* Relying on (2.14), we have the analog of (2.8) - (2.9), namely if

(2.15)
$$|[w < \sigma k_o] \cap [K_{4\rho}(y)\times(\theta(4\rho)^p, \theta(8\rho)^p)]| < \nu |[K_{4\rho}(y)\times(\theta(4\rho)^p, \theta(8\rho)^p)]|$$

then

(2.16) $$w > \frac{1}{2}\sigma k_o \qquad \forall (x,t)\in K_{2\rho}(y)\times(\theta(6\rho)^2, \theta(8\rho)^2),$$

$$\nu = \gamma(\text{data})\,\frac{[\theta(\sigma k_o)^{p-2}]^{N/p}}{[1+\theta(\sigma k_o)^{p-2}]^{1+N/p}}.$$

Remarks

* As for the heat equation, we need a larger cylinder $K_{8\rho}(y)\times(0,\theta(8\rho)^p)$ to get the final information we need, (2.16)

* We need to go from (2.13''),

$$w(x,\tau)\geq k_o \qquad \forall x\in K_\rho(y)$$
$$\forall \tau \geq 0$$

to (2.15). The whole process will fix θ

STEP V Set

$$Q_{4\rho}(\theta) = K_{4\rho}(y) \times (\theta(4\rho)^p, \theta(8\rho)^p)$$

We have the following

Lemma For every $\nu > 0$, there exist $\sigma \in (0,1)$, depending upon the data, and $\theta = \theta(k_o, \sigma)$, such that

$$|[w < \sigma k_o] \cap Q_{4\rho}(\theta)| \le \nu |Q_{4\rho}(\theta)|$$

Remarks

* The statement is similar to what we saw for the heat equation at step III yesterday, but now θ is fixed by this same step
* The proof is similar as well, and we concentrate on the main differences.

Proof of the Lemma

In (2.15) take ζ s.t.

$$\zeta(x,t) = 1 \qquad \text{on } Q_{4\rho}(\theta)$$

$$|D\zeta| \leq \frac{1}{4\rho}$$

$$|\zeta_t| \leq \frac{1}{\theta(4\rho)^p}$$

Moreover take the levels

$$k_j = \frac{1}{2^j} k_0 , \qquad j = 0, 1, \dots j_*$$

$j_* \in \mathbb{N}$ to be chosen.

$$\iint_{Q_{4\rho}(\theta)} |D(w-k_j)_-|^p \, dx\, d\tau \leq$$

$$\leq \gamma \frac{k_j^p}{(4\rho)^p} \left[1 + \frac{1}{\theta k_j^{p-2}} \right] |Q_{4\rho}(\theta)|$$

Lack of homogeneity !

Now choose (main technical point)

$$\theta = \left(\frac{2^{j_*}}{k_0} \right)^{p-2}$$

$$\Longrightarrow \left[1 + \frac{1}{\theta k_j^{p-2}} \right] \leq 2$$

$$\iint_{Q_{4\rho}(\vartheta)} |D(w-k_j)_-|^p\,dx\,d\tau \le \frac{\gamma\, k_j^p}{(4\rho)^p}\,|Q_{4\rho}(\theta)|$$

The other important step is the use of De Giorgi - Poincaré inequality, which we can write as

$$(k_j - k_{j+1})|A_{j+1}(\tau)| \le \gamma \frac{\rho^{N+1}}{|K_{4\rho}\setminus A_j(\tau)|} \int_{[k_{j+1}<w<k_j]} |Dw|\,dx$$

where

$$A_j(\tau) = [w(\cdot,\tau) < k_j] \cap K_{4\rho}(y)$$

$$\tau \in (\vartheta(4\rho)^p, \vartheta(8\rho)^p)$$

By (2.13") (we finally use it!)

$$|K_{4\rho}(y)\setminus A_j(\tau)| \ge |K_\rho| \qquad \forall\tau$$

The rest of the proof is exactly as for the heat equation (except for the exponents of the Hölder inequality) and we conclude

$$|[w < \frac{k_0}{2^{j_*}}] \cap Q_{4\rho}(\theta)]| \le \left(\frac{\gamma}{j_*}\right)^{\frac{p-1}{p}} |Q_{4\rho}(\theta)|$$

For the choices

$$\nu = \left(\frac{\gamma}{j_*}\right)^{\frac{p-1}{p}}$$

$$\sigma = \frac{1}{2^{j_*}}, \quad \theta = \left(\frac{2^{j_*}}{k_0}\right)^{p-2}$$

the Lemma is proved

<u>Step VI</u> Expanding the positivity for w

By (2.15)-(2.16), we have

$$w \ge \frac{1}{2}\sigma k_0 \quad \text{in } K_{2\rho}(y) \times (\theta(6\rho)^p, \theta(8\rho)^p),$$

provided

$$\frac{|[w < \sigma k_0] \cap Q_{4\rho}(\theta)|}{|Q_{4\rho}(\theta)|} \le \gamma \frac{[\theta(\sigma k_0)^{p-2}]^{N/p}}{[1+\theta(\sigma k_0)^{p-2}]^{1+\frac{N}{p}}} = \delta_*$$

$\delta_* = \delta(p, N)$

Choose $\nu = \delta_*$, this determines σ and θ and we have finished.

