in Proposition 2.1, and note that $\max_\mathrm{T} x_1 \leq \tilde{a} + h_\mathrm{T}$. Then, we have

$$||\Pi_h\underline{\tau}||^2_{V_\mathrm{T}} \leq (\tilde{a}+h_\mathrm{T})||\Pi_h\underline{\tau}||^2_{H(\mathrm{div};\mathrm{T})} \ , \tag{2.24}$$

and from Raviart, Thomas (1977)

$$||\Pi_h\underline{\tau}||^2_{H(\mathrm{div};\mathrm{T})} \leq C(||\underline{\tau}||^2_{0,\mathrm{T}} + h^2_\mathrm{T}||D\underline{\tau}||^2_{0,\mathrm{T}} + ||\mathrm{div}\underline{\tau}||^2_{0,\mathrm{T}}) \ , \tag{2.25}$$

where H (div;T)$=\{\underline{\tau} \in (L^2(\mathrm{T}))^2, div\underline{\tau} \in L^2(\mathrm{T})\}$, and V_T denotes the restriction of V to the generic triangle T. Moreover,

$$||\underline{\tau}||^2_{0,\mathrm{T}} = \int_\mathrm{T} |x_1^{-\alpha}\underline{\tau}|^2 x_1^{2\alpha} dx \leq (\tilde{a}+h_\mathrm{T})^{2\alpha}||x_1^{-\alpha}\underline{\tau}||^2_{0,\mathrm{T}} \ , \tag{2.26}$$

$$||D\underline{\tau}||^2_{0,\mathrm{T}} = \int_\mathrm{T} |x_1^{1-\alpha}D\underline{\tau}|^2 x_1^{2\alpha-2} dx \leq \frac{1}{\tilde{a}^{2-2\alpha}}||x_1^{1-\alpha}D\underline{\tau}||^2_{0,\mathrm{T}} \ , \tag{2.27}$$

and

$$||\mathrm{div}\underline{\tau}||^2_{0,\mathrm{T}} \leq \frac{1}{\tilde{a}}||x_1^{1/2}\mathrm{div}\underline{\tau}||^2_{0,\mathrm{T}} \ . \tag{2.28}$$

Using (2.2), from (2.24)-(2.28) we conclude

$$\forall \mathrm{T} \qquad ||\Pi_h\underline{\tau}||^2_{V_\mathrm{T}} \leq C||\underline{\tau}||^2_{\widetilde{V}_{\alpha,\mathrm{T}}} \ . \tag{2.29}$$

More precisely, in the bound of the term coming from (2.27), we used the trivial fact that (2.2) implies $(\tilde{a}+h_\mathrm{T})\tilde{a}^{2\alpha}h^2_\mathrm{T}/\tilde{a}^2 \leq C$. Actually a sharper estimate, that will be useful in the next section, can be derived from (2.2):

$$\frac{h^2_\mathrm{T}(\tilde{a}+h_\mathrm{T})}{\tilde{a}^{2-2\alpha}} = h_\mathrm{T}^{1+2\alpha}(\frac{h_\mathrm{T}}{\tilde{a}})^{1-2\alpha}(\frac{\tilde{a}+h_\mathrm{T}}{\tilde{a}}) \leq Ch_\mathrm{T}^{1+2\alpha} \ . \tag{2.30}$$

Summation of (2.23) and (2.29) over all the elements of T_h gives (2.11).

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$$\int_E P_h v dx = \int_E v dx \qquad \forall E = \text{element of } T_h. \tag{2.31}$$

Notice that (2.31) implies

$$P_h v_{|\mathrm{K}} = x_1 \int_{\mathrm{K}} v dx \Big/ \int_{\mathrm{K}} x_1 dx \quad \forall \mathrm{K} \in T_h, \tag{2.32}$$

$$P_h v_{|\mathrm{T}} = \frac{1}{|\mathrm{T}|} \int_{\mathrm{T}} v dx \qquad \forall \mathrm{T} \in T_h. \tag{2.33}$$

Moreover,by definition of P_h , we deduce, for $v \in W$,

$$b(\underline{\tau}, P_h v - v) = 0 \qquad \forall \underline{\tau} \in V_h. \tag{2.34}$$

