SECOND PART

or

The main serving

Now, back to the p-laplacian. We have

<u>THEOREM</u> Let u be a non-negative solution to

$$u_t - \operatorname{div}(|Du|^{p-2} Du) = 0, \quad p > 2$$

fix $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$. There exist positive constants c and γ depending only upon p and N, such that for all cylinders

$$K_{4\rho}(x_0) \times (t_0 - \vartheta(4\rho)^p, t_0 + \vartheta(4\rho)^p)$$

with $\vartheta = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2}$ contained in E_T, where $K_\rho(x_0)$ is the cube of center x_0 and edge 2ρ, we have

$$u(x_0, t_0) \leq \gamma \inf_{K_\rho(x_0)} u(x, t_0 + \vartheta\rho^p)$$

Remarks

* As for the heat equation, we need a large reference cylinder
* New proof with respect to [8]
* γ and c deteriorate as $p \longrightarrow \infty$, but are stable as $p \to 2$ (but we will not deal with this aspect
* Harnack $\Longrightarrow$ Hölder (as in the classical case)

Strategy

* Recast in the new context the previous work, namely
  - Expansion of positivity
  - Construction of the initial block
* For this second issue, get rid of the Hölder continuity of u.

Proposition 2.1 (Degenerate expansion of positivity) Let u be a non-negative local solution of

$$(2.1) \qquad u_t - \operatorname{div}(|Du|^{p-2} Du) = 0, \qquad p > 2.$$

There exist positive constants b, and $\eta \in (0,1)$, depending only upon the data and independent of (y,s), ρ and M, such that if

$$(2.2) \qquad u(x,s) \geq M \quad \text{for all } x \in K_\rho(y)$$

then

$$(2.3) \qquad u(x,t) \geq \eta M \quad \text{for all } x \in K_{2\rho}(y)$$

for all

$$(2.4) \quad s + \frac{b}{(\eta M)^{p-2}} (2\rho)^p \leq t \leq s + \frac{b}{(\eta M)^{p-2}} (4\rho)^p$$

Remarks

* The height depends on M

* Even if it is not explicitely mentioned, we need more "zoom"

* Stable constants

* The proof relies both on the energy inequality and the equation. For the heat equation we used only the energy inequality

* A straightforward application of the previous method does not work: the cylinder does not have the right size

Energy estimates

Multiply (2.1) by $\pm (u-k)_\pm \zeta^p$ and integrate over $Q_\rho = K_\rho(y) \times (s - \vartheta\rho^p, s)$ to obtain

$$\sup_{s-\vartheta\rho^p < t < s} \int_{K_\rho(y)} (u-k)_\pm^2 \zeta^p(x,t)\,dx + \iint_{Q_\rho} |D(u-k)_\pm \zeta|^p\,dx\,d\tau \tag{2.5}$$

$$\leq \gamma\left[\iint_{Q_\rho} \left[(u-k)_\pm^p |D\zeta|^p + (u-k)_\pm^2 |\zeta_t|\right] dx\,d\tau\right] +$$

$$+ \int_{K_\rho(y)} (u-k)_\pm^2 \zeta^p(x, s - \vartheta\rho^p)\,dx$$

Remarks

* ζ is the usual cut-off function

* Notice that the two terms on the right-hand side are not homogeneous

* As before, we split the proof of Proposition 2.1 in simpler steps.

STEP I A De Giorgi-Type Lemma

Fix $Q^-_{2\rho} \equiv K_{2\rho}(y) \times (s - \theta(2\rho)^p, s) \subset E_T$ and set

$$\mu_+ \geq \sup_{Q^-_{2\rho}} u, \qquad \mu_- \leq \inf_{Q^-_{2\rho}} u$$

$$\omega \geq \mu_+ - \mu_-$$

Let ξ, a be fixed numbers in $(0,1)$. There exists a number ν depending upon the data, and θ, ξ, ω and a such that, if

$$|[u \geq \mu_+ - \xi\omega] \cap Q^-_{2\rho}| \leq \nu |Q^-_{2\rho}| \tag{2.6}$$

then

$$u \leq \mu_+ - a\xi\omega \quad \text{in} \quad Q^-_\rho \tag{2.7}$$

Likewise, if

$$|[u \leq \mu_- + \xi\omega] \cap Q^-_{2\rho}| \leq \nu |Q^-_{2\rho}| \tag{2.8}$$

then

$$u \geq \mu_- + a\xi\omega \quad \text{in} \quad Q^-_\rho \tag{2.9}$$

Remarks

* The proof gives for (2.8) - (2.9)

$$\nu = \left(\frac{1-a}{\gamma(\text{data})} \right)^{N+p} \frac{\left[\theta (\xi \omega)^{p-2} \right]^{N/p}}{\left[1 + \theta (\xi\omega)^{p-2} \right]^{\frac{p+N}{p}}}$$

Analogous result for (2.6) - (2.7)

* Notice the $\theta(\xi\omega)^{p-2}$- term

* We have shrinking both in radius and in height, because we have no control on u for $t = s - \theta(2\rho)^p$. What happens if we have such a control?

