NOTES FOR THE COURSE
“SYSTEMS OF CONSERVATION LAWS IN ONE SPACE VARIABLE”

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Abstract. These are my notes are for a graduate course held in Padova in Spring 2018. I will be very grateful to anybody pointing out mistakes and misprints. These notes contain no new results and almost no references to original research papers. Most of the material is taken from the books [3, 4, 8, 12, 13], which also contain an extended list of references.

Notation. I use standard characters to denote numbers (or real valued functions) and bold characters to denote vectors: in other words, $c \in \mathbb{R}, c \in \mathbb{R}^N$. I also use the following notation
- $\mathbb{R}_+:=[0, +\infty[$
- $c^t$: the transpose of the vector $c$.
- $\mathbf{a} \cdot \mathbf{b}$: the scalar product between the vectors $\mathbf{a}$ and $\mathbf{b}$.

1. Introduction

1.1. Systems of conservation laws in one space variable. A system of conservation laws in one space variable is a partial differential equation in the form
\begin{equation}
\mathbf{u}_t + \left[ f(\mathbf{u}) \right]_x = 0,
\end{equation}
where
- the time variable $t \in \mathbb{R}_+$, the space variable is one dimensional and for the time being we let $x$ vary on the whole $\mathbb{R}$, so $x \in \mathbb{R}$;
- the unknown is $\mathbf{u} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^N$;
- the flux is $f : \mathbb{R}^N \to \mathbb{R}^N$ is smooth (as smooth as needed, as a matter of fact $f \in C^2$ is usually enough).

For the time being we focus on the Cauchy problem obtained by coupling (1.1) with the datum
\begin{equation}
\mathbf{u}(0, x) = \mathbf{u}_0(x),
\end{equation}
where $\mathbf{u}_0$ is a given function $\mathbf{u}_0 : \mathbb{R} \to \mathbb{R}^N$.

1.1.1. Goal of the course. The analysis of systems of conservation laws is very often based on ad-hoc techniques, because (as we will see in the following) many of the most common PDEs techniques fail when applied to systems of conservation laws. In this course I will try to give an introduction to some of the main techniques used to tackle them. Due to time constraints, I will mostly focus on the scalar case $N = 1$, but almost all the techniques I will discuss can be extended to systems (usually at the price of very hard work).

1.1.2. Why “conservation laws”. We want to understand why (1.1) are termed “conservation laws”. Assume that the solution $\mathbf{u}$ of (1.1) is regular, then given $a, b \in \mathbb{R}, a < b$, we have
\begin{equation}
\frac{d}{dt} \int_a^b \mathbf{u}(t, x) dx = \int_a^b \mathbf{u}_t(t, x) dx = - \int_a^b \left[ f(\mathbf{u}) \right]_x dx = -f(\mathbf{u}(b)) + f(\mathbf{u}(a)).
\end{equation}
The heuristic interpretation is that the rate of change in the amount of $\mathbf{u}$ “contained” in the interval $[a, b]$ only depends on the flux entering the domain at $a$ and on the flux leaving the domain at $b$, and does not depend on dissipation phenomena that could in principle occur in between.
Now consider (1.3), let $a \to -\infty$, $b \to +\infty$ and assume for simplicity that $u$ is compactly supported (this assumption can be greatly relaxed). We get that

$$\frac{d}{dt} \int_{\mathbb{R}} u(t,x) dx = 0,$$

which loosely speaking means that “the total amount of $u$” is conserved.

### 1.2. Examples.

1.2.1. *Linear equations.* If $f(u) = Au$ for some constant matrix $A \in \mathbb{M}^{N \times N}$, then (1.1) boils down to

$$u_t + Au_x = 0.$$  

1.2.2. *The Burger’s equation.* It is basically the simplest possible non-linear conservation law, i.e.

$$u_t + \left(\frac{1}{2} u^2\right)_x = 0.$$  

Note that in this case $N = 1$.

1.2.3. *Euler equations.* The Euler equations of fluid dynamics are formally obtained from the compressible Navier-Stokes equations by neglecting the second order terms. They model the motion of particles in a fluid with no viscosity. In the case when the space variable is one dimensional they take the form

$$
\begin{cases}
\rho_t + (\rho v)_x = 0 & \text{conservation of mass} \\
(\rho v)_t + (\rho v^2 + p)_x = 0 & \text{conservation of linear momentum} \\
\left(\rho [e + \frac{1}{2} v^2]\right)_x + (\rho v[e + \frac{1}{2} v^2] + pv)_x = 0 & \text{conservation of energy}.
\end{cases}
$$

The unknowns $u = (\rho, v, e)$ represent the density of the fluid, the velocity of the particle of the fluids and the internal energy, respectively. The function $p = p(\rho, e)$ is the pressure. The law $p = p(\rho, e)$ is the equation of state and depends on the gas under consideration.

1.2.4. *Traffic models.* Conservation laws are used to model both vehicular and pedestrian traffic. The simplest model is

$$u_t + [uv(u)]_x = 0.$$  

In the previous expression, the scalar unknown $u$ represents the density of cars or pedestrian, and $v(u)$ represents their velocity. Note that by applying (1.4) we get that $\int_{\mathbb{R}} u(t,x) dx$ is conserved, i.e. that the total number of cars or pedestrians is conserved.

The actual expression of $v$ depends on the model, but one usually expects that the higher the density, the lower the speed, and hence one takes $v' < 0$. A fairly common choice is $v(u) = 1 - u$.

One can also use much more refined models involving a system of two of more conservation laws. As a matter of fact, there are entire books devoted to traffic models involving conservation laws, see for instance [6].

1.3. *Conservation laws in several space variables.* In this course we focus on conservation laws in one space variable. Here we briefly mention conservation laws in several space variables.
1.3.1. Scalar conservation law. A scalar conservation in several space variables is a partial differential equation in the form

\[(1.9) \quad u_t + \text{div}_x [g(u)] = 0,\]

where \(u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}\) is the unknown and the flux \(g : \mathbb{R} \to \mathbb{R}^d\) is smooth. The symbol \(\text{div}_x\) denotes the divergence with respect to the space variable only.

We argue as in (1.3), but we use the Gauss Divergence Theorem instead of the Fundamental Theorem of Calculus. We get that, if \(u\) is smooth and \(\Omega \subseteq \mathbb{R}^d\) is a regular, bounded open set, then

\[
\frac{d}{dt} \int_\Omega u(t,x)dx \overset{(1.9)}{=} - \int_\Omega \text{div}_x [g(u)] = - \int_{\partial\Omega} g(u) \cdot n d\mathcal{H}^{d-1},
\]

where \(n\) is the outward pointing unit normal vector to \(\partial\Omega\). Namely, the rate of change in the amount of \(u\) “contained” in \(\Omega\) only depends on the inflow or outflow \(g(u) \cdot n\) on \(\partial\Omega\), and \textit{does not} depend on other phenomena that could in principle occur inside \(\Omega\).

1.3.2. Systems of conservation law in several space variables. Systems of conservation laws in several space variables are equations in the form

\[(1.10) \quad u_t + \sum_{i=1}^d [f_i(u)]_{x_i} = 0,\]

where the unknown is \(u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^N\) and the flux functions \(f_i : \mathbb{R}^N \to \mathbb{R}^N, i = 1, \cdots, d\) are smooth. This is the most general case: the cases (1.1) and (1.9) can be obtained by taking \(d = 1\) and \(N = 1\), respectively. It is also the most interesting case from the physical viewpoint. For instance, if we consider the Euler equations in the natural three dimension space we get a system of equations like (1.10) with \(d = 3\), \(N = 5\).

1.4. Well-posedness theory: general picture. Given an equation or a system of equations, I term “well-posedness theory” a set or results that provide existence, uniqueness and possibly stability of some suitable notion of solution.

Very loosely speaking, the state of the art for conservation laws is the following.

- Scalar conservation laws in several state variables (1.9): there is a well-established and quite satisfactory well-posedness theory which dates back to the pioneering work by Kružkov [9]. Kružkov’s theory requires quite reasonable conditions on the initial datum.

- Systems of conservation laws in one space variables (1.1): there is a well-posedness theory, with two milestones due to Glimm [7] (global in time existence of solutions) and Bressan and collaborators [3] (uniqueness and stability of “admissible” solutions). However, this theory is not satisfactory because it requires very restrictive conditions on the initial data, namely it requires that \(\text{TotVar} u_0 \overset{1}{\text{is small}}\) (how small depends on the specific system). There are well-posedness results that require more natural conditions on the initial data, but they only hold for very specific systems.

- Systems of conservation laws in several space variables (1.10): this case is, to a large extent, still very poorly understood. Well-posedness results are only available in very few and special examples and no global existence or uniqueness result is presently available for general systems. Also, recent developments suggest that the situation is much more complex than in the case of one space variable. See [2] for an extended discussion.

2. Classical solutions

In this paragraph we discuss classical solutions of (1.1), i.e. \(C^1\) functions \(u\) that pointwise satisfy the equation. A very powerful tool for the analysis of classical solutions is the method of characteristics.

\(^1\)If you do not know what the total variation of a function is, do not worry: we will come back to this issue later on. Anyhow, keep in mind that in general physical data do not have small total variation.
2.1. The method of characteristics for scalar equations. We consider a scalar conservation law in one space dimension, i.e.

\[ u_t + [f(u)]_x = 0. \]  

Note that (2.1) is (1.1) in the case \( N = 1 \). Assume that \( u \) is a classical solution, then we can rewrite (2.1) in the quasilinear form

\[ u_t + f'(u)u_x = 0. \]

A characteristic line of (2.1) is a solution of the ODE

\[ \frac{dX}{dt} = f'(u(t, X)) \]

The key observation is that classical solutions are constant along the characteristic lines. Indeed,

\[ \frac{d}{dt}u(t, X) = u_t + u_x \frac{dX}{dt} = u_t + u_x f'(u) = 0. \]

In the following, we term \( X(t, y) \) the solution of the Cauchy problem

\[ \begin{cases} \frac{dX}{dt} = f'(u(t, X)) \\ X(0, y) = y. \end{cases} \]

In other words, \( X(t, y) \) is the characteristic line starting at \( y \).

2.1.1. Example: linear equations. Consider the scalar, linear equation

\[ u_t + a u_x = 0 \quad a \in \mathbb{R}. \]

In this case \( f'(u) = a \) and hence the all the characteristic lines have slope \( a \). This means that by solving (2.4) we arrive at \( X(t, y) = y + at \). Now consider the Cauchy problem by coupling (2.5) with the initial datum

\[ u(0, x) = u_0(x). \]

Since \( u \) is constant along the characteristic lines, to determine the value of \( u \) at the point \( (t, x) \) we simply have to determine the value at \( t = 0 \) of the characteristic line passing through \( (t, x) \), which is \( y = x - at \). This implies that the solution of the Cauchy problem (2.5), (2.6) is \( u(t, x) = u_0(x - at) \).

2.1.2. Example: Burgers equation. We now construct the characteristic lines of the Burgers equation (1.6). Note that in this case \( f'(u) = u \). We remark that \( u \) is constant along the characteristic line \( X(t, y) \), which gives

\[ u(t, X(t, y)) = u(0, X(0, y)) = u_0(y). \]

This implies that \( X(t, y) = y + u_0(y)t \). We would like to proceed as before and construct the solution of the Cauchy problem (1.6)-(2.6) by using the method of characteristics (i.e., by using the characteristic lines).

Given a point \( (t, x) \), we should determine the characteristic line passing through \( (t, x) \), i.e. the point \( y \) such that \( X(t, y) = y + u_0(y)t = x \). In general, this is not possible. However, we have the following result. For simplicity, we assume that the initial datum is compactly supported, but this assumption can be greatly relaxed.

2.2. Lemma. Let \( u_0 \in C_c^1(\mathbb{R}) \), then there is \( T > 0 \) such that for every \( (t, x) \in [0, T] \times \mathbb{R} \) there is exactly one characteristic line \( X(t, y) \) passing through \( (t, x) \).
Proof. We fix $t > 0$ and we consider the map $X(t, \cdot) : \mathbb{R} \to \mathbb{R}$, given by $X(t,y) = y + f'(u_0(y))t$.

**STEP 1:** we show that the map is surjective. First, we point out that, since $u_0$ is continuous, then $X(t, \cdot)$ is continuous and hence the image of an interval is an interval. Since $u_0$ is also compactly supported (and hence bounded) we get

$$\lim_{y \to -\infty} X(t,y) = \lim_{y \to -\infty} y + u_0(y)t = -\infty \quad \text{and} \quad \lim_{y \to +\infty} X(t,y) = +\infty.$$ 

and this shows that the image is the whole $\mathbb{R}$.

**STEP 2:** we show that the map is strictly increasing provided that $t$ is sufficiently small. We compute the derivative

$$\left( y + u_0(y)t \right)' = 1 + u_0'(y)t.$$ 

Since $u_0'$ is continuous and compactly supported, it is also bounded and hence $1 + u_0'(y)t > 0$ for every $y$ provided that $t$ is sufficiently small.

**STEP 3:** by combining **STEP 1** and **STEP 2** we get that the map $X(t, \cdot)$ is invertible and hence for every $(t,x)$ there is exactly one $y$ such that $X(t,y) = x$. 

By using Lemma 2.2 we can construct a local in time classical solution of the Cauchy problem (1.6)-(2.6). Indeed, in the proof of Lemma 2.2 we have shown that the map $X(t, \cdot)$ is invertible provided $t$ is sufficiently small. We term $Y(t, \cdot)$ its inverse. We set $u(t,x) := u_0(Y(t,x))$. By construction, $u$ is constant along the characteristic lines.

**2.3. Exercise.** Show that $u$ is of class $C^1$ and pointwise satisfies the Burgers equation (1.6).

**2.4. Exercise.** Construct the characteristic lines of a general scalar equation (2.1). Show that, if $u_0 \in C^1_c(\mathbb{R})$ and $t$ is sufficiently small, for every point $(t,x)$ there is exactly one characteristic line passing through $(t,x)$. Conclude that the Cauchy problem (2.1),(2.6) has a local in time classical solution.

**2.5. Exercise.** Show that the Cauchy problem (2.1)-(2.6) has at most one classical solution.

**2.6. Finite propagation speed.** By using the method of characteristics, we obtain a very important property of the conservation law (2.1): equation (2.1) has finite propagation speed.

To understand what this means, we first consider the linear equation (2.5) and we assume that the initial datum $u_0$ satisfies $\text{supp } u_0 \subseteq -R, R[ for some $R > 0$. Since the solution of the Cauchy problem (2.5),(2.6) is $u(t,x) = u_0(x-at)$, then

$$u(t,x) \equiv 0 \text{ if } x < -R + at \text{ and } x > R + at.$$ 

Next, we consider the Burgers equation (1.6), we assume as before that the initial datum $u_0$ satisfies $\text{supp } u_0 \subseteq -R, R[ and we term $M := \max_{x \in [-R,R]} |u_0(x)|$. Since the solution is constant along the characteristic lines $X(t,y) = y + u_0(y)t$, then we get

$$u(t,x) \equiv 0 \text{ if } x < -R - Mt \text{ and } x > R + Mt.$$ 

In general, consider the Cauchy problem (2.1),(2.6) and assume that $u_0$ satisfies $\text{supp } u_0 \subseteq -R, R[. Let $M := \max_{x \in [-R,R]} |f'(u_0(x))|$. By using the method of characteristics we arrive at (2.7). Loosely speaking, we can say that the “information” $u_0 \neq 0$ travels with finite propagation speed and does not instantaneously propagate to the whole $\mathbb{R}$.