Since $v \in W$, using (2.32) and Cauchy-Schwarz inequality we deduce, on a generic rectangle K,

$$\int_{\mathrm{K}} \frac{P_h v^2}{x_1} dx = (\int_{\mathrm{K}} x_1^{-1/2} v\, x_1^{1/2} dx)^2 \Big/ \int_{\mathrm{K}} x_1 dx \leq ||x_1^{-1/2} v||^2_{0,\mathrm{K}}. \tag{2.35}$$

Consider now a generic triangle $\mathrm{T} \in T_h$. From (2.33) we have (with the notation of Proposition 2.1)

$$\int_{\mathrm{T}} \frac{P_h v^2}{x_1} dx = \frac{1}{|\mathrm{T}|^2} (\int_{\mathrm{T}} v dx)^2 \int_{\mathrm{T}} \frac{1}{x_1} dx \leq \frac{1}{|\mathrm{T}|\tilde{a}} (\int_{\mathrm{T}} v dx)^2. \tag{2.36}$$

Since $\max_{\mathrm{T}} x_1 \leq \tilde{a} + h_{\mathrm{T}}$, via Cauchy-Schwarz inequality, we have

$$(\int_{\mathrm{T}} v dx)^2 = (\int_{\mathrm{T}} x_1^{-1/2} v\, x_1^{1/2} dx)^2 \leq (\tilde{a} + h_{\mathrm{T}}) |\mathrm{T}| ||x_1^{-1/2} v||^2_{0,\mathrm{T}}. \tag{2.37}$$

Hence, using (2.2) in (2.36)-(2.37), we obtain

$$\int_{\mathrm{T}} \frac{P_h v^2}{x_1} dx \leq C ||x_1^{-1/2} v||^2_{0,\mathrm{T}}. \tag{2.38}$$

Finally, (2.35) and (2.38) give the following Proposition

Proposition 2.3 *Let* $P_h : W \longrightarrow W_h$ *be the interpolant operator defined by* (2.31)*. Then, there exists a constant* C *independent of* h *, such that*

$$||P_h v||_W \leq C\, ||v||_W \qquad \forall\ v \in W\ . \tag{2.39}$$

■

$$\exists\overline{\beta} > 0 \ : \ \forall v \in W_h\backslash\{0\}, \ \exists\underline{\tau} \in V_h\backslash\{0\} \ : \ \frac{b(\underline{\tau}, v)}{||\underline{\tau}||_V ||v||_W} \geq \overline{\beta}, \qquad (2.40)$$

with $\overline{\beta}$ independent of h.

Since $W_h \subset W$, the discrete *Inf-Sup* condition follows from (1.12), (2.10), and (2.11) with $\overline{\beta} = \beta/\gamma$ (see Brezzi, Fortin (1991), e.g.).

3. Error Estimates

The first theorem in this section follows by standard arguments (as in Brezzi, Fortin (1991)), using the properties of the interpolant operators Π_h and P_h. Nevertheless for completeness we present the proof.

Theorem 3.1 *Problem (2.6) has a unique solution* $(\underline{\sigma}_h, u_h)$ *, and the following estimates hold*

$$||x_1^{1/2}(\underline{\sigma} - \underline{\sigma}_h)||_{0,\Omega} \ \leq \ ||x_1^{1/2}(\underline{\sigma} - \Pi_h\underline{\sigma})||_{0,\Omega}, \qquad (3.1)$$

$$||u - u_h||_W \ \leq \ C \ (||u - P_h u||_W + ||x_1^{1/2}(\underline{\sigma} - \Pi_h\underline{\sigma})||_{0,\Omega}) \ , \qquad (3.2)$$

with $(\underline{\sigma}, u)$ *solution of (1.5), and* C *a constant independent of* h.