STEP VII Expanding the positivity of u

We now go back to u, taking into account the definitions of $f(\tau)$, w and k_0

In particular

$$b_1 \overset{\text{def}}{=} \exp\left\{\frac{2^{p-2}\, 6^p}{(p-2)\sigma^{p-2}\delta}\right\} \leq f(\tau) \leq \exp\left\{\frac{2^{p-2}\, 8^p}{(p-2)\sigma^{p-2}\delta}\right\} \overset{\text{def}}{=} b_2$$

As for u, correspondingly we have

$$u(x,t) \geq \frac{\sigma \xi M}{4 b_2} \overset{\text{def}}{=} \eta M \qquad \forall x \in K_{2\rho}(y)$$

for all

$$s + \left(\frac{\bar{b}}{\eta M}\right)^{p-2} (2\rho)^p \leq t \leq s + \left(\frac{\bar{b}}{\eta M}\right)^{p-2} (4\rho)^p$$

for a proper $\bar{b}$, that depends only on N, p.

Remarks

* The proof shows that $\eta(p) \longrightarrow 0$ as $p \to 2$ and a stabilization is needed

PROOF OF THE HARNACK INEQUALITY

Once we have the intrinsic expansion of positivity, the proof is very much the same as given for the heat equation, provided we have an initial positivity set.

Fix $(x_0, t_0) \in E_T$ and $\rho > 0$ s.t.

$$K_{8\rho}(x_0) \times \left(t_0 - \vartheta (8\rho)^p,\ t_0 + \vartheta (8\rho)^p\right) \subseteq E_T$$

where

$$u(x_0, t_0) > 0,$$

$$\vartheta = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2} \qquad c \geq 1 \text{ to be determined}$$

The change of variables

$$x \longrightarrow \frac{x - x_0}{\rho}, \qquad t \longrightarrow u(x_0,t_0)^{p-2}\,\frac{t-t_0}{\rho^p}$$

maps the previous cylinder into

$$K_8(0) \times \left(-8^p c^{p-2},\ 8^p c^{p-2}\right)$$

We denote again by (x, t) the transformed variables.

The transformed function

$$v(x,t) = \frac{1}{u(x_0,t_0)}\, u\left(x_0 + \rho x,\ t_0 + \frac{\rho^p t}{[u(x_0,t_0)]^{p-2}}\right)$$

solves

$$\begin{cases} v_t - \operatorname{div}\left(|Dv|^{p-2} Dv\right) = 0 \\ v(0,0) = 1 \end{cases}$$

* We have to prove that there exist two positive constants $c_1, c_2 = c_1, c_2$ (data) such that

$$\inf_{K_1(0)} v(x, c_1) \geq c_2$$

* For $\tau \in (0,1)$ define

$Q_\tau = K_\tau(0) \times (-\tau^p, 0)$, $M_\tau = \sup_{Q_\tau} u$

$N_\tau = (1-\tau)^{-\beta}$, $\beta > 1$ to be chosen

* Let τ_0 be the largest root of $M_\tau = N_\tau$

* Let $(x_*, t_*) \in Q_{\tau_0}$ such that

$$u(x_*, t_*) = (1-\tau_0)^{-\beta}$$

* The box

$$\tilde{Q} = K_{\frac{1-\tau_0}{2}}(x_*) \times \left(t_* - \left(\frac{1-\tau_0}{2}\right)^p,\ t_*\right) \subseteq Q_{\frac{1+\tau_0}{2}}$$

$$\Longrightarrow \sup_{\tilde{Q}} v \leq N_{\frac{1+\tau_0}{2}} = 2^\beta (1-\tau_0)^{-\beta}$$

* We can now apply the Hölder continuity and conclude

$$v(x, t_*) \geq \frac{1}{2}(1-\tau_0)^{-\beta} \quad \forall x \in K_{\varepsilon\frac{1-\tau_0}{2}}(x_*)$$

* From here on we conclude as with the heat equation

Fundamental Question: Can we single out an initial positivity set <u>without</u> using the Hölder continuity?