Finite propagation speed is a very important feature of conservation laws. Note that the solutions of other classes of equations have a completely different behavior. For instance, consider the Cauchy problem for the heat equation

$$\begin{cases}
    v_t = v_{xx} \\
    v(0,x) = v_0(x)
\end{cases}$$

and assume that $v_0 \in C^1_c(\mathbb{R})$ and that $v_0(x) \geq 0$ for every $x \in \mathbb{R}$. If $v_0$ is not identically zero, then the solution $v$ satisfies $v(t,x) > 0$ for every $x \in \mathbb{R}$ and every $t > 0$. In the case of the heat equation, the “information” $v_0 > 0$ instantaneously propagates to the whole $\mathbb{R}$.
2.7. The method of characteristics for systems. Despite higher technical difficulties, the method of characteristics is also useful in the analysis of a system of conservation laws (1.1). We start by introduction a very common assumption in the analysis of conservation laws.

2.8. Definition. We term system (1.1) strictly hyperbolic if, for every \( u \in \mathbb{R}^N \), the Jacobian matrix \( Df(u) \) has \( N \) real and distinct eigenvalues

\[
\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_N(u).
\]

Remarks:

- if \( N = 1 \), strict hyperbolicity is trivially satisfied.
- one can show that the Euler equations (1.7) are strictly hyperbolic for physically sounded choices of the equation of state. For instance, it is strictly hyperbolic in the case of a so-called polytropic gas, where the pressure is given by \( p(\rho, e) = C \rho e \), where \( C \) is a constant depending on the gas.

2.8.1. Linear strictly hyperbolic systems. Consider the Cauchy problem obtained by coupling the linear equation (1.5) with the initial datum (1.2). Assume that \( A \) is strictly hyperbolic, i.e. that it has \( N \) real and distinct eigenvalues. This implies that \( A \) is diagonalizable. Let \( \ell_1, \ldots, \ell_N \) denote the left eigenvectors of \( A \), i.e.

\[
(2.8) \quad \ell_i \in \mathbb{R}^N, \quad \ell_i \neq 0, \quad \ell_i \cdot A = \lambda_i \ell_i, \quad i = 1, \ldots, N.
\]

Next, we set \( u_i := \ell_i \cdot u \). By left multiplying (2.5) times \( \ell_i \) we arrive at

\[
(2.9) \quad u_{it} + \lambda_i u_{ix} = 0 \quad i = 1, \ldots, N.
\]

This is the same equation as (2.5) provided that \( a = \lambda_i \) and hence by arguing as in \( \S \) 2.1.1 we get \( u_i(t, x) = u_{i0}(x - \lambda_i t) \), where \( u_{i0} = \ell_i \cdot \mathbf{u}_0 \).

Summing up: assume that \( \mathbf{u}_0 \in C^1_c(\mathbb{R}) \), then with the method of characteristics we can construct classical solutions the Cauchy problem obtained by coupling (1.5) with (1.2). However, there is an important difference with the scalar equation: in the scalar case case (2.1) there is a single characteristic line emanating from the point \((0, y)\) and the solution is transported along the characteristic lines. In the case of the linear system (1.5) there are \( N \) characteristic lines emanating from each point \((0, y)\), and the different components \( u_1, \ldots, u_N \) are constant along the different characteristic lines.

2.8.2. Semilinear strictly hyperbolic systems. A semilinear hyperbolic system is a system of equations in the form

\[
(2.10) \quad u_t + A(t, x)u_x = 0,
\]

where the matrix \( A \in \mathbb{M}^{N \times N} \) depends on \((t, x)\), but not on \( u \). Assume that \( A \in C^1(\mathbb{R}_+ \times \mathbb{R}) \) and that \( u_0 \in C^1_c(\mathbb{R}) \) and that \( A \) is strictly hyperbolic, then one can use the method of characteristics to establish global in time existence of the solution of the Cauchy problem obtained by coupling (2.10) with (1.2). We refer to [3, pp. 46-64] for the detailed construction.

2.8.3. General strictly hyperbolic system. We now consider the general case. We consider the Cauchy problem obtained by coupling (1.1) with (1.2) and we assume that \( \mathbf{u}_0 \in C^1_c(\mathbb{R}) \). We want to use the method of characteristics to construct classical solutions. As in the case of the Burgers equation (see \( \S \) 2.1.2) we can only hope for local in time existence.

By using the chain rule, we can re-write (1.1) in the so-called quasilinear form

\[
(2.11) \quad u_t + Df(u) \cdot u_x = 0.
\]

To construct classical solutions we can rely on an iteration method: we set \( \mathbf{u}^0(t, x) := \mathbf{u}_0(x) \) and we inductively define \( \mathbf{u}^{m+1} \) as the classical solution of the Cauchy problem obtained by coupling (1.2) with

\[
(2.12) \quad u_t^{m+1} + A(u^m)u_x^{m+1} = 0.
\]
Note that $u^m$ is defined by the $m$-th inductive step and hence (2.12) is a semilinear equation, which can be solved by the method of characteristics as we pointed out in § 2.8.2. It turns out that, if the time interval is sufficiently small, then the functions $u^m$ converge uniformly on compact sets to some limit $u$, which is a classical solution of (1.1). We refer to [3, pp. 64-71] for the detailed construction.

3. **Weak solutions**

3.1. **Breakdown of classical solutions.** We now show that, in general, classical solutions are only defined on a finite time interval.

Consider the Cauchy problem obtained by coupling the Burger's equation (1.6) with the initial datum

\[ u_0(x) = \frac{1}{1 + x^2}. \]

We recall the analysis in § 2.1.2 and, in particular, that the characteristic line starting at $y$ is

\[ X(t, y) = y + u_0(y)t \]

This implies that

\[ X(t, 0) = t, \quad X(t, 1) = 1 + \frac{t}{2}. \]

Note that the above lines intersect at $t = 2$. This apparently yields a contradiction since $u = 1$ along the characteristic line $X(t, 0)$ and $u = 1/2$ along the characteristic line $X(t, 1)$. The contradiction is solved by recalling that to show that $u$ is constant along the characteristic line we had to assume that the function is sufficiently regular. The fact that the characteristic intersect means that the solution is not regular at the time when they intersect. This phenomenon is known as the **breakdown of classical solutions**: even if the initial datum is very regular, in general classical solution of the Cauchy problem are only defined on a finite time interval. This implies that we cannot hope for results providing global in time existence of classical solutions.

3.2. **Remark.** As a matter of fact, classical solutions of (1.6), (3.1) develop discontinuities in finite time. Note that this is a very severe loss of regularity: indeed, if $u \in W^{1,1}_{\text{loc}}(\mathbb{R})$, then $u$ is a continuous function\(^2\). Since the solution of (1.6), (3.1) develops discontinuities in finite time, the solution does not belong to any Sobolev space $W^{1,p}$, $p \geq 1$. This implies that we cannot hope for global in time existence results in Sobolev spaces. From the technical viewpoint, this is a big challenge since Sobolev spaces are a very useful and common tool to study partial differential equations (PDEs). As a matter of fact, many functional analytic tools that have been successfully applied to analysis of other classes of PDEs cannot be applied to conservation laws.

3.3. **Weak solutions.** Since in general classical solutions are only defined on a finite time interval it is very natural to introduce a notion of weak solution.

3.4. **Definition.** A **weak solution of the system of conservation laws** (1.1) is a locally summable function $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^N$ such that $f(u)$ is also locally summable and the following two requirements are satisfied.

i) **The function $u$ is a distributional solution of (1.1).** In other words, for every test function $\varphi \in C_c^\infty([0, +\infty[ \times \mathbb{R})$, we have

\[ \int_0^T \int_\mathbb{R} u \varphi_t + f(u) \varphi_x \, dx \, dt = 0. \]

ii) **The function $t \mapsto u(t, \cdot)$ is continuous from $\mathbb{R}_+$ to $L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N)$, endowed with the strong topology.**

Note that any classical solution is a weak solution: in particular, condition (3.2) holds true because of the Integration by Parts Formula.

\(^2\)We recall that $u \in W^{1,1}_{\text{loc}}(\mathbb{R})$ if and only if $u \in W^{1,1}_{\text{loc}}(I)$ for every bounded interval $I \subseteq \mathbb{R}$.
3.4.1. The Cauchy problem. We have so far given the definition of weak solution of the conservation law (1.1). We now want to give the definition of weak solution of the Cauchy problem obtained by coupling (1.1) with the initial datum (1.2).

Note that, if \( u \) is only a locally summable function, the pointwise value \( u(t, x) \) is not well-defined. However, owing to condition ii) in Definition 3.4, we can give a meaning to \( u(t, \cdot) \) for every \( t \in [0, +\infty[ \). This justifies the following definition.

3.5. Definition. The locally summable function \( u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^N \) is a weak solution of the Cauchy problem (1.1), (1.2) if it is a weak solution of (1.1) and furthermore \( u(0, \cdot) = u_0 \).

3.6. Exercise. Show that, if \( u \) is a weak solution of the Cauchy problem (1.1) and \( \psi \in C^\infty_c([0, +\infty[ \times \mathbb{R}) \), we have

\[
\int_0^T \int_\mathbb{R} u\psi_t + f(u)\psi_x \, dxdt + \int_\mathbb{R} \psi(0,x)u_0(x) \, dx = 0
\]

3.7. Rankine-Hugoniot conditions. To investigate the structure of weak solutions, we investigate the simplest possible “irregular solution”, which only attains two values.

3.8. Lemma (Rankine-Hugoniot conditions). Fix \( \lambda \in \mathbb{R}, u^+, u^- \in \mathbb{R}^N \). The function

\[
w(t, x) := \begin{cases} u^+ & x > \lambda t \\ u^- & x < \lambda t \end{cases}
\]

is a weak solution of (1.1) if and only if the so-called Rankine-Hugoniot conditions hold, i.e.

\[
f(u^+) - f(u^-) = \lambda (u^+ - u^-).
\]

Proof. It is easy to show that \( w \) is locally summable, \( f(w) \) is locally summable and the map \( t \mapsto w(t, \cdot) \) is continuous from \( \mathbb{R} \) to \( L^1_{\text{loc}}(\mathbb{R}) \). We are left to discuss (3.2). We decompose the set \( \mathbb{R}^+ \times \mathbb{R} \) as \( \mathbb{R}^+ \times \mathbb{R} = \Omega^- \cup \Omega^+ \cup \Lambda \), where

\[
\Omega^- = \{(t, x) : x < \lambda t\}, \quad \Omega^+ = \{(t, x) : x > \lambda t\}, \quad \Lambda := \{(t, x) : x = \lambda t\}.
\]

The function \( w \) is a weak solution of (1.1) if and only if it satisfies (3.2) for every \( \varphi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}) \). Since the set \( \Lambda \) is negligible, then (3.2) can be rewritten as

\[
\int\int_{\Omega^-} w\varphi_t + f(w)\varphi_x \, dxdt + \int\int_{\Omega^+} w\varphi_t + f(w)\varphi_x \, dxdt = 0.
\]

By recalling the explicit expression of \( w \) (3.3), we can rewrite the above equation as

\[
\int\int_{\Omega^-} u^-\varphi_t + f(u^-)\varphi_x \, dxdt + \int\int_{\Omega^+} u^+\varphi_t + f(u^+)\varphi_x \, dxdt = 0.
\]

To avoid some technicalities, in the remaining part of the proof we assume that \( N = 1 \). However, the argument easily extends to the general case (one can basically repeat the same argument for each component of \( u \)). We consider the function

\[
g : \Omega^- \to \mathbb{R}^2 \\
(x, t) \mapsto (f(u^-)\varphi, u^-\varphi).
\]

Note that \( g \in C^\infty_c(\mathbb{R}^2; \mathbb{R}^2) \) and furthermore

\[
\text{div}_{x,t} g = f(u^-)\varphi_x + u^-\varphi_t.
\]

By recalling that \( \varphi \) is compactly supported (and hence the integral over \( \Omega^- \) reduces to the integral over a bounded set) and applying the Gauss-Green formula we get

\[
\int\int_{\Omega^-} f(w)\varphi_x + w\varphi_t \, dxdt = \int\int_{\Omega^-} \text{div}_{x,t} g \, dxdt = \int_{\Lambda} g \cdot n \, d\mathcal{H}^1,
\]
where \( \mathbf{n} \) is the unit outward pointing normal vector to \( \Lambda \). Since \( \mathbf{n} = (1, -\lambda)/\sqrt{1 + \lambda^2} \), we eventually get
\[
\int_{\Omega^-} f(w)\varphi_x + w\varphi_t \, dx \, dt = \frac{1}{\sqrt{1 + \lambda^2}} \int_{\Lambda} \varphi \left( f(u^-) - \lambda u^- \right) \, dH^1.
\]
We now compute the integral over \( \Lambda^+ \) by following exactly the same argument as before, the only difference is that the outward pointing unit normal vector is now \( (-1, \lambda)/\sqrt{1 + \lambda^2} \). We eventually get
\[
\int_{\Omega^+} f(w)\varphi_x + w\varphi_t \, dx \, dt = \frac{1}{\sqrt{1 + \lambda^2}} \int_{\Lambda} \varphi \left( -f(u^+) + u^+ \right) \, dH^1.
\]
By plugging (3.6) and (3.7) into (3.5) we get
\[
\int_{\Lambda} \varphi \left( f(u^-) - f(u^+) \right) - \lambda (u^- - u^+) \, dH^1 = 0.
\]
If conditions (3.4) are satisfied, then (3.8) holds and hence \( w \) is a distributional solution of (1.1). Conversely, if \( w \) is a distributional solution then by (3.8) and the arbitrariness of the function \( \varphi \) we get (3.4). This concludes the proof of the lemma.

3.9. Exercise. Exhibit a weak solution of the Cauchy problem obtained by coupling the Burgers equation (1.6) with the initial datum
\[
u(0, x) = \begin{cases} 
1 & x > 0 \\
0 & x < 0
\end{cases}
\]

3.10. Exercise. Exhibit a weak solution of the Cauchy problem
\[
\begin{aligned}
& u_t + |u^3|_x = 0 \\
& u(0, x) = \begin{cases} 
2 & x > 0 \\
-1 & x < 0
\end{cases}
\end{aligned}
\]

3.11. Non-uniqueness of weak solutions. We now show that a given Cauchy problem has, in general, infinitely many weak solutions. Consider the Cauchy problem
\[
\begin{aligned}
& u_t + \left( \frac{u^2}{2} \right)_x = 0 \\
& u(0, x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x < 0
\end{cases}
\end{aligned}
\]
We now fix \( \alpha \in]0, 1[ \) and consider the function \( u_\alpha \) which is defined by setting
\[
u_\alpha(t, x) = \begin{cases} 
0 & x < \frac{\alpha t}{2} \\
\alpha & \frac{\alpha t}{2} < x < \frac{1 + \alpha}{2} t \\
1 & x > \frac{1 + \alpha}{2} t.
\end{cases}
\]

3.12. Lemma. For every \( \alpha \in]0, 1[ \), the function \( u_\alpha \) defined as in (3.10) is a weak solution of the Cauchy problem (3.9).

