Theorem 3.1 (An alternative form of the Harnack inequality)

Let u be a non-negative solution of (3.1) - (3.3), choose $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$.

Under the usual assumptions on the cylinders, set

$$\theta = \left(\frac{c}{u(x_0, t_0)} \right)^{p-2}$$

$$(\bar{x}, \bar{t}) = (x_0, t_0 + \theta \rho^p)$$

and let $\mathcal{D}(t)$ the region previously defined for such a choice of $(\bar{x}, \bar{t})$

There exist $\underbrace{\text{constants}}^{\text{positive}}$ γ_1, γ_2, depending only upon the data and independent of (x_0, t_0), and $\bar{t}$, such that if $\mathcal{D}(t) \subset E_T$ then

$$u(x_0, t_0) \leq \gamma_1 \left(\frac{\rho^p}{t - t_0} \right)^{\frac{1}{p-2}} + $$

$$+ \gamma_2 \left(\frac{t - t_0}{\rho^p} \right)^{\nu/p} \left[\inf_{B_\rho(x_0)} u(x, t) \right]^{\lambda/p}$$

Remarks

* Once more a <u>family</u> of alternative Harnack inequalities.
* They are all <u>mutually equivalent</u> and also equivalent to the intrinsic Harnack inequality
* The value of ν does not yield any further information on the structure of u, other than possibly the optimal constants γ_1 and γ_2.
* We can now deal with the optimal growth of the initial data as $|x| \to \infty$ for the solvability of the Cauchy Problem for the quasi-linear case.

Theorem 3.2 (Decay in the Space Variables)
Let u be a non-negative solution of (3.1) - (3.3), satisfying

$$u(\cdot, \bar{t}) \geq k \quad \text{in} \quad B_\rho(\bar{x})$$

for some $(\bar{x}, \bar{t}) \in E_T$.
Then for all $(x,t) \in E_T$ with $x \neq \bar{x}$ and $0 < t - \bar{t} < \frac{1}{2}\bar{t}$

$$(3.5) \quad u(x,t) \geq \frac{k\rho^\nu}{S^{\nu/\lambda}(t)}\left[1 - \gamma_0 \lambda^{\frac{1}{p-1}} \left(\frac{|x-\bar{x}|}{S^{1/\lambda}(t)}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}}$$

provided $\lambda k^{p-2}\rho^{\nu(p-2)} \geq 2\rho^\lambda$; γ_0 is a proper parameter that depends only on the data; $S(t)$, λ, and ν are defined as we have seen previously.

Remarks

* (3.5) shows that $\Gamma_{\nu,p}(x,t;\bar{x},\bar{t})$ are universal sub-potentials for equations (3.1) – (3.3)

* When $p \longrightarrow 2$ we recover Moser's result, answering the first question we raised before

* We are back to the starting point of 20 years ago. In fact

  From $\Gamma_{N,p}(x,t;\bar{x},\bar{t})$ to the intrinsic Harnack inequality for the p-laplacian [DiBenedetto, 1988]

  Now
  From the intrinsic Harnack inequality for quasi-linear equations to $\Gamma_{\nu,p}(x,t;\bar{x},\bar{t})$

* All the previous results can be extended to the full quasi-linear structure (i.e. with lower order terms)

* All the previous results hold also for porous media-like equations, again with the full quasi-linear structure.

In order to prove Theorem 3.2, it is enough to prove the following Proposition 3.2, and then combine it with Proposition 3.1

Proposition 3.2

Let u be a non-negative solution to (3.1) - (3.3) and let $(x_0, t_0) \in E_T$ be such that $u(x_0,t_0) > 0$. Then for all $(x,t) \in E_T$ with $x \neq x_0$ and $0 < t - t_0 < \frac{1}{4} t_0$

$$u(x,t) \geq u(x_0,t_0) \left[1 - \gamma_0 \left(\frac{|x-x_0|^p}{[u(x_0,t_0)]^{p-2}(t-t_0)} \right)^{\frac{1}{p-1}} \right]_+^{\frac{p-1}{p-2}}$$

where γ_0 depends only on the constants appearing in the intrinsic Harnack inequality.