* $Q^+_{2\rho}$ will also do

STEP II A variant of the previous Lemma

Fix $Q_{2\rho}^{+} = K_{2\rho}(y) \times (s, s+\vartheta(2\rho)^{p})$ and assume

(2.10) $$u(x,s) \geq \xi M \qquad \forall x \in K_{2\rho}(y)$$

for some $M>0$ and $\xi \in (0,1]$. Then for a fixed $a \in (0,1)$, if

(2.11) $$\left|[u<\xi M] \cap Q_{2\rho}^{+}\right| \leq \frac{\delta(\text{data}, a)}{\vartheta(\xi M)^{p-2}} \left|Q_{2\rho}^{+}\right|,$$

$\delta \in (0,1)$, then

(2.12) $$u(x,t) \geq a\xi M \qquad \forall (x,t) \in K_{\rho}(y) \times (s, s+\vartheta(2\rho)^{p})$$

Remarks

* $\dfrac{\delta}{\vartheta(\xi M)^{p-2}}$ replaces ν.

* The height remains the same.

First Key remark

Take $\theta = \delta(\xi M)^{2-p}$: then (2.11) is satisfied and

(2.13) $$u\left(x, s + \frac{\delta \rho^p}{(\xi M)^{p-2}}\right) \geq a \xi M \quad \forall x \in K_\rho(y)$$

Second Key remark

If (2.10) holds for $\xi \in (0,1)$, it holds for any $\xi_\tau \leq \xi$ and we have the analogous of (2.13), provided we choose θ accordingly, i.e.

(2.13') $$u\left(x, s + \frac{\delta \rho^p}{(\xi_\tau M)^{p-2}}\right) \geq a \xi_\tau M \quad \forall x \in K_\rho(y)$$

Now for $\tau > 0$, let

$$\xi_\tau = \frac{\xi}{f(\tau)} \qquad f(\tau) = e^{\frac{\tau}{p-2}}$$

Then for all $\tau \geq 0$

(2.13'') $$u\left(x, s + \left[\frac{f(\tau)}{\xi M}\right]^{p-2} \delta \rho^p\right) \geq a \frac{\xi M}{f(\tau)}$$
$$\forall x \in K_\rho(y)$$

Third key remark

(2.13″) gives a time-decay of u in $K_\rho(y)$ [smaller cube]. Roughly speaking

$$u(t) \simeq \frac{1}{t^{(p-2)^{-1}}} \qquad t \gg 0$$

Question In the proof of the Harnack inequality for the heat equation, the key fact is a control (uniform control!) on the measure of the level sets. Can we get it here?

IDEA. Change of variables: STEP III

Set

$$w(x,\tau) \overset{\text{def}}{=} \frac{f(\tau)}{\xi M}(\delta\rho^p)^{\frac{1}{p-2}}\, u\Big(x,\, s+\Big[\frac{f(\tau)}{\xi M}\Big]^{p-2}\delta\rho^p\Big)$$

Then, by (2.13″) $\forall x \in K_\rho(y)$, $\forall \tau \geq 0$

$$w(x,\tau) \geq a\,(\delta\rho^p)^{\frac{1}{p-2}} \overset{\text{def}}{=} k_0$$

Trivial remarks

* We have rescaled the time, but the space is unchanged

* We have a uniform bound below on w for any time

New question : what kind of equation does w satisfy ?

STEP IV Relating w to the Evolution Equation

(Formal) calculations

$$w_\tau = \frac{1}{p-2} \frac{e^{\frac{\tau}{p-2}}}{\xi M} (\delta \rho^p)^{\frac{1}{p-2}} \, u +$$

$$+ \left(\frac{e^{\frac{\tau}{p-2}}}{\xi M} (\delta \rho^p)^{\frac{1}{p-2}} \right)^{p-1} u_t$$

$$= \frac{1}{p-2} w + \left(\frac{e^{\frac{\tau}{p-2}}}{\xi M} (\delta \rho^p)^{\frac{1}{p-2}} \right)^{p-1} \left(\operatorname{div} (|Du|^{p-2} Du) \right)$$

$$= \frac{1}{p-2} w + \operatorname{div} (|Dw|^{p-2} Dw)$$

But $\quad w \geq 0, \quad p > 2 \quad \Longrightarrow$

$$(2.14) \qquad w_\tau - \operatorname{div}\left(|Dw|^{p-2} Dw\right) \geq 0$$

w is a supersolution of the parabolic p-laplacian!

Two consequences

* We use (2.14) in $K_{8\rho}(y) \times (0, \theta(8\rho)^2) \equiv Q_{8\rho}$ ($\tau \geq 0$!) and we get

$$\sup_{0<\tau<\theta(8\rho)^2} \int_{K_{8\rho}(y)} (w-k)_-^2 \zeta^p(x,\tau)\,dx + \iint_{Q_{8\rho}} |D(w-k)_- \zeta|^p \,dx\,d\tau \tag{2.15}$$

$$\leq \gamma \iint_{Q_{8\rho}} \left[(w-k)_-^p |D\zeta|^p + (w-k)_-^2 |\zeta_t| \right] dx\,d\tau$$

for a smooth, non-negative cut-off function ζ that vanishes on the parabolic boundary of $Q_{8\rho}$