Proof. We point out that \( u_\alpha \) satisfies condition ii) in Definition 3.4. Also, one can show that \( u_\alpha \) satisfies (3.2) if and only if the Rankine-Hugoniot condition (3.4) are satisfied at both discontinuity lines. At the discontinuity line \( x = \alpha t/2 \), the Rankine-Hugoniot condition (3.4) boil down to
\[
\frac{1}{2}(\alpha^2 - 0^2) = \frac{\alpha}{2}(\alpha - 0),
\]
which is satisfied. At the discontinuity $x = (1 + \alpha)/2$, the Rankine-Hugoniot condition boils down to

$$\frac{1}{2}(1 - \alpha^2) = \frac{1 + \alpha}{2}(1 - \alpha),$$

which is also satisfied. This implies that, for every $\alpha \in [0, 1]$, $u_\alpha$ is a weak solution of (3.10). □

3.13. **Exercise.** Show that the function $u_\alpha$ in (3.10) satisfies (3.2) if and only if the Rankine-Hugoniot conditions (3.4) are satisfied at both discontinuity lines.

3.14. **Exercise.** Show that the function (3.11)

$$u(t, x) = \begin{cases} 
0 & x < -\frac{1}{2}t \\
-1 & -\frac{1}{2}t < x < 0 \\
1 & 0 < x < \frac{1}{2}t \\
0 & x > \frac{1}{2}t.
\end{cases}$$

is a solution of the Cauchy problem obtained by coupling (1.6) with the initial datum $u_0 \equiv 0$. Conclude that finite propagation speed can be violated.

### 4. Entropy admissible solutions

#### 4.1. Introduction

Let us sum up what we have seen so far. We want to obtain a well-posedness theory for the Cauchy problem (1.1)-(1.2). In particular, we want to obtain existence and uniqueness of a suitable notion of solution. We have first considered classical solutions, but we cannot hope for global in time existence of classical solutions, since in general classical solutions breakdown in finite time. Next, we have introduced a notion of weak solution by interpreting (1.1) in the sense of distributions. We have pointed out that weak solutions are, in general not unique.

To overcome this problem, various admissibility criteria have been introduced in the literature (see in particular Dafermos [4] for an extended discussion). Introducing an admissibility criterion amounts to augmenting the definition of weak solution with additional conditions in an attempt at selecting uniqueness. In this section we go over one of the most popular admissibility criteria, the entropy admissibility criterion.

In the next section we will see that one can indeed establish existence and uniqueness of entropy admissible solutions of the Cauchy problem, at least in the scalar case.

#### 4.2. The entropy admissibility criterion

4.3. **Definition.** An entropy-entropy flux pair for system (1.1) is a couple of $C^2$ functions $(\eta, q)$, $\eta : \mathbb{R}^N \to \mathbb{R}$, $q : \mathbb{R}^N \to \mathbb{R}$, such that

$$\nabla \eta(u) Df(u) = \nabla q(u) \quad \forall u \in \mathbb{R}^N.$$  

Some remarks are here in order:

- Scalar equations have plenty of entropy-entropy flux pairs. Indeed, consider the scalar case (2.1). Fix any $C^2$ function $\eta : \mathbb{R} \to \mathbb{R}$, then we can construct an entropy flux $q : \mathbb{R} \to \mathbb{R}$ by setting

$$q(u) = \int_0^u \eta'(z)f'(z)dz.$$

- In general, (4.1) is a system of $N$ equations in 2 unknowns ($\eta$ and $q$). Hence, when $N > 2$ it is overdetermined and in general has no solution. However, systems with physical applications typically do have entropy-entropy flux pairs. One should keep in mind that the mathematical entropy is usually the opposite of the physical entropy.

- In the following we will mostly focus on the case of entropy-entropy flux pairs $(\eta, q)$ where $\eta$ is convex.
Another important remark is the following property of classical solutions.

4.4. **Lemma.** Assume that system (1.1) has an entropy-entropy flux pair. Let \( u \in C^1 \) be a classical solution of (1.1). Then

\[
(4.2) \quad \eta(u)_t + q(u)_x = 0.
\]

**Proof.** It suffices to apply the chain rule:

\[
\eta(u)_t + q(u)_x = \nabla \eta(u)u_t + \nabla q(u)u_x \overset{(4.1)}{=} \nabla \eta(u)u_t + \nabla \eta(u)Du_x = \nabla \eta(u)\left[ u_t + Du_x \right]
\]

\[
= \nabla \eta(u)\left[ u_t + f(u)_x \right] \overset{(1.1)}{=} 0.
\]

\[\Box\]

Here is the definition of entropy admissible solution of (1.1).

4.5. **Definition.** Assume that system (1.1) has at least one entropy-entropy flux pair \((\eta, q)\) with \( \eta \) convex. A locally bounded function \( u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^N \) is an entropy admissible solution of (1.1) if it is a weak solution of (1.1) and furthermore the inequality

\[
(4.3) \quad \eta(u)_t + q(u)_x \leq 0
\]

is satisfied in the sense of distributions for every entropy-entropy flux pair \((\eta, q)\) with \( \eta \) convex. In other words,

\[
(4.4) \quad \int_0^{+\infty} \int_{\mathbb{R}} \eta(u)(u)_t \varphi + q(u)(u)_x \geq 0
\]

for every \( \varphi \in C_c^\infty([0, +\infty[ \times \mathbb{R}) \) such that \( \varphi(t, x) \geq 0 \) for every \((t, x)\).

Note that by space-integrating (4.3) (and by assuming that \( u \) is for instance compactly supported) we formally arrive at

\[
\frac{d}{dt} \int_{\mathbb{R}} \eta(u)(t, x) \, dx \overset{(4.3)}{=} - \int_{\mathbb{R}} q(u)_x(t, x) \, dx = 0,
\]

i.e. the total amount of entropy is monotone non-increasing. By recalling that the physical entropy is the opposite of \( \eta \), we infer that the definition of entropy admissible solution is consistent with the Second Principle of Thermodynamics.

Another way of justifying the definition of entropy admissible solution is through the following lemma.

4.6. **Lemma.** Assume that (1.1) has an entropy-entropy flux pair. Consider the family of equations

\[
(4.5) \quad u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}.
\]

Assume that the solutions \( u^\varepsilon \) are smooth, that \( u^\varepsilon \to u \) strongly in \( L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}) \) when \( \varepsilon \to 0^+ \) and that

\[
(4.6) \quad \|u^\varepsilon\|_{L^\infty} \leq M, \quad \text{for every } \varepsilon > 0
\]

for some constant \( M > 0 \). Then \( u \) satisfies (3.2) for every test function \( \varphi \) and furthermore satisfies (4.4) for every test function \( \varphi \geq 0 \) and every entropy-entropy flux pair \((\eta, q)\) with \( \eta \) convex.

Loosely speaking, the above lemma expresses the fact that the limit of the second order approximation (4.5) (which is called vanishing viscosity approximation) is a distributional solution, which moreover satisfies the entropy inequality (4.3) for every entropy-entropy flux pair \((\eta, q)\) with \( \eta \) convex.

**Proof of Lemma 4.6.** By combining (1.1) with the integration by parts formula, we conclude that

\[
(4.7) \quad \int_0^T \int_{\mathbb{R}} u^\varepsilon \varphi_t + f(u^\varepsilon)\varphi_x \, dx \, dt = -\varepsilon \int_0^T \int_{\mathbb{R}} u^\varepsilon \varphi_{xx} \, dx \, dt
\]
for every test function $\varphi$. We recall that $u^\varepsilon \to u$ strongly in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^N)$ when $\varepsilon \to 0^+$ and we remark that, owing to (4.7),

\[ \int_0^T \int_\mathbb{R} u\varphi_t + f(u)\varphi_x \, dx \, dt = \int_0^T \int_\mathbb{R} |u - u^\varepsilon|\varphi_t + |f(u) - f(u^\varepsilon)|\varphi_x \, dx \, dt - \varepsilon \int_0^T \int_\mathbb{R} u^\varepsilon \varphi_x \, dx \, dt \tag{4.8} \]

We term $L$ a Lipschitz constant for $f$ on the ball of radius $M$ in $\mathbb{R}^N$ and we point out that

\[ \int_0^T \int_\mathbb{R} |f(u) - f(u^\varepsilon)|\varphi_x \, dx \, dt \leq \int_0^T \int_\mathbb{R} |f(u) - f(u^\varepsilon)|\varphi_x \, dx \, dt \tag{4.9} \]

By using the fact that $\varphi_x$ is bounded and compactly supported and recalling that $u^\varepsilon \to u$ strongly in $L^1$ on compact sets of $\mathbb{R}^+ \times \mathbb{R}$ we conclude that the right hand side of (4.9) converges to 0 as $\varepsilon \to 0^+$. By analogous, but easier, computations we conclude that the right hand side of (4.8) converges to 0, which means that $u$ satisfies (3.2), i.e. that it is a distributional solution of (1.1).

To establish the entropy inequality, we focus on the case $N = 1$, but the argument extends to the general case $N > 1$. We write (4.5) in the quasilinear form

\[ u^\varepsilon_t + f'(u^\varepsilon)u^\varepsilon_x = \varepsilon \ u^\varepsilon_{xx}, \]

we left multiply times $\eta'(u^\varepsilon)$ and we use (4.1). We obtain

\[ \eta'(u^\varepsilon)u^\varepsilon_t + \eta'(u^\varepsilon)f'(u^\varepsilon)u^\varepsilon_x = \varepsilon \eta'(u^\varepsilon)u^\varepsilon_{xx} = \varepsilon \eta(u^\varepsilon)_{xx} - \varepsilon \eta''(u^\varepsilon)(u^\varepsilon_x)^2. \]

We now fix a test function $\varphi \geq 0$, we multiply the above equation times $\varphi$ and we use the integration by parts formula. We arrive at

\[ \int_0^T \int_\mathbb{R} \eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x \, dx \, dt = -\varepsilon \int_0^T \int_\mathbb{R} \eta(u^\varepsilon)\varphi_{xx} + \varepsilon \eta''(u^\varepsilon)(u^\varepsilon_x)^2 \varphi \, dx \, dt + \int_0^T \int_\mathbb{R} \eta(u^\varepsilon)\varphi_{xx} \, dx \, dt. \]

To establish the last inequality, we have used the fact that $\eta$ is convex, and hence $\eta'' \geq 0$, and that $\varphi \geq 0$. By arguing as in (4.9) and letting $\varepsilon \to 0^+$ we arrive at (4.4) (the details are left as an exercise).

4.7. Entropy admissible discontinuities. Lemma 4.4 ensures that classical solutions are always entropy admissible. It is not, in general, immediate to determine whether or not a given weak solution is entropy admissible. The following lemma establishes a simple criterium valid in the case when the weak solution is in the form (3.3).

4.8. Lemma. Fix $\lambda \in \mathbb{R}$, $u^+, u^- \in \mathbb{R}^N$ and let $w$ be the function defined as in (3.3). Then the entropy inequality (4.4) is satisfied for every test function $\varphi \geq 0$ if and only if

\[ \lambda [\eta(u^+) - \eta(u^-)] \geq q(u^+) - q(u^-). \tag{4.10} \]


By relying on Lemma 4.8 we obtain the following.

4.10. Lemma. Fix $\alpha \in [0, 1]$ and let $u_\alpha$ be the same as in (3.10). Then $u_\alpha$ is not an entropy admissible solution.

Proof. First, we set

\[ \eta(u) := \frac{1}{2} u^2, \quad q(u) := \frac{1}{3} u^3. \]

Since

\[ \eta'(u)f'(u) = u \cdot u = q'(u), \]

then $(\eta, q)$ is an entropy-entropy flux pair with $\eta$ convex. Next, we point out (the proof is left as an exercise) that $u_\alpha$ is entropy admissible if and only if condition (4.10) is satisfied at both the
discontinuities lines \( x = \alpha t/2 \) and \( x = (1 + \alpha)t/2 \). Let us verify that condition (4.10) is violated at the discontinuity line \( x = \alpha t/2 \):

\[
\alpha^3/4 = \frac{\alpha}{2} \left( \frac{1}{2} \alpha^2 - \frac{1}{2} \alpha^0 \right) \nless \frac{1}{3} \alpha^3 - \frac{1}{3} \alpha^0 = \frac{\alpha^3}{3}.
\]

This implies that \( u_\alpha \) is not an entropy admissible solution. \( \square \)

4.11. Exercise. Show that the function \( u \) defined as in (3.11) is not an entropy admissible solution of (1.6).

5. Well-posedness theory for scalar conservation laws

5.1. Functions of bounded total variation in one space variable. Before discussing the well-posedness results, we have to define the functional space we use. We refer to the book [1, § 3.2] and [5] for an extended discussion about functions of bounded variation, in both one and more space variables.

5.1.1. Definition. Let \( u : \mathbb{R} \to \mathbb{R}^N \) be a function. Consider a set of points \( x_1, \ldots, x_M \in \mathbb{R} \) such that \( x_1 < x_2 < \cdots < x_M \) and consider the quantity

\[
\sum_{i=1}^{M-1} |u(x_{i+1}) - u(x_i)|.
\]

Next, consider the supremum of the above quantity among all the possible sets of points \( x_1, \ldots, x_M \in \mathbb{R} \). In this way we obtain a quantity that we term “pointwise variation”,

\[
pV(u) := \sup_{x_1 < x_2 < \cdots < x_M} \sum_{i=1}^{M-1} |u(x_{i+1}) - u(x_i)|.
\]

5.2. Exercise. Show that, if \( u : \mathbb{R} \to \mathbb{R} \) is a monotone function, then

\[
pV(u) = \left| \lim_{x \to +\infty} u(x) - \lim_{x \to -\infty} u(x) \right|.
\]

Note that in general we can have \( pV(u) = +\infty \).

5.3. Exercise. Exhibit a discontinuous function \( u : \mathbb{R} \to \mathbb{R} \) such that \( pV(u) < +\infty \).

There is one last issue we have to tackle to define the “total variation” of a function: the pointwise variation changes if we change the value of \( u \) on a zero measure set. To address this issue, we define the total variation of \( u \) by setting

\[
\text{TotVar} u := \inf \left\{ pV(v) : \ u(x) = v(x) \text{ for a.e. } x \in \mathbb{R} \right\}.
\]

5.4. Definition. A function of bounded variation is a function \( u \in L^1(\mathbb{R}; \mathbb{R}^N) \) such that \( \text{TotVar} u < +\infty \). We also introduce the space of functions of bounded total variation by setting

\[
BV(\mathbb{R}; \mathbb{R}^N) := \left\{ u \in L^1(\mathbb{R}; \mathbb{R}^N) : \text{TotVar} u < +\infty \right\}.
\]

If \( N = 1 \), we write \( BV(\mathbb{R}) \) instead of \( BV(\mathbb{R}; \mathbb{R}^N) \). One can show that, if \( u \in C^1(\mathbb{R}; \mathbb{R}^N) \), then

\[
\text{TotVar} u = \int_{\mathbb{R}} |u'(x)| \, dx.
\]

However, \( BV(\mathbb{R}; \mathbb{R}^N) \) contains also more irregular functions. For instance, there are discontinuous functions that belong to \( BV(\mathbb{R}; \mathbb{R}^N) \) (see also Exercise 5.1.1). This is important for us because in general solutions of conservation laws develop discontinuities in finite time, so we cannot work with functional spaces containing only continuous functions (see also Remark 3.2).
5.4.1. Some properties of BV functions. A BV function has right and left limits at every point.