Proof of Proposition 3.1

The starting point is the Harnack inequality

$$u(x_0,t_0) \le \gamma \inf_{B_\rho(x_0)} u(\cdot,\ t+\theta\rho^p)$$

$$\theta = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2}$$

Fix (x_0,t_0), (x,t) such that $t > t_0$, $x \ne x_0$

Line through them

$$y(s) - x_0 = \frac{x - x_0}{t - t_0}(s-t)$$

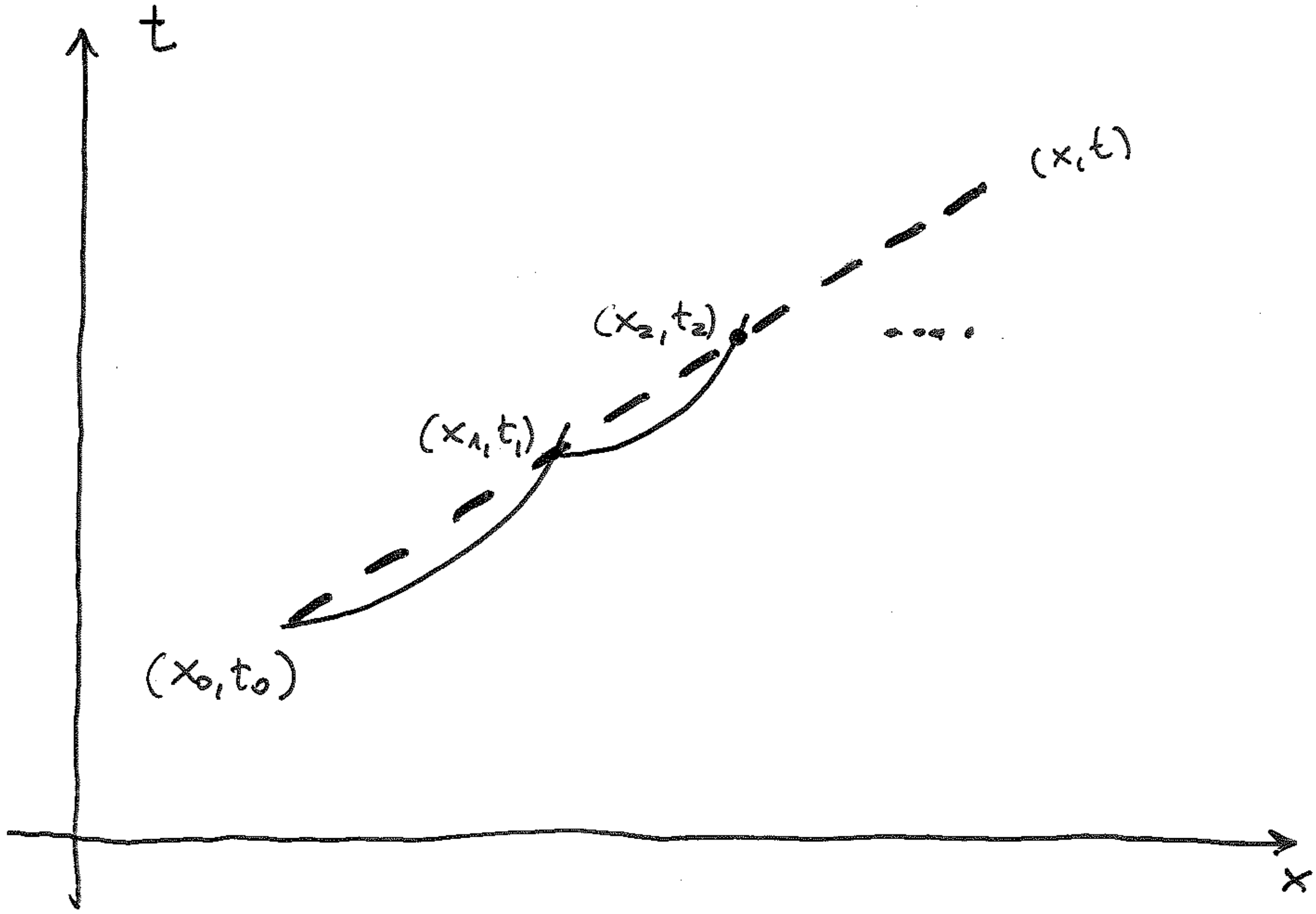
p-paraboloid with vertex at (x_0,t_0)