Proof Uniqueness follows from the discrete *Inf-Sup* condition (2.40). By subtracting (2.6) from (1.5) we obtain the error equation

$$\begin{cases} a(\underline{\sigma} - \underline{\sigma}_h, \underline{\tau}) \ - \ b(\underline{\tau}, u - u_h) \ = \ 0 & \forall \underline{\tau} \in V_h \ , \\ b(\underline{\sigma} - \underline{\sigma}_h, v) \ = \ 0 & \forall v \in W_h \ . \end{cases} \qquad (3.3)$$

We have

$$a(\underline{\sigma} - \underline{\sigma}_h, \underline{\sigma} - \underline{\sigma}_h) = a(\underline{\sigma} - \underline{\sigma}_h, \underline{\sigma} - \Pi_h\underline{\sigma}) + a(\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h) \ . \qquad (3.4)$$

The first error equation and the property (2.34) of P_h give

$$a(\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h) = b(\Pi_h\underline{\sigma} - \underline{\sigma}_h, u - u_h) = b(\Pi_h\underline{\sigma} - \underline{\sigma}_h, P_h u - u_h) \ . \qquad (3.5)$$

Using next the property (2.10) of Π_h and the second error equation we deduce

$$b(\Pi_h\underline{\sigma} - \underline{\sigma}_h, P_h u - u_h) = b(\underline{\sigma} - \underline{\sigma}_h, P_h u - u_h) = 0 \ . \qquad (3.6)$$

Hence, from (3.4) we obtain (3.1).

The *Inf-Sup* condition (2.40), the property (2.34) of P_h and the first error equation imply

$$||P_h u - u_h||_W \leq \overline{\beta}^{-1} \sup_{\underline{\tau} \in V_h\backslash\{0\}} \frac{b(\underline{\tau}, P_h u - u_h)}{||\underline{\tau}||_V} = \overline{\beta}^{-1} \sup_{\underline{\tau} \in V_h\backslash\{0\}} \frac{b(\underline{\tau}, u - u_h)}{||\underline{\tau}||_V}$$

$$= \overline{\beta}^{-1} \sup_{\underline{\tau} \in V_h\backslash\{0\}} \frac{a(\underline{\sigma} - \underline{\sigma}_h, \underline{\tau})}{||\underline{\tau}||_V} \ . \qquad (3.7)$$

$$||P_h u - u_h||_W \le C||x_1^{1/2}(\underline{\sigma} - \Pi_h \underline{\sigma})||_{0,\Omega}\ , \tag{3.8}$$

which gives (3.2), by triangle inequality.

∎

It remains to estimate the interpolation errors. We can state the following

Theorem 3.2 *Let* $(\underline{\sigma},\, u)$ *be the solution of (1.5), and let* P_h, Π_h *be defined by (2.31), (2.7)-(2.9). Then, there exist two positive constants* C_1, C_2 *independent of* h *such that:*

$$||u - P_h u||_W \ \le\ C_1\, h||f||_W, \tag{3.9}$$

$$||x_1^{1/2}(\underline{\sigma} - \Pi_h \underline{\sigma})||_{0,\Omega} \ \le\ C_2\, h^{1-\epsilon}||f||_W \qquad 0 < \epsilon < 1. \tag{3.10}$$

Proof Consider first the contribution of a generic triangle $\mathrm{T} \in T_h$, and recall that, on T, P_h is the classical L^2 - projection on the constants, so that we have (see Ciarlet (1978), e.g.)

$$\int_{\mathrm{T}} \frac{|u(x) - P_h u(x)|^2}{x_1} dx \le C \frac{h_{\mathrm{T}}^2}{\tilde{a}} ||Du||_{0,\mathrm{T}}^2, \tag{3.11}$$

with $\tilde{a} = \min_{\mathrm{T}} x_1$. Using (1.4) and $\max_{\mathrm{T}} x_1 \le \tilde{a} + h_{\mathrm{T}}$, we obtain, for $0 \le \alpha < 3/2$,

$$||Du||_{0,\mathrm{T}}^2 = \int_{\mathrm{T}} (x_1^{-\alpha} Du)^2 x_1^{2\alpha} dx \le C(\tilde{a} + h_{\mathrm{T}})^{2\alpha} ||x_1^{-\alpha} Du||_{0,\mathrm{T}}^2 \quad . \tag{3.12}$$

From (2.2) it follows that $(\tilde{a} + h_{\mathrm{T}})^{2\alpha}/\tilde{a} \le C$, for $\alpha \ge 1/2$. Hence, from (3.11) and (3.12) we obtain

$$\forall \mathrm{T} \quad \int_{\mathrm{T}} \frac{|u(x) - P_h u(x)|^2}{x_1} dx \le C h^2 ||x_1^{-\alpha} Du||_{0,\mathrm{T}}^2, \qquad \frac{1}{2} \le \alpha < \frac{3}{2}. \tag{3.13}$$