5.5. Lemma. Assume that \(u \in BV(\mathbb{R}; \mathbb{R}^N)\), then \(u\) has a representative such that for every \(x_0 \in \mathbb{R}\) the right and left limits

\[
\lim_{x \to x_0^+} u(x), \quad \lim_{x \to x_0^-} u(x)
\]

both exist and are finite.

In the following we will use this characterization.

5.6. Theorem (Finite differences characterization of BV functions). Assume that \(L^1(\mathbb{R}; \mathbb{R}^N)\), then we have \(u \in BV(\mathbb{R}; \mathbb{R}^N)\) if and only if there is \(C > 0\) such that

\[
\int_{\mathbb{R}} |u(x + h) - u(x)| dx \leq C|h|, \quad \text{for every } h \in \mathbb{R}.
\]

Also, the smallest constant \(C > 0\) satisfying (5.6) is exactly \(\text{TotVar}\ u\).

We now state the Helly’s Compactness Theorem, which is actually a particular case of the Kolmogorov’s Compactness Theorem.

5.7. Theorem (Helly’s Compactness Theorem). Assume that \(\{u_k\} \subseteq BV(\mathbb{R}; \mathbb{R}^N)\) is a sequences of functions such that

\[
\|u_k\|_{L^\infty(\mathbb{R}, \mathbb{R}^N)} \leq M, \quad \text{TotVar}\ u_k \leq M
\]

for every \(k\) and for some \(M > 0\). Then there is a function \(u \in L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N)\) and a subsequence \(\{u_{k_j}\}\) such that \(u_{k_j} \to u\) strongly in \(L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N)\) as \(j \to +\infty\).

5.8. Well-posedness of the Cauchy problem for a scalar conservation law.

5.9. Theorem (Kružkov). Assume that \(u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})\), then there is a unique bounded, entropy admissible solution of the Cauchy problem (2.1), (2.6). The entropy admissible solution is defined globally in time.

The above theorem is due to Kružkov [9]. As a matter of fact, Kružkov’s analysis applies to a general scalar conservation law in several space dimensions, i.e. to (1.9). However, to simplify the exposition here we focus on the case of a single space variable.

6. The vanishing viscosity method

We now establish the existence part of Theorem 5.9.

6.1. Theorem. Assume that \(u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})\), then there is a bounded, entropy admissible solution of the Cauchy problem (2.1), (2.6).

6.2. Proof road map. To construct an entropy admissible solution of the Cauchy problem (2.1), (2.6) we consider the vanishing viscosity approximation

\[
u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}
\]

and we couple it with the initial datum

\[
u^\varepsilon(0, x) = u_0(x).
\]

We give for granted that, if \(u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})\), then the Cauchy problem (6.1), (6.2) has a global in time solution, which is smooth on \([0, +\infty[ \times \mathbb{R}\).

The basic idea underpinning the proof of Theorem 6.1 is the following:
we fix \( \varepsilon_n \to 0^+ \) and we show that, up to subsequences, \( u^{\varepsilon_n} \) converges to some limit \( u \) strongly in \( L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}) \). To establish strong convergence we rely on the Helly Compactness Theorem, so we establish uniform bounds on the \( L^\infty \) norm (see Lemma 6.5) and on the total variation (see equation (6.11)). As a matter of fact, the Helly Compactness Theorem alone does not suffice to establish convergence in \( L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}) \), but we will come back to this point in the following.

- We recall Lemma 4.6 and we use the \( L^1_{\text{loc}} \) convergence and the \( L^\infty \) bounds to show that the limit \( u \) is a distributional solution of (2.1), which furthermore satisfies the entropy inequality (4.3) for every entropy-entropy flux pair \( (\eta,q) \) with \( \eta \) convex.

- To conclude, we have to show that the limit \( u \) is continuous with respect to time in the \( L^1_{\text{loc}}(\mathbb{R}) \) topology. To show this we rely on Lemma 6.11.

6.3. Remark. Besides providing an existence proof, the analysis of the limit \( \varepsilon \to 0^+ \) of (6.1) is also interesting from other points of view. For instance, it is relevant in view of numerical applications, see the book by LeVeque [11] for an extended discussion about numerical methods for systems of conservation laws.


6.5. Lemma (Maximum principle). Let \( u^\varepsilon \) be the solution of (6.1), (6.2), then
\[
\|u^\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty}, \quad \text{for every} \ \varepsilon > 0 \ \text{and} \ t \geq 0.
\]

We first provide an heuristic justification of the reason why equation (6.1) should satisfy a maximum principle. Assume that \( u^\varepsilon \) is smooth, fix \( t > 0 \) and assume that \( x_0 \) is a point at which \( u^\varepsilon(t,\cdot) \) attains its maximum. Then
\[
u^\varepsilon_t(t,x_0) = -f'(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x} + u_{xx}^\varepsilon \leq 0.
\]

This implies that in a point of maximum the value of \( u^\varepsilon(t,x) \) is decreasing and hence it is reasonable to expect that the equation (6.1) has a maximum principle.

Proof of Lemma 6.5. We provide a formal proof, which could be rigorously justified at the price of higher technicalities. We set \( A := \|u_0\|_{L^\infty} \) and we introduce a function \( \beta : \mathbb{R} \to \mathbb{R} \) by setting
\[
\beta(z) := \begin{cases} 0 & \text{if} \ z \leq A \\ (z - A)^2 & \text{if} \ z \geq A 
\end{cases}
\]

Note that \( \beta \geq 0 \) and that \( \beta(z) > 0 \Leftrightarrow z > A \). Next, we consider equation (6.1) and we multiply it times \( \beta'(u^\varepsilon) \). We obtain
\[
\frac{\partial u^\varepsilon}{\partial t} + \beta'(u^\varepsilon) f'(u^\varepsilon) u^\varepsilon_x + \varepsilon \beta'(u^\varepsilon) u^\varepsilon_{xx} = \varepsilon \beta'(u^\varepsilon) u^\varepsilon_{xx} - \varepsilon \beta''(u^\varepsilon)(u^\varepsilon_x)^2.
\]

provided \( p' = \beta' f' \). By integrating (6.5) with respect to \( x \) and recalling that \( \beta'' \geq 0 \) we arrive at
\[
\frac{d}{dt} \int_{\mathbb{R}} \beta(u^\varepsilon)dx \leq 0.
\]

Note that
\[
\int_{\mathbb{R}} \beta(u_0)dx = 0
\]

because \( \beta(u_0) \equiv 0 \) since \( u_0(x) \leq A \) for a.e. \( x \). By combining the above two inequalities we arrive at
\[
\int_{\mathbb{R}} \beta(u^\varepsilon(t,x))dx = 0 \quad \text{for every} \ t \geq 0,
\]

which in turn implies that \( u^\varepsilon(t,x) \leq A \) for a.e. \( (t,x) \). By arguing in a similar way one can show that \( u^\varepsilon(t,x) \geq -A \) for a.e \( (t,x) \). \( \square \)
6.6. Exercise. By introducing a suitable function similar to the function $\beta$ in (6.4), complete the proof of Lemma 6.5 by showing that $u^\varepsilon(t,x) \geq -A$ for a.e. $(t,x)$.

6.7. Space stability estimates.

6.8. Lemma. Assume $u_0, v_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and consider the Cauchy problems

\begin{equation}
(6.6) \begin{cases}
  u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \\
  u^\varepsilon(0,x) = u_0(x)
\end{cases}
\begin{cases}
  v^\varepsilon_t + f(v^\varepsilon)_x = \varepsilon v^\varepsilon_{xx} \\
  v^\varepsilon(0,x) = v_0(x)
\end{cases}
\end{equation}

Then we have

\begin{equation}
(6.7) \int_\mathbb{R} |u^\varepsilon - v^\varepsilon|(t,x)dx \leq \int_\mathbb{R} |u_0 - v_0|(x)dx, \text{ for every } t \geq 0.
\end{equation}

Proof of Lemma 6.8. We only provide a very formal proof, which can be made rigorous by relying on suitable approximation arguments (see for instance [4]). By taking the difference between the two equations in (6.6) we get

\[ (u^\varepsilon - v^\varepsilon)_t + (f(u^\varepsilon) - f(v^\varepsilon))_t = \varepsilon(u^\varepsilon - v^\varepsilon)_{xx}. \]

We now multiply the above equation times $\beta'(u^\varepsilon - v^\varepsilon)$, where $\beta : \mathbb{R} \to \mathbb{R}$ is a suitable function, to be determined in the following. We arrive at

\[ \beta'(u^\varepsilon - v^\varepsilon)(u^\varepsilon - v^\varepsilon)_t + \beta'(u^\varepsilon - v^\varepsilon)(f(u^\varepsilon) - f(v^\varepsilon))_x = \varepsilon \beta'(u^\varepsilon - v^\varepsilon)(u^\varepsilon - v^\varepsilon)_{xx} \]

\[ = \varepsilon \beta(u^\varepsilon - v^\varepsilon)_{xx} - \beta''(u^\varepsilon - v^\varepsilon)[u^\varepsilon_x - v^\varepsilon_x]^2. \]

Next, we assume that $\beta$ is convex, which implies that $-\beta''(u^\varepsilon - v^\varepsilon)[u^\varepsilon_x - v^\varepsilon_x]^2 \leq 0$ and we integrate with respect to $x$. We get

\begin{equation}
(6.8) \frac{d}{dt} \int_\mathbb{R} \beta(u^\varepsilon - v^\varepsilon)dx + \int_\mathbb{R} \beta'(u^\varepsilon - v^\varepsilon)(f(u^\varepsilon) - f(v^\varepsilon))_x dx \leq \varepsilon \int_\mathbb{R} \beta(u^\varepsilon - v^\varepsilon)_{xx} dx = 0.
\end{equation}

We rewrite the second term as

\begin{equation}
(6.9) \int_\mathbb{R} \beta'(u^\varepsilon - v^\varepsilon)(f(u^\varepsilon) - f(v^\varepsilon))_x dx = -\int_\mathbb{R} \beta''(u^\varepsilon - v^\varepsilon)(u^\varepsilon_x - v^\varepsilon_x)(f(u^\varepsilon) - f(v^\varepsilon))dx.
\end{equation}

We now choose $\beta(z) = |z|$, which formally implies $\beta'(z) = \text{sign } z$, $\beta''(z) = \delta_{z=0}$. In particular,

\[ \int_\mathbb{R} \beta''(u^\varepsilon - v^\varepsilon)(u^\varepsilon_x - v^\varepsilon_x)(f(u^\varepsilon) - f(v^\varepsilon))dx = \int_\mathbb{R} \delta_{u^\varepsilon=v^\varepsilon_x}(u^\varepsilon_x - v^\varepsilon_x)(f(u^\varepsilon) - f(v^\varepsilon))dx = 0,
\]

which implies that the left hand side of (6.9) is 0 and by recalling (6.8) we arrive at

\[ \frac{d}{dt} \int_\mathbb{R} |u^\varepsilon - v^\varepsilon|(t,x)dx \leq 0,
\]

which eventually implies (6.7).

Important corollaries.

6.9. Corollary. Assume $u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and let $u^\varepsilon$ be the solution of the Cauchy problem (6.1), (6.2). We have

\begin{equation}
(6.10) \|u^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} \quad \text{for every } t \geq 0, \varepsilon > 0.
\end{equation}

and

\begin{equation}
(6.11) \text{TotVar } u^\varepsilon(t,\cdot) \leq \text{TotVar } u_0, \quad \text{for every } t \geq 0, \varepsilon > 0.
\end{equation}
Proof of Corollary 6.9. To establish (6.10), it suffices to apply (6.7) with \( v_0 \equiv 0 \), which implies that \( v^\varepsilon \equiv 0 \). To establish (6.11), we fix \( h \in \mathbb{R} \) and we apply (6.7) with \( v_0(x) := u_0(x + h) \), which implies that \( v^\varepsilon(t,x) = u^\varepsilon(t,x + h) \). We get
\[
\int_\mathbb{R} |u^\varepsilon(t,x+h) - u^\varepsilon(t,x)| \, dx \leq \int_\mathbb{R} |u_0(x+h) - u_0(x)| \, dx, \quad \text{for every } t \geq 0, \varepsilon > 0. \tag{6.12}
\]
We now recall Theorem 5.6: since \( u_0 \in BV(\mathbb{R}) \), then
\[
\int_\mathbb{R} |u_0(x+h) - u_0(x)| \, dx \leq (\text{TotVar} u_0) |h|. \]
By plugging the above inequality into (6.12), we arrive at
\[
\int_\mathbb{R} |u^\varepsilon(t,x+h) - u^\varepsilon(t,x)| \, dx \leq (\text{TotVar} u_0) |h|, \quad \text{for every } t \geq 0, \varepsilon > 0
\]
and by using again Theorem 5.6, this implies (6.11). \( \square \)

6.10. Time stability bounds.

6.11. Lemma. Assume that \( u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}) \) and consider the Cauchy problem (6.1), (6.2). There is a non-decreasing function \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lim_{z \to 0^+} \omega(z) = 0 \) and
\[
\int_\mathbb{R} |u^\varepsilon(t+h,x) - u^\varepsilon(t,x)| \, dx \leq \omega(h), \quad \text{for every } t \geq 0, h > 0, \varepsilon > 0. \tag{6.13}
\]
Note that the function \( \omega \) in general depends on \( u_0 \), but does not depend on \( t \) or \( \varepsilon \). The proof of Lemma 6.11 is omitted, and can be found for instance in [4].

6.12. Conclusion of the proof of Theorem 6.1. We first discuss the basic idea underpinning the proof. We fix \( u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}) \) and we consider the family of Cauchy problems (6.1), (6.2). We want to show that the family \( \{u^\varepsilon\} \) is compact with respect to the strong topology of \( L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}) \). The basic idea is that we want to use the Helly Theorem to establish this compactness. However, we only have established total variation bounds for \( t \) fixed, see (6.11). By directly applying the Helly Theorem we get that, for every \( t \geq 0 \) the family \( \{u^\varepsilon(t,\cdot)\} \) is compact with respect to the strong topology of \( L^1_{\text{loc}}(\mathbb{R}) \), but a priori this does not provide sufficient information on the compactness of \( \{u^\varepsilon(t,\cdot)\} \) in \( L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}) \). To tackle this issue, we use the stability with respect to time, i.e. Lemma 6.11. To provide a formal proof, we fix a sequence \( \varepsilon_n \to 0^+ \) and we term \( u^n \) the corresponding sequence of solutions of the Cauchy problem (2.1), (2.6). Next, we proceed according to the following steps.