$$s - t_0 = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2} |y - x_0|^p$$

They intersect at (x_1,t_1) where

$$|x_1 - x_0|^{p-1} = \left[\frac{u(x_0,t_0)}{c}\right]^{p-2} \frac{t - t_0}{|x - x_0|}$$

$$t_1 - t_0 = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2} |x_1 - x_0|^p$$



Iteration of this process gives a sequence $\{(x_j,t_j)\}$ such that

(3.6) $$|x_{j+1}-x_j|^{p-1} = \left(\frac{u(x_j,t_j)}{c}\right)^{p-2} \frac{t-t_0}{|x-x_0|}$$

(3.7) $$t_{j+1} - t_j = \left(\frac{c}{u(x_j,t_j)}\right)^{p-2} |x_{j+1}-x_j|^p$$

We can apply the Harnack inequality if we have enough zoom and this occurs if

$$t_j - 4(t_{j+1}-t_j) \geq t_0 - 4(t-t_0) \geq 0$$

Hence if

$$0 < t - t_0 < \frac{1}{4} t_0$$

we can say that

$$u(x_j, t_j) \leq \gamma\, u(x_{j+1}, t_{j+1}) \qquad j = 0, 1, \dots, n$$

where n is to be chosen. By iteration

$$(3.8) \qquad u(x_n, t_n) \geq \gamma^{-n}\, u(x_0, t_0)$$

From (3.6)–(3.7)

$$|x - x_0| \geq \sum_{j=0}^{n-1} |x_{j+1} - x_j| \geq$$

$$\geq \left(\frac{1}{C^{p-2}} \frac{t - t_0}{|x - x_0|} \right)^{\frac{1}{p-1}} \sum_{j=0}^{n-1} [u(x_j, t_j)]^{\frac{p-2}{p-1}}$$

$$\geq \left(\frac{t - t_0}{|x - x_0|} \right)^{\frac{1}{p-1}} \left(\frac{u(x_0, t_0)}{C} \right)^{\frac{p-2}{p-1}} \sum_{j=0}^{n-1} \left(\gamma^{-\frac{p-2}{p-1}} \right)^j$$

$$= \left(\frac{t - t_0}{|x - x_0|} \right)^{\frac{1}{p-1}} \left(\frac{u(x_0, t_0)}{C} \right)^{\frac{p-2}{p-1}} \frac{1 - q^n}{1 - q}$$

where

$$q = \gamma^{-\frac{p-2}{p-1}}$$

From this

$$(3.9)\quad (\gamma^{-n})^{\frac{p-2}{p-1}} \geq 1 - \gamma_0 \left(\frac{|x-x_0|^p}{[U(x_0,t_0)]^{p-2}(t-t_0)} \right)^{\frac{1}{p-1}}$$

where

$$\gamma_0 = \left(\frac{\gamma^{\frac{p-2}{p-1}} - 1}{\gamma^{\frac{p-2}{p-1}}} \right) c^{\frac{p-2}{p-1}}$$

Without loss of generality we can assume $(x_n, t_n) = (x, t)$

Combining (3.8) and (3.9) we have finished.