Let us now consider a rectangle K. We have from (2.35)

$$7), e.g.) \int_{\mathrm{K}} \frac{|u(x) - P_h u(x)|^2}{x_1} dx \le 4 \int_{\mathrm{K}} \frac{|u(x)|^2}{x_1} dx. \tag{3.14}$$

Since the regularity of u implies that $u = 0$ for $x_1 = 0$, the fundamental theorem of calculus gives

$$u(x_1, x_2) \ = \ \int_0^{x_1} \frac{\partial u}{\partial x_1}(t, x_2) t^{\alpha} t^{-\alpha}\, dt, \qquad a.e. \ . \tag{3.15}$$

$$u(x_1,x_2)^2 \le C x_1^{2\alpha+1} \int_0 \left(\frac{\partial u}{\partial x_1}(t,x_2)t^{-\alpha}\right)^2 dt, \tag{3.16}$$

giving

$$\int_{\mathrm{K}} \frac{|u(x)|^2}{x_1} dx \le C \int_{\mathrm{K}} x_1^{2\alpha} \int_0^{h_1} \left(\frac{\partial u}{\partial x_1}(t,x_2)t^{-\alpha}\right)^2 dt dx$$
$$\le C h_1^{2\alpha+1} ||x_1^{-\alpha} Du||_{0,\mathrm{K}}^2, \qquad -\frac{1}{2} < \alpha < \frac{3}{2}. \tag{3.17}$$

Substituting in (3.14) yields

$$\forall \mathrm{K} \quad \int_{\mathrm{K}} \frac{|u(x) - P_h u(x)|^2}{x_1} dx \le C h^{2\alpha+1} ||x_1^{-\alpha} Du||_{0,\mathrm{K}}^2, \ -\frac{1}{2} < \alpha < \frac{3}{2}. \tag{3.18}$$

Summation of (3.13) and (3.18) over all the elements of T_h , and (1.4b) give (3.9). In order to prove (3.10), consider first a generic triangle T. Using a similar argument as in Proposition 2.2 (and same notation), we obtain

$$\int_{\mathrm{T}} x_1 |\underline{\sigma} - \Pi_h \underline{\sigma}|^2 dx \le C(\tilde{a} + h_{\mathrm{T}}) h_{\mathrm{T}}^2 \int_{\mathrm{T}} |D\underline{\sigma}|^2 dx. \tag{3.19}$$

Since $\underline{\sigma} \in \widetilde{V}_{\alpha,\Omega}$, with $0 \le \alpha < 1/2$, we deduce

$$\int_{\mathrm{T}} |D\underline{\sigma}|^2 dx \le \frac{1}{\tilde{a}^{2-2\alpha}} \int_{\mathrm{T}} |x_1^{1-\alpha} D\underline{\sigma}|^2 dx \ , \tag{3.20}$$

and therefore

$$\int_{\mathrm{T}} x_1 |\underline{\sigma} - \Pi_h \underline{\sigma}|^2 dx \le C \frac{(\tilde{a} + h_{\mathrm{T}}) h_{\mathrm{T}}^2}{\tilde{a}^{2-2\alpha}} \int_{\mathrm{T}} |x_1^{1-\alpha} D\underline{\sigma}|^2 dx \ . \tag{3.21}$$

Using (2.30) in (3.21)we conclude

$$\forall \mathrm{T} \qquad \int_{\mathrm{T}} x_1 |\underline{\sigma} - \Pi_h \underline{\sigma}|^2 dx \le C h^{1+2\alpha} ||\underline{\sigma}||_{\widetilde{V}_{\alpha,\mathrm{T}}}^2 \qquad 0 \le \alpha < 1/2 \ . \tag{3.22}$$

On a rectangle K we apply Lemma 2.1 to obtain

$$\forall \mathrm{K} \qquad \int_{\mathrm{K}} x_1 |\underline{\sigma} - \Pi_h \underline{\sigma}|^2 dx \le C h^{1+2\alpha} ||\underline{\sigma}||_{\widetilde{V}_{\alpha,\mathrm{K}}}^2 \qquad 0 \le \alpha < 1/2 \ . \tag{3.23}$$

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