Step 1: we fix \( t > 0 \) and we recall (6.3) and (6.11). We apply the Helly Theorem 5.7 and we conclude that we can extract a subsequence \( u^{n_k} \) such that \( \{u^{n_k}(t,\cdot)\} \) converges to some locally summable function \( u(t,\cdot) \) in \( L^1_{\text{loc}}(\mathbb{R}) \) as \( k \to +\infty \).

Step 2: we apply a diagonal argument and we conclude that from \( \{u^n\} \) we can extract a sequence (that for simplicity we still call \( \{u^{n_k}\} \)) such that the following holds. For every \( q \in \mathbb{Q} \cap \mathbb{R}_+ \), the function \( \{u^{n_k}(q,\cdot)\} \) converges to some locally summable function \( u(q,\cdot) \) in \( L^1_{\text{loc}}(\mathbb{R}) \) as \( k \to +\infty \). Also, by passing to the limit in the time stability estimate (6.13) we conclude that
\[
\int_\mathbb{R} |u(q,x) - u(p,x)| \, dx \leq \omega(|q - p|), \quad \text{for every } q,p \in \mathbb{Q} \cap \mathbb{R}_+. \tag{6.14}
\]
By passing to the limit in the estimate (6.10) we also have
\[
\|u(q,\cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1}, \quad \text{for every } q \in \mathbb{Q} \cap \mathbb{R}_+. \tag{6.15}
\]

Step 3: we take \( t \in \mathbb{R}_+ \) and we consider a sequence \( \{q_j\} \) such that \( q_j \to t \) as \( j \to +\infty \). Owing to (6.7), the sequence \( \{u(q_j,\cdot)\} \) is a Cauchy sequence in \( L^1_{\text{loc}}(\mathbb{R}) \) and hence it has limit, that we term \( \{u(t,\cdot)\} \). By using again (6.14) we can conclude that the value \( \{u(t,\cdot)\} \) does not depend on the choice of the sequence \( \{q_j\} \). Also, we have
\[
\int_\mathbb{R} |u(t,x) - u(s,x)| \, dx \leq \omega(|t - s|), \quad \text{for every } t,s \in \mathbb{R}_+. \tag{6.16}
\]
and
\[ \|u(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1}, \quad \text{for every } t \in \mathbb{R}_+. \]

**Step 4:** so far we have constructed a “candidate limit function” \( u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \). We now want to show that \( u^{nk} \to u \) in \( L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}) \). In other words, we fix \( R > 0 \) and \( T > 0 \) and we want to show that
\[ \int_0^T \int_{-R}^R |u^{nk} - u|(t,x)dxdt \to 0 \quad \text{as } k \to +\infty. \]

We define the function
\[ f_{nk}(t) := \int_{-R}^R |u^{nk} - u|(t,x)dx. \]

Note that, owing to (6.17), for every \( t \in [0, T] \) we have \( 0 \leq f_{nk}(t) < 2\|u_0\|_{L^1(\mathbb{R})} \). Owing to the Lebesgue Dominated Convergence Theorem, to establish (6.18) it suffices to show that, for a.e. \( t \in [0, T] \), \( f_{nk}(t) \to 0 \) as \( k \to +\infty \). For every \( q \in \mathbb{R}_+ \cap \mathbb{Q} \), we can write
\[ 0 \leq f_{nk}(t) = \int_{-R}^R |u^{nk} - u|(t,x)dx \]
\[ \leq \int_{-R}^R |u^{nk}(t,x) - u^{nk}(q,x)|dx + \int_{-R}^R |u^{nk} - u|(q,x)dx + \int_{-R}^R |u(t,x) - u(q,x)|dx \]
\[ \leq 2\omega(|t - q|) + \int_{-R}^R |u^{nk} - u|(q,x)dx. \]

Fix \( \delta > 0 \): we can choose \( q \) in such a way that \( \omega(|t - q|) < \delta \). Next, we choose \( k(\delta, q) \) in such a way that, if \( k > k(\delta, q) \), then
\[ \int_{-R}^R |u^{nk} - u|(q,x)dx \leq \delta. \]

By plugging these inequalities into (6.19), we get that \( |f_{nk}(t)| \leq 3\delta \), and by the arbitrariness of \( \delta \) we conclude that \( f_{nk}(t) \to 0 \) as \( k \to +\infty \). By passing to the limit in the inequality (6.3), we also obtain that \( u \) is globally bounded.

**Step 5:** we recall Lemma 4.6 and the uniform \( L^\infty \) bound (6.3). Owing to the previous steps, we conclude that the limit \( u \) satisfies the distributional formulation (3.2) for every test function \( \varphi \) and the entropy inequality (4.4) for every rest function \( \varphi \geq 0 \) and every entropy-entropy flux pair \((\eta, q)\) with \( \eta \) convex. To show that \( u \) is an entropy admissible solution of (2.1), (2.6) we are left to prove that the map \( t \mapsto u(t, \cdot) \) is continuous with values in \( L^1_{\text{loc}} \). This follows from (6.16) and concludes the proof of Theorem 6.1.

7. **Uniqueness of entropy admissible solutions**

In this section we establish the uniqueness part of Theorem 5.9, namely

7.1. **Theorem.** Assume that \( u_0 \in L^\infty(\mathbb{R}) \). Then there is at most one bounded, entropy admissible solution of the Cauchy problem (2.1)-(2.6).

To establish the proof of the above theorem we first have to introduce the notion of Kružkov entropy.

7.2. **Kružkov’s entropy-entropy flux pairs.** Fix \( k \in \mathbb{R} \), and define \( \tilde{\eta}_k : \mathbb{R} \to \mathbb{R} \) by setting \( \tilde{\eta}_k(u) := |u - k| \). Plug \( \tilde{\eta}_k \) into (4.1), i.e. into the definition of entropy-entropy flux pair. We obtain
\[ \tilde{q}_k' = \text{sign}(u - k) f'(u) \implies \tilde{q}_k(u) = \text{sign}(u - k) [f(u) - f(k)]. \]

Note that \( \tilde{\eta}_k \) is convex but, technically speaking, the couple \((\tilde{\eta}_k, \tilde{q}_k)\) does not satisfy Definition 4.3 of entropy-entropy flux pair because \( \tilde{\eta}_k \) and \( \tilde{q}_k \) are not \( C^2 \). However, by relying on an approximation argument one can establish the following lemma.
7.3. Lemma. Assume that $u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R})$ satisfies the entropy inequality (4.4) for every test function $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$ such that $\varphi \geq 0$ and every entropy-entropy flux pair $(\eta, q)$ with $\eta$ convex. Then for every $k \in \mathbb{R}$

$$
(7.1) \quad |u - k|_t + \left[ \text{sign}(u - k) \left[ f(u) - f(k) \right] \right]_x \leq 0 \quad \text{in the sense of distributions, namely}
$$

$$
(7.2) \quad \int_0^{+\infty} \int_{\mathbb{R}} |u - k| \varphi_t \, dx \, dt + \int_0^{+\infty} \int_{\mathbb{R}} \text{sign}(u - k) \left[ f(u) - f(k) \right] \varphi_x \, dx \, dt \geq 0
$$

for every test function $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$ such that $\varphi(t, x) \geq 0$ for every $(t, x) \in \mathbb{R} \times \mathbb{R}$.

The couples $(\tilde{\eta}_k, \tilde{q}_k), k \in \mathbb{R},$ are sometimes termed Kružkov’s entropy-entropy flux pairs.

7.4. Kružkov’s uniqueness result. The main ingredient in the proof of Theorem 7.1 is the following result.

7.5. Lemma. Assume $u, v$ are two bounded, weak solutions of the conservation law (2.1) such that the inequality (7.2) holds for every test function $\varphi \geq 0$ and for every $k \in \mathbb{R}$. Then for every $t > 0$ and $R > 0$ we have

$$
(7.3) \quad \int_{-R}^{R} |u(t, x) - v(t, x)| \, dx \leq \int_{-R-Lt}^{R-Lt} |u(0, x) - v(0, x)| \, dx,
$$

where $L$ is a constant satisfying

$$
(7.4) \quad |f(a) - f(b)| \leq L|a - b|, \quad \text{for every } a, b \in \mathbb{R} \text{ such that } |a|, |b| \leq \min \{ \|u\|_{L^\infty}, \|v\|_{L^\infty} \}.
$$

Note that (7.3) is consistent with the finite propagation speed property: what happens at time $t$ on the interval $[-R, R]$ is only determined by what happens at time $t = 0$ on the interval $[-R - Lt, R + Lt]$.

Proof of Theorem 7.1. It follows from Lemma 7.5. Assume that there are two bounded, entropy admissible solutions of the Cauchy problem (2.1)-(2.6). We call them $u$ and $v$. Owing to Lemma 7.3, both $u$ and $v$ satisfy (7.1) for every $k \in \mathbb{R}$. We can then apply Lemma 7.5: fix $R > 0$ and $t > 0$, then by using (7.3) and the fact that $u$ and $v$ coincide at $t = 0$ we conclude that $u(t, x) = v(t, x)$ for a.e. $x \in ]-R, R[$. By the arbitrariness of $R$ we conclude that $u \equiv v$, which establishes uniqueness. □

7.6. Remark. What one actually shows in the proof of Theorem 7.1 is that there is a unique bounded weak solution of the Cauchy problem (2.1)-(2.6) that satisfies (7.1) for every $k \in \mathbb{R}$. On the other hand, if $u_0 \in L^\infty \cap BV(\mathbb{R})$, then there is a bounded entropy admissible solution and by Lemma 7.3 it satisfies (7.1) for every $k \in \mathbb{R}$. We conclude that, if $u_0 \in L^\infty \cap BV(\mathbb{R})$ and $u$ is the bounded weak solution satisfying (7.1) for every $k \in \mathbb{R}$, then $u$ is the entropy admissible solution. In other words, testing on Kružkov’s entropies suffices to single out the entropy admissible solution.\(^3\)

We only outline the main steps of the proof of Lemma 7.5 and we refer to [3, pp. 114-117] for the complete proof.

7.7. Proof of Lemma 7.5: heuristic strategy. We now describe the heuristic ideas underpinning the proof.

We fix $u$ and $v$ as in the statement of Lemma 7.5. The starting point is the entropy inequality (7.1). Assume for a moment that in (7.1) we can replace the constant $k$ with the function $v$ (as a matter of fact we cannot do this, because $v$ is not constant). Then we obtain

$$
(7.5) \quad |u - v|_t + \left[ \text{sign}(u - v) \left[ f(u) - f(v) \right] \right]_x \leq 0.
$$

Next, fix $R > 0, \tau > 0$ and consider the function

$$
(7.6) \quad \tau \mapsto \int_{-R-L(\tau-t)}^{R-L(\tau-t)} |u - v|(t, x) \, dx.
$$

\(^3\)One can actually directly show this fact by relying on the properties of convex functions, see for instance [12, p.35]
We formally compute its derivative and use (7.5):

\[
\frac{d}{dt} \int_{-R-L(t) t + L}^{R+L(t-t)} |u - v|(t, x) dx = \int_{-R-L(t) t + L}^{R+L(t-t)} |u - v|(t, x) dx + \\
- L|u - v|(t, R + L(t-t)) - L|u - v|(t, -R - L(t-t)) \\
\leq - \int_{-R-L(t) t + L}^{R+L(t-t)} \left[ \text{sign}(u - v) [f(u) - f(v)] \right] x (t, x) dx \\
- L|u - v|(t, R + L(t-t)) + L|u - v|(t, -R - L(t-t)) \\
= - \left[ \text{sign}(u - v) [f(u) - f(v)] + L|u - v| \right] (t, x = R + L(t-t)) \\
+ \left[ \text{sign}(u - v) [f(u) - f(v)] - L|u - v| \right] (t, x = -R - L(t-t))
\]

(7.7)

Owing to (7.4),

\[-L|u - v| \leq \text{sign}(u - v)[f(u) - f(v)] \leq L|u - v|\]

and hence the right hand side of (7.7) is \(\leq 0\), which implies that the map in (7.6) is non-increasing, and by evaluating it at \(t=\tau\) and \(t=0\) this eventually gives (7.3).

We now have to rigorously justify the above argument. There are two main challenges we have to tackle:

i) the entropy inequality (7.1) holds when \(k\) is a constant, but \(v\) in general in not a constant;

ii) the computation in (7.7) is formal and must be rigorously justify.

To tackle the first issue we rely on the so-called doubling of variable technique, which we discuss in the next paragraph. The doubling of variable technique allows us to rigorously establish the distributional formulation of (7.5), i.e. that for every test function \(\psi \geq 0\) we have

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} |u - v|\psi_t \ dx dt + \int_{\mathbb{R}^+} \int_{\mathbb{R}} \text{sign}(u - v) \left[ f(u) - f(v) \right] \psi_x \ dx dt \geq 0.
\]

(7.8)

Once we have obtained (7.8) we can tackle the second issue and rigorously justify the computations in (7.7) by relying on a suitable choice of the test functions (we omit the details of this part and instead refer to [3, p.116]).

7.8. The doubling of variable technique. We fix \(u, v\) as in the statement of Lemma 7.1. As mentioned before, we want to show that for every test function \(\psi \geq 0\) we have (7.8). The basic idea is that we rely on the distributional formulation of the entropy inequality, i.e. (7.2), but, instead of working in the space \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\), we “double the variables” and work in the space \((t, s, x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}\).

We assume that \(u\) depends on \((t, x)\) and \(v\) depends on \((s, y)\). We fix \(\phi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})\), \(\phi \geq 0\). Since \(v\) is constant with respect to \((t, x)\), by using (7.2) we get

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} |u - v(s,y)| \phi_t \ dx dt + \int_{\mathbb{R}^+} \int_{\mathbb{R}} \text{sign}(u - v(s,y)) \left[ f(u) - f(v(s,y)) \right] \phi_x \ dx dt \geq 0
\]

for every \((s, y) \in \mathbb{R}^+ \times \mathbb{R}\). By integrating with respect to \((s, y)\) we arrive at

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}} |u - v| \phi_t \ dx dt dy ds + \int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \text{sign}(u - v) \left[ f(u) - f(v(s,y)) \right] \phi_x \ dx dt dy ds \geq 0
\]

(7.9)

We now repeat the same argument by first integrating with respect to \(s\) and \(y\) and pointing out that \(u\) is constant with respect to \((s, y)\). We infer that

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}} |u - v| \phi_y \ dy ds dx dt + \int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \text{sign}(u - v) \left[ f(u) - f(v(s,y)) \right] \phi_y dy ds dx dt \geq 0
\]

(7.10)
By adding (7.9) and (7.10) we get
\[ \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}} |u-v|(\phi_t + \phi_x) \, dydsdxdt + \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \text{sign}(u-v) \left[ f(u) - f(v(s,y)) \right] (\phi_y + \phi_x) \, dydsdxdt \geq 0. \]

Very loosely speaking, this would be basically (7.8) if we could choose \( \phi = \psi \delta_{s=x} \delta_{x=y} \), where \( \delta \) represents the Dirac delta. The rigorous proof is established by approximating the Dirac delta by a sequence of smooth, compactly supported functions and can be found in [3, p.117].