Answer: YES, with pure measure-theoretic arguments

We rely on the following

<u>Lemma</u> Let $u \in W^{1,1}(K_\rho)$ satisfy

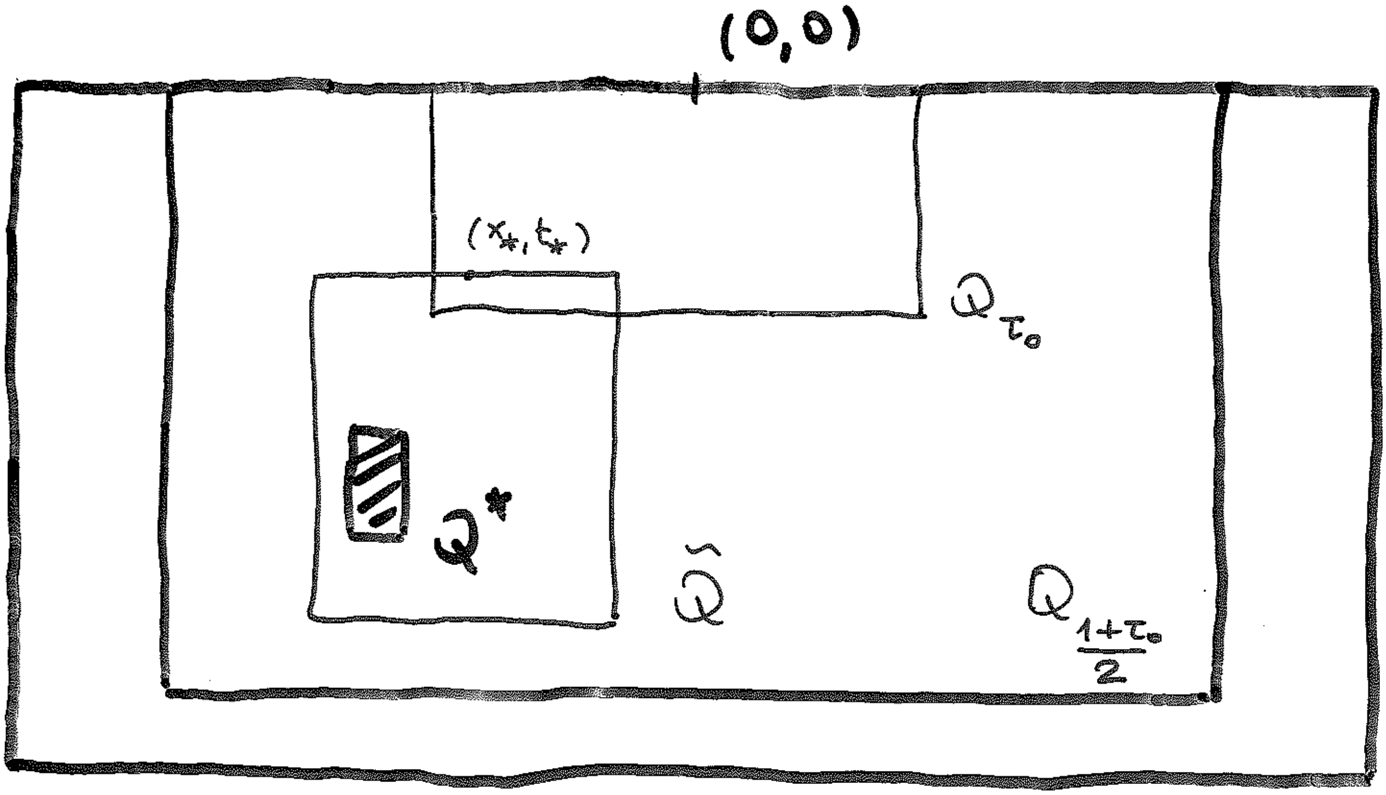
$$\|u\|_{W^{1,1}(K_\rho)} \leq \gamma \rho^{N-1},$$

$$|[u>1] \cap K_\rho| \geq \alpha |K_\rho|$$

for some $\alpha \in (0,1)$, $\gamma > 0$. Then, for every $\delta \in (0,1)$ and $\lambda \in (0,1)$, there exist $x_0 \in K_\rho$ and $\eta = \eta(\alpha, \gamma, \delta, \lambda, N)$, $\eta \in (0,1)$ such that

$$|[u>\lambda] \cap K_{\eta\rho}(x_0)| > (1-\delta) |K_{\eta\rho}(x_0)|$$

* It is a statement about the clustering of positivity.
* Full proof in [9]
* We need a parabolic version of this statement.



Set

$$M_0 = 2^{\beta}(1-\tau_0)^{-\beta}$$

$$R_0 = \frac{1-\tau_0}{2}, \qquad \vartheta_0 = M_0^{2-p}$$

$$\tilde{Q} = K_{R_0}(x_*) \times (t_* - \vartheta_0 R_0^p,\ t_*)$$

* It is not difficult to show that

$$|[v > 2^{-(1+\beta)} M_0] \cap \tilde{Q}| > \nu(\beta)\, |\tilde{Q}|$$

* Relying on the previous Lemma, we can show that $\exists\, Q^* \subset \tilde{Q}$ such that

$$|[v < \lambda 2^{-(\beta+1)} M_0] \cap Q^*| < \nu_0 |Q^*|$$

(Namely, $\forall \lambda, \nu_0 \in (0,1)$ $\exists Q^*$ s.t. ....)

By (2.8) – (2.9) this is enough to conclude that

$$v(x,s) \geq \frac{1}{8}(1-\tau_0)^{-\beta} \quad \forall (x,s) \ Q^*/2$$

and we have what we need to start the iteration process.

MAIN CONSEQUENCE

As in the classical case, the Harnack inequality implies the Hölder continuity

Final Remarks

* The same results hold for porous media equation and their fully quasi-linear extension
* The results are purely local, that is initial and/or boundary conditions play <u>no</u> role
* Purely measure-theoretic arguments
* No covering
* No cross-over Lemma
* The equation is used only to derive the energy estimates and to analyze the behavior of w.