8. The Riemann problem for a scalar conservation law

8.1. The Riemann problem. The so-called Riemann problem is a particular type of Cauchy problem, which is obtained by coupling the system of conservation laws (1.1) with an initial datum in the form
\[ u(0, x) = \begin{cases} u^+ & x > 0 \\ u^- & x < 0, \end{cases} \]

where \( u^+ \) and \( u^- \) are given states in \( \mathbb{R}^N \). Note that the Cauchy problem (1.1)-(8.1) has a solution in the form (3.3), provided \( \lambda \in \mathbb{R} \) satisfies the Rankine-Hugoniot conditions (3.4). Also, we recall Lemma 4.8, which says that \( w \) in (3.3) is entropy admissible if and only if (4.10) is satisfied for every entropy-entropy flux pair \( (\eta, q) \) with \( \eta \) convex.

In this section we focus on the scalar case, and hence we use the Kružkov’s entropy-entropy flux pairs introduced in § 7.2. By arguing as in the proof of Lemma 3.8 we get the following extension of Lemma 4.8.

8.2. Lemma. Assume that \( N = 1 \), fix \( u^+, u^-, \lambda \in \mathbb{R} \) and set
\[ u(t, x) := \begin{cases} u^+ & x > \lambda t \\ u^- & x < \lambda t. \end{cases} \]

Assume that the Rankine-Hugoniot conditions (3.4) are satisfied, then for every \( k \in \mathbb{R} \) \( u \) satisfies (7.1) if and only if
\[ \lambda \left| |u^+ - k| - |u^- - k| \right| \geq \text{sign}(u^+ - k) \left[ f(u^+) - f(k) \right] - \text{sign}(u^- - k) \left[ f(u^-) - f(k) \right]. \]

Note that (8.3) can be formally obtained from (4.10) by taking \( \eta(u) := |u - k| \) and \( q(u) = \text{sign}(u - k) \left[ f(u) - f(k) \right] \).

8.3. Entropy admissible shocks in the convex case. We now focus on the case where the flux is convex, namely we assume \( f'' > 0 \). In this case weak solutions in the form (8.2) are termed shocks. We have the following easy, albeit important, characterization of entropy admissible shocks.

8.4. Lemma. Assume \( u^- \neq u^+, f'' > 0 \). The function \( u \) in (8.2) is a weak entropy admissible solution of (2.1) if and only if
\[ \lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} \]

and \( u^+ < u^- \).

Proof. Assume that \( u \) in (8.2) is an entropy admissible solution of (2.1). In particular, it is a weak solution, which implies the Rankine-Hugoniot condition, i.e. (8.4). Owing to Lemma 8.2, the fact that the solution is entropy admissible implies (8.3). Assume by contradiction that \( u^- < u^+ \) and take \( k \in ]u^-, u^+[ \), then (8.3) boils down to
\[ \lambda \left[ u^+ + u^- - 2k \right] \geq f(u^+) + f(u^-) - 2f(k), \]

which owing to (8.4) becomes
\[ [f(u^+) - f(u^-)] \left[ u^+ + u^- - 2k \right] \geq \left[ f(u^+) + f(u^-) - 2f(k) \right] [u^+ - u^-]. \]
Next, we recall that the function $f'' > 0$ and hence, if $u^- < k < u^+$, then
\begin{equation}
\frac{f(u^+) - f(k)}{u^+ - k} > \frac{f(u^+) - f(u^-)}{u^+ - u^-} \iff [f(u^+) - f(k)][u^+ - u^-] > [f(u^+) - f(u^-)][u^+ - k]
\end{equation}
and
\begin{equation}
\frac{f(k) - f(u^-)}{k - u^-} < \frac{f(u^+) - f(u^-)}{u^+ - u^-} \iff [f(u^-) - f(k)][u^+ - u^-] > [f(u^+) - f(u^-)][u^- - k]
\end{equation}
By adding (8.6) and (8.7) we arrive at
\[ [f(u^+) + f(u^-) - 2f(k)][u^+ - u^-] > [f(u^+) - f(u^-)][u^+ + u^- - 2k], \]
which contradicts (8.5). This shows that, if $u$ is entropy admissible, then $u^- > u^+$.

Conversely, assume that (8.4) holds and $u^- > u^+$. Owing to Lemma 3.8, the function $u$ in (8.2) is a weak solution of (2.1). We are left to show that it is entropy admissible. It suffices to establish (8.3) for every $k \in \mathbb{R}$. We separately consider the following cases:

Case 1: $k < u^+$. Then (8.3) boils down to
\[ \lambda [u^+ - u^-] \geq f(u^+) - f(u^-), \]
which is satisfied owing to (8.4).

Case 2: $k > u^-$. Then (8.3) boils down to
\[ \lambda [-u^+ + u^-] \geq -f(u^+) + f(u^-), \]
which again is satisfied owing to (8.4).

Case 3: $u^+ < k < u^-$. Then (8.3) boils down to
\[ \lambda [2k - u^+ - u^-] \geq 2f(k) - f(u^+) - f(u^-), \]
which one can establish by using the inequalities that replace (8.6) and (8.7) in the case $u^+ < k < u^-$. Case 4 and 5: the cases $k = u^+$ and $k = u^-$ are left as an exercise.
We have eventually established that (8.3) holds true for every $k \in \mathbb{R}$. This concludes the proof of the lemma.

8.5. **Centered rarefaction waves.** We now want to describe the entropy admissible solution of the Riemann problem obtained by coupling (2.1) with the initial datum
\begin{equation}
u_0(0, x) := \begin{cases} u^+ & x > 0 \\ u^- & x < 0 \end{cases}
\end{equation}
in the case where $f'' > 0$ and $u^- < u^+$. First, we point out that, since $f'' > 0$, then the equality $u^- < u^+$ implies that $f'(u^-) < f'(u^+)$. Also, the function $f'$ is invertible and we denote by $(f')^{-1}$ its inverse. We introduce the function $v$ by setting
\begin{equation}
\text{for every } t > 0, x \in \mathbb{R}, \quad v(t, x) := \begin{cases} u^- & x < f'(u^-)t \\ (f')^{-1}(x/t) & f(u^-)t < x < f'(u^+)t \\ u^+ & x > f'(u^+)t \end{cases}
\end{equation}
The function $v$ is termed **centered rarefaction wave**. Note that, when $t \to 0^+$, $v(t, \cdot)$ converges in $L^1_{\text{loc}}(\mathbb{R})$ to the initial datum in (8.8). It turns out that $v$ is an entropy admissible solution of the Cauchy problem (2.1), (8.8). More precisely, we have the following result.

8.6. **Lemma.** Assume $f'' > 0$, $u^- < u^+$ and let $v$ be the same function as in (8.9). Then $v$ is a bounded, locally Lipschitz continuous function on $\mathbb{R}_+ \times \mathbb{R}$ and it is an entropy admissible solution of (2.1), (8.8).
8.7. Exercise. Write down the entropy admissible solution of the Cauchy problem
\[
\begin{cases}
  u_t + \frac{u^2}{2}x = 0 \\
  u(0, x) = \begin{cases} 
    1 & x > 0 \\
    0 & x < 0.
  \end{cases}
\end{cases}
\]

8.8. Summing up (convex case). Consider the Riemann problem (2.1), (8.1) and assume \( f'' > 0 \). If \( u^- > u^+ \), then the entropy admissible solution is the shock (8.2), where \( \lambda \) is given by (8.4). If \( u^- < u^+ \), then the entropy admissible solution is the centered rarefaction wave (8.9).

8.9. Some remarks about the non convex case. In this paragraph we make some handwaving remark about the solution of the Riemann problem (2.1)-(8.8) in the case where the flux \( f \) in (2.1) is not convex. We do not state a general result, but only consider a more specific example.

First, we point out that, if \( f'' < 0 \), then one can repeat the same argument as in the previous paragraphs and conclude that if \( u^- < u^+ \), then the solution is a shock as in (8.2), if \( u^+ > u^- \) then the solution is a centered rarefaction wave. This can be also obtained by pointing out that, if \( u \) is an entropy solution of (2.1), then \( w(t, x) := u(t, -x) \) is an entropy solution of
\[
w_t - f(w)_x = 0.
\]
If \( f'' < 0 \), then \( f'' > 0 \) and hence the results for a concave flux can be loosely speaking obtained by taking the results for convex fluxes and then switching the roles of \( u^- \) and \( u^+ \).

Let us now consider the general case when \( f \) is neither convex, nor concave. We assume for simplicity that \( u^- < u^+ \) and we construct the solution of the Riemann problem (2.1)-(8.8). We consider the convex envelope
\[
\text{conv}_{[u^-, u^+]} f := \sup \left\{ g : [u^-, u^+] \rightarrow \mathbb{R}, g \text{ convex, } g(v) \leq f(v) \text{ for every } v \in [u^-, u^+] \right\}.
\]
Known results on convex functions imply that \( \text{conv}_{[u^-, u^+]} f \) is a convex function. Also, by definition, \( \text{conv}_{[u^-, u^+]} f \geq f \). In particular, there will be intervals where \( \text{conv}_{[u^-, u^+]} f > f \), and a closed set where the two coincide. Assume for instance that
\[
\text{conv}_{[u^-, u^+]} f(v) = \begin{cases}
  f(v) & v \in [u^-, u_1] \\
  > f(v) & v \in ]u_1, u_2[ \\
  f(v) & v \in [u_2, u^+]
\end{cases}
\]
for some \( u_1, u_2 \) such that \( u^- < u_1 < u_2 < u^+ \). See for instance Figure 1. In this case one can show that
\[
f'(u_1) = f'(u_2) = \frac{f(u_1) - f(u_2)}{u_1 - u_2}
\]
Assume furthermore that \( f'' > 0 \) on \( ]u^-, u_1[ \) and on \( ]u_2, u^+[ \) (see Figure 1). The basic heuristic idea to construct the solution of the Riemann problem (2.1), (8.8) is that the values where \( \text{conv}_{[u^-, u^+]} f \) and
We have constructed an entropy admissible solution of the Riemann problem (2.1), (8.8) (recalling (8.11))
\[
u(t, x) := \begin{cases} 
  u^- & x < f'(u^-)t \\
  (f'|_{u^-,u_1})^{-1}(x/t) & f'(u^-)t < x < f'(u_1)t \\
  (f'|_{u_2,u^+})^{-1}(x/t) & f'(u_2)t < x < f'(u^+)t
\end{cases}
\]

9. The wave-front tracking algorithm

We now want to discuss another proof of Theorem 6.1. In § 6 we have constructed an entropy admissible solution of (2.1)-(2.6) by taking the limit \( \varepsilon \to 0^+ \) of the family \( u^\varepsilon \), obtained by using the vanishing viscosity approximation (6.1). In this section we construct an entropy admissible solution by taking the limit \( n \to +\infty \) of a sequence of approximate solution \( u^n \) constructed by using the so-called wave-front tracking algorithm. The basic idea of the wave-front tracking algorithm is that we construct a piecewise constant approximate solution by solving a finite number of Riemann problems. In this section we do not actually provide all the details of the construction of wave-front tracking approximation, and we refer to [3, § 6] for the complete analysis. Also, for simplicity we focus on the case where \( f'' > 0 \), but the method extends to the general case \( f \in C^2 \).

The wave-front tracking algorithm can be adapted to the analysis of systems. Some of the key results concerning systems of conservation laws have actually been established by using the wave-front tracking approximation, see [3, 8] for an extended discussion. Also, as we will see in the following, the wave-front tracking approximation is constructed by working with solutions of Riemann problems and hence as such it very much “preserves the structure” of the conservation law. As a matter of fact, the wave-front tracking algorithm gives in many cases a very good heuristic intuition of the structure of the solution of the Cauchy problem (2.1)-(2.6).

9.1. Approximation of the initial datum. We term \( w : \mathbb{R} \to \mathbb{R} \) a piecewise constant function if
i) it only attains a finite number of values,
ii) it has a finite number of discontinuities.

We now consider the Cauchy problem (2.1)-(2.6) and we construct a suitable piecewise approximation of the initial datum \( u_0 \).

9.2. Lemma. Assume that \( u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}) \), then for every \( n \in \mathbb{N} \) there is a piecewise constant function \( u_0^n \) such that
\[
\begin{align*}
(9.1) \quad & \text{TotVar } u_0^n \leq \text{TotVar } u_0 \\
(9.2) \quad & \|u_0^n\|_{L^\infty} \leq \|u_0\|_{L^\infty} \\
(9.3) \quad & u_0^n(x) \in \{k2^{-n} \mid k \in \mathbb{Z}\}, \text{ for a.e. } x \in \mathbb{R}
\end{align*}
\]
and furthermore
\[
(9.4) \quad \lim_{n \to +\infty} \|u_0^n - u_0\|_{L^1} = 0.
\]

The proof of Lemma 9.2 can be obtained by arguing as in the proof of [3, Lemma 2.2].

9.3. Construction of the approximate solution at time \( t = 0 \). We now want to fix \( n \in \mathbb{N} \) and construct an approximate solution \( u^n \) of the Cauchy problem obtained by coupling (2.1) with the initial datum \( u(0, x) = u_0^n(x) \), where \( u_0^n \) is the same as in the statement of Lemma 9.2. We make the following remarks:

• the function \( u_0^n \) is piecewise constant
• since the entropy admissible solution of (2.1)-(2.6) enjoys finite propagation speed owing to (7.3), we can impose that the approximate solution has also finite propagation speed.
We conclude that (locally in time) we can construct the approximate solution $u^n$ of the Cauchy problem with initial datum $u(0, x) = u^n_0(x)$ by patching together the solution of several Riemann problem, each of them corresponding to a discontinuity point of $u^n_0$.

To construct the local in time approximate solution it thus suffices to describe the approximate solution of the Riemann problem (2.1)-(8.8) in the case where $u^- = k2^{-n}$, $u^+ = h2^{-n}$ for some $k, h \in \mathbb{Z}$. We separately consider two cases:

**Case 1:** if $u^- > u^+$, then since $f'' > 0$ the entropy admissible solution of the Riemann problem (2.1)-(8.8) is the shock (8.2), with $\lambda$ given by (8.4). In this case we take as approximate solution $u^n$ the entropy admissible solution. Note that $u^n$ satisfies the following properties:

\begin{align}
\|u^n\|_{L^\infty} &= \|u^n_0\|_{L^\infty}, \quad \text{TotVar } u^n(t, \cdot) = \text{TotVar } u^n_0 \text{ for every } t > 0, \\
\text{TotVar } u^n(t, x) &= \{k2^{-n}, k \in \mathbb{Z}\} \text{ for a.e. } (t, x) \in \mathbb{R}^+ \times \mathbb{R}
\end{align}

**Case 2:** if $u^- < u^+$, then we have that $u^- = k2^{-n}$ and $u^+ = (k + j)2^{-n}$ for some $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Since $f'' > 0$, in this case the entropy admissible solution of the Riemann problem (2.1)-(8.8) is a centered rarefaction wave. Since we want to have a piecewise constant approximate solution, we cannot take the centered rarefaction wave as the approximate solution. To construct the approximate solution, we
Figure 3. The collision of two wave fronts

consider the states

\[ u_1 := (k + 1)2^{-n}, \quad u_2 := (k + 2)2^{-n}, \quad \ldots, \quad u_{j-1} := (k + j - 1)2^{-n} \]

and the speeds

\[ \lambda_1 := \frac{f(u_1) - f(u_0)}{u_1 - u_0}, \quad \lambda_2 := \frac{f(u_2) - f(u_1)}{u_2 - u_1}, \quad \ldots, \quad \lambda_j := \frac{f(u^+) - f(u_{j-1})}{u^+ - u_{j-1}}. \]

Note \( \lambda_1 \) satisfies the Rankine-Hugoniot conditions (8.4) between \( u^- \) and \( u_1 \), \( \lambda_2 \) satisfies the Rankine-Hugoniot conditions between \( u_2 \) and \( u_1 \), and so on. Note furthermore that, since \( f'' > 0 \), then

\[ \lambda_1 < \lambda_2 < \cdots < \lambda_j. \]

We can eventually define the approximate solution by setting

\[
\begin{align*}
u^n(t, x) := \begin{cases} 
  u_- & x < \lambda_1 t \\
  u_1 & \lambda_1 t < x < \lambda_2 t \\
  \vdots & \vdots \\
  u_{j-1} & \lambda_{j-1} t < x < \lambda_j t \\
  u^+ & x > \lambda_j t
\end{cases}
\end{align*}
\]

Note that:

- \( u^n \) satisfies (9.5) and (9.6).
- \( u^n \) is a weak solution of (2.1), but it is not entropy admissible, because it is obtained by patching together shocks where the left state is < than the right state (recall Lemma 8.4). However, when \( n \to +\infty \) the points \( u^-, u_1, \ldots, u_{j-1}, u_j \) get closer and closer and \( u^n \) approaches a centered rarefaction wave.

To define the approximate solution of the Cauchy problem obtained by coupling (2.1) with the initial datum \( u^n_0 \) given by Lemma 9.2 we patch together the solutions of each of the Riemann problems corresponding to the discontinuity points of \( u^n_0 \) (see Figure 2). Note that the resulting approximate solution is piecewise constant and discontinuous along a finite number of lines, which are termed wave-fronts. The solution is defined locally in time as long as two distinct wave-front do not collide.

9.4. Analysis of the interactions. Let \( \tau \) be the first time at which two wave-fronts collide. For simplicity, let us assume that only two wave-fronts collide at time \( t = \tau \). We term \( u_\ell \) and \( u_m \) the left and right state of the left wave-front, and \( u_m \) and \( u_r \) the left and right state of the right wave-front (see Figure 3). To define the approximate solution for \( t > \tau \) we construct the approximate solution of the Riemann problem between \( u_\ell \) (on the left) and \( u_r \) (on the right) by using the algorithm described in § 9.3. Note that in this way \( u^n(\tau^+, \cdot) \) (i.e., the solution right after the interaction time) satisfies

\[ u^n(\tau^+, x) \in \{ k2^{-n}, \ k \in \mathbb{Z} \}, \text{ for a.e. } x \in \mathbb{R}. \]
We now want to describe how the total variation of the approximate solution \( u^n \) changes at the interaction time \( \tau \), i.e. we want to compare \( \text{TotVar} u^n(\tau^-) \) with \( \text{TotVar} u^n(\tau^+,\cdot) \). Note that

\[
\text{TotVar} u^n(\tau^-,\cdot) = |u_m - u_l| + |u_m - u_r| + \text{total variation of the other wave-fronts}
\]

Since \( u^n(\tau^+,\cdot) \) is obtained by constructing the approximate solution of the Riemann problem, we get

\[
\text{TotVar} u^n(\tau^+,\cdot) = |u_l - u_r| + \text{total variation of the other wave-fronts} \leq \text{TotVar} u^n(\tau^-,\cdot)
\]

and hence that at any interaction time the total variation does not increase.

9.5. **Exercise.** Show that at any interaction time the \( L^\infty \) norm does not increase, i.e.

\[
\text{TotVar} u^n(\tau^+,\cdot) \leq \text{TotVar} u^n(\tau^-,\cdot).
\]

Finally, note that the total number of wave-fronts always decreases at the interaction time \( \tau \).

9.6. **Conclusion of the proof.** By arguing as in § 9.4, we can extend the approximate solution \( u^n \) after the first interaction time \( \tau \). The approximate solution \( u^n \) is piecewise constant, has discontinuities along a finite numbers of wave-fronts and it is defined locally in time as long as two wave-fronts do not collide. By arguing as in § 9.4, we can actually extend the approximate solution after the second interaction time and, by iteration, we can actually extend the solution \( u^n \) globally in time \(^4\).

Since the estimates (9.9) and (9.10) hold true at every interaction time, and by recalling (9.5) and (9.6), we conclude that for every \( t > 0 \)

\[
\text{TotVar} u^n(t,\cdot) \leq \text{TotVar} u^n_0 \leq \text{TotVar} u_0
\]

and

\[
\|u^n(t,\cdot)\|_{L^\infty} \leq \|u^n_0\|_{L^\infty} \leq \|u_0\|_{L^\infty}.
\]

This implies that, for every fixed \( t > 0 \), the sequence \( \{u^n(t,\cdot)\}_{n \in \mathbb{N}} \) satisfies the hypotheses of Helly’s Compactness Theorem. As a matter of fact, one can also establish a uniform time stability estimate in the same spirit as (6.13). We can then argue as in § 6.12 and show that the sequence \( \{u^n\}_{n \in \mathbb{N}} \) is compact in \( L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}) \) and hence (up to subsequences) converges to some limit \( u \). After some more work we can also show that \( u \) is an entropy admissible solution of the Cauchy problem (2.1)-(2.6) and hence conclude the proof of Theorem 6.1.

9.7. **Exercise.** Show that the approximate function \( u^n \) satisfies the following estimate:

\[
\int_{\mathbb{R}} |u^n(t,x) - u^n(s,x)|dx \leq 2LT\text{TotVar}(u_0)|t-s|
\]

for every \( t,s \geq 0, n \in \mathbb{N} \). In the previous expression,

\[
L := \max \left\{ \|f'(w)\| : \|w\| \leq \|u_0\|_{L^\infty} \right\}.
\]

10. **The Riemann problem for systems**

In this section we discuss the solution of the Riemann problem (1.1)-(8.1) in the case of systems, i.e. \( N > 1 \). We first have to introduce a new admissibility criterion, which replaces the definition of entropy admissible solution. Indeed, there are systems that do not have any entropy-entropy flux pair and hence the definition of entropy admissible solution is not suited to deal with general systems. The admissibility criterion we now discuss is due to Lax [10]. As we will see in the following, the Lax admissibility criterion only applies at discontinuity curves, but it is nevertheless very useful.

\(^4\)There is actually a subtle technical point we are neglecting: we can iterate the construction of the approximate solution because the total number of wave-fronts is decreasing with respect to time and hence, in particular, does not blow up in finite time.
10.1. Lax admissibility criterion (scalar case). We first consider the scalar case \( N = 1 \).

10.2. Definition. Assume that \( N = 1 \) and fix \( \lambda \in \mathbb{R}, \ u^-, \ u^+ \in \mathbb{R}, \ u^- \neq u^+ \) and assume that the Rankine-Hugoniot condition (3.4) holds true. Then the function (8.2) is Lax admissible if
\[
f'(u^+) \leq \lambda \leq f'(u^-).
\]

An heuristic justification of the Lax admissibility criterion is the following. We recall that the characteristic lines of (2.1) have slope \( f'(u) \) and are lines along which the smooth solution is constant. In other words, the “information” (i.e., the solution \( u \)) is transported along the characteristic lines. Note that requiring Lax condition (10.1) amounts to impose that the characteristic lines are running into the discontinuity line \( x = \lambda t \). Hence, heuristically speaking Lax condition expresses the fact that, in physical systems, information can be only destroyed and cannot be created out of nowhere.

In the case where the flux function \( f \) in (2.1) is convex, the Lax admissibility criterion 10.2 has a very simple interpretation.

10.3. Lemma. Assume that \( f'' > 0 \) and that the values \( \lambda, u^-, u^+, \ u^- \neq u^+ \) satisfy the Rankine-Hugoniot condition (3.4). Then (10.1) holds if and only if \( u^- > u^+ \).

Proof. First, we assume (10.1) and we establish the inequality \( u^- > u^+ \). We assume by contradiction that \( u^- < u^+ \). By using the Rankine-Hugoniot conditions (3.4), we get
\[
\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = f'(\xi) \quad \text{for some } \xi \in ]u^-, u^+[.
\]

To establish the second equality, we have used the Mean Value Theorem (Lagrange’s Theorem). Since \( f' \) is strictly increasing, then \( f'(u^-) < f'(\xi) < f'(u^+) \) and this contradicts (10.1). This shows that \( u^- > u^+ \).

Conversely, assume \( u^- > u^+ \), then we recall (10.2) and we use again that \( f' \) is strictly increasing to conclude that \( f'(u^-) > \lambda > f'(u^+) \), which implies (10.1). \( \square \)

10.4. Remark. By combining Lemma 10.3 and Lemma 8.4, we conclude that, if \( N = 1 \) and \( f'' > 0 \), the shock (8.2) is entropy admissible if and only if it is Lax admissible.

10.5. Lax admissibility criterion (systems). We now introduce the Lax admissibility criterion in the case of systems. We need the following preliminary result.

10.6. Lemma. Fix \( \lambda \in \mathbb{R}, \ u^-, \ u^+ \in \mathbb{R}^N \) and assume that the Rankine-Hugoniot conditions (3.4) hold true. Then there is a matrix \( A(u^-, u^+) \in \mathbb{M}^{N \times N} \) such that \( \lambda \) is an eigenvector of \( A(u^-, u^+) \), and \( u^+ - u^- \) a corresponding eigenvector.

Proof. We set
\[
h(t) := f((1 - t)u^- + tu^+), \quad t \in [0, 1]
\]
and we point out that \( h(0) = f(u^-), \ h(1) = f(u^+) \). By using the Fundamental Theorem of Calculus, we re-write (3.4) as
\[
\lambda[u^+ - u^-] = f(u^+) - f(u^-) = h(1) - h(0) = \int_0^1 h'(t)dt = \int_0^1 Df((1 - t)u^- + tu^+)dt [u^+ - u^-] = A(u^-, u^+).
\]
The above equation implies that \( \lambda \) is an eigenvector of \( A(u^-, u^+) \), and \( u^+ - u^- \) a corresponding eigenvector. \( \square \)

From now on we assume that system (1.1) is strictly hyperbolic (see Definition 2.8) and we term \((r_1(u), \lambda_1(u)), \ldots, (r_N(u), \lambda_N(u))\) the eigencouples of the matrix \( A(u) \). Also, we recall the explicit expression of the matrix \( A(u^-, u^+) \) in Lemma 10.6, i.e.
\[
A(u^-, u^+) := \int_0^1 Df((1 - t)u^- + tu^+)dt
\]
and we point out that, when \( u^+ = u^- \), then \( A(u^-, u^-) = Df(u^-) \). By continuity, when \( u^+ \) is sufficiently close to \( u^- \) each of the eigenvalues of \( A(u^-, u^-) \) is close to an eigenvalue of \( Df(u^-) \). This implies that, if \( \lambda \in \mathbb{R} \), \( u^+, u^- \in \mathbb{R}^N \) satisfy (3.4) and \( u^+ \) and \( u^- \) are sufficiently close, then \( \lambda \) is close to an eigenvalue of \( Df(u^-) \). In the following definition we assume that \( \lambda \) is close to the \( i \)-th eigenvalue.

We can now introduce the definition of Lax admissible discontinuity.

10.7. Definition. Fix \( \lambda \in \mathbb{R} \), \( u^+, u^- \in \mathbb{R}^N \) and assume that the Rankine-Hugoniot conditions (3.4) hold true. The function \( w \) in (3.3) is Lax admissible if

\[
\lambda_i(u^+) \leq \lambda \leq \lambda_i(u^-).
\]

In the following paragraphs we show that, if \( |u^+ - u^-| \) is sufficiently small, then we can construct a solution of the Riemann problem (1.1)-(8.1) that is Lax admissible at every discontinuity line. The construction is fairly technical and we have to introduce some preliminary results: in § 10.8 we introduce the rarefaction curves and shock curves, in § 10.11 we discuss the definition of genuinely nonlinear and linearly degenerate vector field, in § 10.17 we introduce further technical results and in § 10.24 we eventually complete the construction.


10.8.1. The \( i \)-th shock curve. The definition of \( i \)-th shock curve is provided by the following lemma.

10.9. Lemma. Fix \( \hat{u} \in \mathbb{R}^N \) and \( i = 1, \ldots, N \), then there are a constant \( \nu > 0 \) and a curve \( S_i(\hat{u}, \cdot) : ]-\nu, \nu[ \to \mathbb{R}^N \) such that, for every \( \sigma \in ]-\nu, \nu[ \), the states \( u^- := \hat{u} \) and \( u^+ := S_i(\hat{u}, \sigma) \) satisfy the Rankine-Hugoniot conditions (3.4) for some \( \lambda \) close to \( \lambda_i(u^-) \). Also, \( S_i(\hat{u}, 0) = \hat{u} \), the curve is continuously differentiable and

\[
\frac{dS_i(\hat{u}, \sigma)}{d\sigma} \bigg|_{\sigma=0} = r_i(\hat{u}),
\]

where \( r_i(\hat{u}) \) is an eigenvector of \( Df(\hat{u}) \) associated to the eigenvalue \( \lambda_i(\hat{u}) \).

The curve \( S_i(\hat{u}, \cdot) \) is usually term \( i \)-th shock curve through \( \hat{u} \). The proof of Lemma 10.9 can be obtained by combining Lemma 10.6 with the Implicit Function Theorem.

10.9.1. The \( i \)-th rarefaction curve. Fix \( \hat{u} \in \mathbb{R}^N \) and \( i = 1, \ldots, N \). Consider the integral curve of the vector field \( r_i \) passing through \( \hat{u} \), i.e. the solution of the Cauchy problem

\[
\begin{cases}
\frac{dR_i}{d\sigma} = r_i(R_i) \\
R_i(0) = \hat{u}
\end{cases}
\]

We term this curve \( i \)-th rarefaction curve through \( \hat{u} \) and we denote it by the symbol \( R_i(\hat{u}, \sigma) \).

10.10. Remark. Note that, owing to (10.4), the \( i \)-th shock curve \( S_i \) and the \( i \)-th rarefaction curve \( R_i \) through \( \hat{u} \) are tangent at \( \hat{u} \).

10.11. Genuinely nonlinear vector fields and linearly degenerate. We can now introduce the definition of genuinely nonlinear vector field.

10.12. Definition. We say that the \( i \)-th vector field is genuinely nonlinear if

\[
\nabla \lambda_i(u) \cdot r_i(u) > 0 \quad \text{for every } u \in \mathbb{R}^N.
\]

Some remarks are here in order.

10.13. Remark. In the case where \( N = 1 \), we have \( \lambda_i = f' \) and the inequality (10.6) boils down to \( f'' > 0 \). In the general case \( N > 1 \), inequality (10.6) very loosely speaking expresses the fact that the function \( f : \mathbb{R}^N \to \mathbb{R}^N \) is “convex along the \( i \)-th vector field”. Indeed, \( \lambda_i \) can be regarded as “the derivative of \( f \) in the direction \( r_i \)”, and the expression at the left hand side of (10.6) can be regarded as a second derivative.
\textbf{Remark.} Note that the expression at the left hand side of (10.6) changes sign when we change the orientation of $r_i$, which is arbitrary. However, what actually matters in our discussion is that the left hand side of (10.6) has a sign, and that the sign does not depend on $u$. The results we discuss in the following can be adapted to the case where the left hand side of (10.6) has strictly negative sign. However, in the following, to fix the ideas, if the $i$-th vector field is genuinely nonlinear we choose the orientation of $r_i$ in such a way that (10.6) holds true.

We can now give the definition of linearly degenerate vector field.

\textbf{Definition.} We say that the $i$-th vector field is linearly degenerate if
\begin{equation}
\nabla \lambda_i(u) \cdot r_i(u) = 0 \quad \text{for every } u \in \mathbb{R}^N.
\end{equation}

\textbf{Remark.} In the scalar case $N = 1$, condition (10.7) reads $f'' = 0$, which means that $f(u) = au + b$ for some $a, b \in \mathbb{R}$ and the scalar conservation law (2.1) boils down to the linear transport equation (2.5). In the general case, condition (10.6) very loosely speaking expresses the fact that the function $f : \mathbb{R}^N \to \mathbb{R}^N$ is “linear in the direction of $r_i$.” Keep in mind, however, that there are nonlinear systems where every characteristic field is linearly degenerate, i.e. (10.7) holds true for every $i = 1, \ldots, N$.

In the following, we will mostly focus on strictly hyperbolic systems where every vector field is either linearly degenerate or genuinely nonlinear, i.e. for every $i = 1, \ldots, N$ either (10.6) or (10.7) holds true. This is a somehow restrictive assumption, but it is satisfied by many interesting systems, for instance by the Euler equations (1.7).

\textbf{The $i$-th admissible wave fan curve.}

10.17.1. \textit{Genuinely nonlinear vector fields.} Assume that the $i$-th vector field is genuinely nonlinear, i.e. (10.6) holds true. This implies that
\begin{equation}
\frac{d\lambda_i(\tilde{R}_i(\tilde{u}, \sigma))}{d\sigma} = \nabla \lambda_i(\tilde{R}_i(\tilde{u}, \sigma)) = \nabla \lambda_i(\tilde{R}_i(\tilde{u}, \sigma)) = 0,
\end{equation}
i.e. that the function $\sigma \mapsto \lambda_i(\tilde{R}_i(\tilde{u}, \sigma))$ is strictly increasing, for every $\tilde{u} \in \mathbb{R}^N$.

Fix $\sigma > 0$ and consider the Riemann problem (1.1)-(8.1) with $u^- := \tilde{u}$, $u^+ = \tilde{R}_i(\tilde{u}, \sigma)$. Since $\sigma > 0$, we can define the function
\begin{equation}
u(t, x) := \left\{ \begin{array}{ll} \tilde{u} & x < \lambda_i(u^-)t \\ \tilde{R}_i(\tilde{u}, s) & x = \lambda_i(\tilde{R}_i(\tilde{u}, s))t, \ s \in [0, \sigma], \text{ i.e.} \\ \tilde{R}_i(\tilde{u}, \sigma) & x > \lambda_i(\tilde{R}_i(\tilde{u}, \sigma))t \\ \end{array} \right.
\end{equation}
Note that the above function is well defined since the map $\lambda_i \circ \tilde{R}_i$ is invertible owing to (10.8).

In the scalar case $N = 1$, this boils down to (8.9). One can show that the function $\nu$ is locally Lipschitz continuous in $\mathbb{R}_+ \times \mathbb{R}$ and satisfies (1.1) for a.e. $(t, x)$. We now recall Lemma 10.9 and the definition of shock curve. We have the following result.

\textbf{Lemma.} Fix $\tilde{u} \in \mathbb{R}^N$, $i = 1, \ldots, N$ and let $S_i(\tilde{u}, \cdot) = -\nu, \nu[\to \mathbb{R}^N$ be the same curve as in the statement of Lemma 10.9. Under assumption (10.6), consider the function $\nu$ in (3.3) with $u^- := \tilde{u}$ and $u^+ := S_i(\tilde{u}, \sigma)$. If $\nu$ is sufficiently small, then $\nu$ is Lax admissible if and only if $\sigma \leq 0$.

\textbf{Sketch of the proof.} We only provide the rough idea underpinning the proof of Lemma (10.18). First, we consider the function $\sigma \mapsto \lambda_i(S_i(\tilde{u}, \sigma))$ and we compute its derivative at $\sigma = 0$:
\begin{equation}
\frac{d\lambda_i(S_i(\tilde{u}, \sigma))}{d\sigma} \bigg|_{\sigma=0} = \nabla \lambda_i(S_i(\tilde{u}, 0)) \cdot S_i'(0) = \nabla \lambda_i(\tilde{u}) \cdot r_i(\tilde{u}) > 0.
\end{equation}
Since the derivative of $\lambda_i(S_i(\tilde{u}, \sigma))$ is strictly positive at $\sigma = 0$, then it is strictly positive in a neighborhood. In particular, if $\sigma > 0$ then
\begin{equation}
\lambda_i(S_i(\tilde{u}, 0)) = \lambda_i(\tilde{u}) < \lambda_i(S_i(\tilde{u}, \sigma))
\end{equation}
and hence \( u^- = \tilde{u} \) and \( u^+ = S_i(\tilde{u}, \sigma) \) violate the Lax condition (10.3). Conversely, if \( \sigma < 0 \) then 
\[
\lambda_i(S_i(\tilde{u}, \sigma)) < \lambda_i(\tilde{u})
\]
and one can show that the speed of the shock between \( u^- = \tilde{u} \) and \( u^+ = S_i(\tilde{u}, \sigma) \) satisfies the Lax condition (10.3). \[\square\]

Given \( \tilde{u} \in \mathbb{R}^N \), we can now define the so-called \( i \)-th admissible wave fan curve \( T_i(\cdot)(\tilde{u}) \): \( -\nu, \nu[ \rightarrow \mathbb{R}^N \) by setting
\[
T_i(\sigma)(\tilde{u}) := \begin{cases} 
R_i(\tilde{u}, \sigma) & \sigma \geq 0 \\
S_i(\tilde{u}, \sigma) & \sigma < 0.
\end{cases}
\] (10.10)

10.19. Remark. Note that, by construction, for every \( \sigma \in ]-\nu, \nu[ \) the solution of the Riemann problem (1.1)-(8.1) with \( u^- = \tilde{u} \) and \( u^+ = T_i(\sigma)(\tilde{u}) \) has a very simple structure: if \( \sigma > 0 \) it is the centered rarefaction wave (10.9), if \( \sigma < 0 \) it is the shock (3.3). This can be regarded as a generalization of the discussion in § 8.8 concerning the solution of the scalar Riemann problem with convex flux.

10.20. Remark. Owing to Remark 10.10, we have
\[
\left. \frac{dT_i(\sigma)(\tilde{u})}{d\sigma} \right|_{\sigma=0} = r_i(\tilde{u}),
\] (10.11)
and the curve \( T^i \) is \( C^1 \) regular.

10.20.1. Linearly degenerate vector fields. We now construct the \( i \)-th admissible wave fan curve in the case where the \( i \)-th vector field is linearly degenerate, i.e. (10.7) holds true. We first need a preliminary result.

10.21. Lemma. Fix \( \tilde{u} \in \mathbb{R}^N, \ i = 1, \ldots, N \) and assume the \( i \)-the vector field is linearly degenerate, i.e. (10.6) holds true. Then the \( i \)-th shock curve \( S_i \) (defined as in Lemma 10.9) and the \( i \)-th rarefaction curve \( R_i \) (defined as in (10.5)) coincide. Also, if \( \sigma \in ]-\nu, \nu[ \) and \( \lambda \) is given by the Rankine-Hugoniot conditions (3.4) with \( u^- = \tilde{u} \) and \( u^+ = R_i(\tilde{u}, \sigma) \), then
\[
\lambda_i(\tilde{u}) = \lambda = \lambda_i(R_i(\tilde{u}, \sigma))
\]
and hence the Lax condition (10.3) is verified.

10.22. Remark. Assume the \( i \)-th vector field is linearly degenerate, fix \( \tilde{u} \in \mathbb{R}^N, \ \sigma \in \mathbb{R} \) and let \( \lambda \) be as in the statement of Lemma 10.21, then the discontinuous function \( w \) in (3.3) is termed contact discontinuity. Note that \( w \) is a Lax admissible solution of the Riemann problem (1.1)-(8.1).

If the \( i \)-the vector field is linearly degenerate, then the \( i \)-th admissible wave fan curve is defined as the \( i \)-th rarefaction curve (10.5), i.e.
\[
T_i(\sigma)(\tilde{u}) := R_i(\tilde{u}, \sigma) \quad \text{Lemma} \ 10.21 \quad S_i(\tilde{u}, \sigma)
\] (10.12)

Note that, owing to Lemma 10.21, if the \( i \)-the vector field is linearly degenerate and \( u^+ = T_i(\sigma)(u^-) \) for some \( \sigma \in \mathbb{R} \), then the Riemann problem (1.1)-(8.1) has a simple, Lax admissible solution given by the contact discontinuity. In the case \( N = 1 \), we recover the fact that, if the system is linear, then the discontinuous function \( w \) is always an entropy admissible solution of the Riemann problem (2.5)-(8.8), provided \( a = \lambda \), as the next exercise shows.

10.23. Exercise. Show that, when \( N = 1 \), for every \( u^+, u^- \in \mathbb{R} \), the discontinuous function (8.2) is an entropy admissible solution of the linear Riemann problem (2.5)-(8.8), provided \( a = \lambda \). Hint: use (4.10).

10.24. The Lax solution of the Riemann problem. We now complete the construction of the solution of a Lax admissible solution of the Riemann problem (1.1)-(8.1). We assume that every vector field is either linearly degenerate or genuinely nonlinear, i.e. for every \( i = 1, \ldots, N \) either (10.6) or (10.7) holds true. We fix \( u^- \in \mathbb{R}^N \) and consider the function \( \psi \) defined by setting
\[
\psi(\sigma_1, \ldots, \sigma_N) := T_N(\sigma_N) \circ T_{N-1}(\sigma_{N-1}) \circ \cdots \circ T_2(\sigma_2) T_1(\sigma_1)(u^-),
\] (10.13)
where $T_i$ is defined by (10.10) if the $i$-th vector field is genuinely nonlinear and by (10.12) if the $i$-th vector field is linearly degenerate. The meaning of the above expression is the following: the function $T_2(\sigma_2)(\tilde{u})$ is evaluated at the point $\tilde{u} = T_1(\sigma_1)(u^-)$, and so on. By using Remark 10.20, one can show that the map $\psi$ is of class $C^1$ and the columns of the Jacobian matrix $D\psi(0,\ldots,0)$ are the eigenvectors $r_1(u^-),\ldots,r_N(u^-)$. By combining strict hyperbolicity (see Definition 2.8) with the Local Invertibility Theorem, we conclude that $\psi$ is invertible in a sufficiently small neighborhood of $(0,\ldots,0)$.

We are now ready to construct a Lax admissible solution of the Riemann problem (1.1)-(8.1), provided $u^-$ and $u^+$ are sufficiently close. Since the function $\psi$ is locally invertible, then, if $|u^- - u^+|$ is sufficiently small, from the relation $u^+ = \psi(\sigma_1,\ldots,\sigma_N)$ we can uniquely determine the values $(\sigma_1,\ldots,\sigma_N)$. Let us now consider the states

$$u_0 := u^-, \quad u_1 := T_1(\sigma_1)(u^-), \quad u_2 := T_2(\sigma_2)(u_1), \ldots \quad u_N := T_N(\sigma_N)(u_{N-1}).$$

For every $i = 0,\ldots,N-1$, the solution of the Riemann problem with left state $u_i$ and right state $u_{i+1}$ is either a contact discontinuity (if the $i$-th vector field is linearly degenerate), or a Lax admissible shock (if the $i$-th vector field is genuinely non linear and $\sigma_i < 0$), or a centered rarefaction curve (if the $i$-th vector field is genuinely non linear and $\sigma_i > 0$). Also, the solution of the Riemann problem with left state $u_i$ and right state $u_{i+1}$ is constant outside either a discontinuity line (contact discontinuities and shocks) or the plane sector $x/t \in [\lambda_i(u_i),\lambda_i(u_{i+1})]$. Either case, if $|u^+ - u^-|$ is sufficiently small the various interval where the solution is not constant do not overlap as $i$ runs from 1,\ldots,N.

A Lax admissible solution of the Riemann problem (1.1)-(8.1) can then be obtained by juxtaposing the solution of the $N$ Riemann problems with left state $u_i$ and right state $u_{i+1}$, $i = 0,\ldots,N-1$. The solution obtained in this way is sometimes termed \textit{Lax solution of the Riemann problem (1.1)-(8.1)}. Consider for instance the following example: $N = 3$, the 1- and the 3-characteristic field are genuinely nonlinear, and the 2-characteristic field is linearly degenerate. This occurs for instance in the case of the Euler equations (1.7). Fix $u^-, u^+ \in \mathbb{R}^3$ sufficiently close and assume that by solving the equation $\psi(\sigma_1,\sigma_2,\sigma_3) = u^+$ we obtain $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 = 0$. The Lax solution of the Riemann
problem (1.1)-(8.1) is then
\[
\mathbf{u}(t,x) = \begin{cases} 
\mathbf{u}^- & x < \lambda_1(\mathbf{u}^-)t \\
\mathbf{R}_1 \left( \left[ \lambda_1 \circ \mathbf{R}_1 \right]^{-1} (x/t) \right) & \lambda_1(\mathbf{u}^-)t < x < \lambda_1(\mathbf{u}_1)t \\
\mathbf{u}_1 & \lambda_1(\mathbf{u}_1)t < x < \lambda_2(\mathbf{u}_1)t \\
\mathbf{u}_2 & \lambda_2(\mathbf{u}_1)t < x < s_3(\sigma_3)t \\
\mathbf{u}^+ & x > s_3(\sigma_3)t 
\end{cases}
\]

where \( s_3(\sigma_3) \) is the speed of the shock between \( \mathbf{u}_2 \) and \( \mathbf{u}^+ \), namely it satisfies
\[
f(\mathbf{u}_2) - f(\mathbf{u}^+) = s_3(\sigma_3) \left[ \mathbf{u}_2 - \mathbf{u}^+ \right].
\]

See also Figure 4.

10.25. Remark. In this section we have discussed the solution of the Riemann problem (1.1)-(8.1) in the case where every characteristic field is either genuinely nonlinear or linearly degenerate. As a matter of fact, the construction extends to general strictly hyperbolic systems. Very loosely speaking, the basic ideas underpinning the extension are the same that we discussed in § 8.9, but the construction requires severe technicalities.

References


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