Indeed

$$(*) \quad u(x,t) = (T-t)_+^{\frac{N+2}{4}} \left[a + b|x|^{\frac{2N}{N-2}} \right]^{-\frac{N}{2}}, \quad N>2$$

with

$a>0$, $T>0$ arbitrary

$$b = \frac{N-2}{N^2} \left(\frac{N+2}{4Na} \right)^{\frac{N+2}{N-2}}$$

is a solution for $p = \frac{2N}{N+2} < \frac{2N}{N+1}$ in all $\mathbb{R}^N \times \mathbb{R}$, but does not satisfy the H.I. in any of the forward, backward or elliptic forms

$$\Downarrow$$

Case $\overline{\underline{\text{III}}}$ No backward in time H.I. for p strictly subcritical,

Case $\overline{\underline{\text{IV}}}$ What about $p = p_*$?

Answer: Nope !

Take $T=1$, $a=1$, $(x_0, t_0) = (0,0)$ in $(*)$

We have

$$u(0,0) = 1$$

$$\inf_{B_\rho(0)} u(x,0) = \left[1 + b\rho^{\frac{2N}{N-2}}\right]^{-N/2}$$

and

$$\inf_{B_\rho(0)} u(x,0) \xrightarrow[|x|\to\infty]{} 0$$

Moreover

$$\inf_{B_\rho(0)} u(x, -\delta_* \rho^p) = u(|x|=\rho, -\delta_* \rho^{\frac{2N}{N+2}})$$

$$= \left[1 + b\rho^{\frac{2N}{N-2}}\right]^{-N/2} \left(1 + \delta_* \rho^{\frac{2N}{N+2}}\right)^{\frac{N+2}{4}}$$

and when $\rho >> 1$

$$\inf_{B_\rho(0)} u(x, -\delta\rho^p) \simeq \rho^{-N/2} \xrightarrow[\rho\to\infty]{} 0$$

Indeed

$$(**) \qquad u(x,t) = \left[|x|^{\frac{2N}{N-1}} + e^{bt} \right]^{-\frac{N-1}{2}}$$

$$b = \frac{2N^{\frac{2N}{N+1}}}{N-1}, \qquad N \geq 2$$

is a solution in $\mathbb{R}^N \times \mathbb{R}$ but satisfies no Harnack inequality in any of the previous forms.

__Question__ What form, if any, does the Harnack inequality take for $1 < p \leq p_*$?

__Remark__

* Comparison with Bonforte–Vazquez's results.
* Tools used in the proof:
  - Measure-theoretical arguments
  - A "novel" iteration technique

* Difference with respect to the usual size of the reference cylinder (even if it is always larger)
* The proof does not rely on the Hölder continuity of the solution; hence, as for $p>2$, even here Harnack $\Longrightarrow$ Hölder
* Decay in the space variables. We have the following

Proposition 4.1 Let u be a non-negative solution of (4.1)-(4.2) and let $(x_0,t_0)\in E_T$ be such that $u(x_0,t_0)>0$. Then for all $(x,t)\in E_T$ with $x\neq x_0$ and $0<t-t_0<\frac{(c^q\delta_*)^{\frac{2-p}{p}}}{8}t_0$

$$\frac{u(x,t)}{u(x_0,t_0)}\geq\left[1+\gamma\left(\frac{[u(x_0,t_0)]^{2-p}|x-x_0|^p}{t-t_0}\right)^{\frac{1}{p-1}}\right]^{\frac{p-1}{p-2}}$$

where

$$\gamma=(c^{\frac{p-2}{p-1}}-1)\,\delta_*^{\frac{1}{p-1}}$$

Remarks

* Proof similar to the $p>2$ case
* Optimal decay in space, but not in time
* The backward in time Harnack Inequality gives a similar upper bound

What about the Porous Medium Equation?

We deal with

$$u_t - \Delta(|u|^{m-1}u) = 0, \qquad (4.8)$$

$$\frac{(N-2)_+}{N} < m < 1 \qquad \text{(supercritical range)} \qquad (4.9)$$

If $(x_o,t_o) \in E_T$ is such that $u(x_o,t_o) > 0$ we set

$$Q_\rho(x_o,t_o) = B_\rho(x_o) \times \left\{ t_o - \left(\frac{u(x_o,t_o)}{c^4}\right)^{1-m}\rho^2 < t < t_o + \left(\frac{u(x_o,t_o)}{c^4}\right)^{1-m}\rho^2 \right\}$$

<u>Theorem 4.2</u> Let u be a non-negative solution to (4.8)-(4.9). There exist positive constants δ_* and c, depending only upon the data, such that, for all cylinders $Q_{8\rho}(x_o,t_o) \subset E_T$, we have

$$c\, u(x_o,t_o) \le \inf_{B_\rho(x_o)} u(\cdot,t)$$

for all times

$$t_o - \delta[u(x_o,t_o)]^{1-m}\rho^2 \le t \le t_o + \delta[u(x_o,t_o)]^{1-m}\rho^2$$

The situation is very similar to the one we have for the p-laplacian (stability of constants, optimality of the range, etc.), but we have an extra result

<u>Theorem 4.3</u> There exists a positive constant γ depending only upon the data and independent of u, such that for every multiindex α

$$|D^\alpha u(x_0,t_0)| \leq \frac{\gamma^{|\alpha|+1}\,|\alpha|!}{\rho^{|\alpha|}}\; u(x_0,t_0)$$

and for any $k \in \mathbb{N}$

$$\left|\frac{\partial^k}{\partial t^k} u(x_0,t_0)\right| \leq \frac{\gamma^{2k+1}\,(k!)^2}{\rho^{2k}}\,[u(x_0,t_0)]^{1-(1-m)k}$$

provided $u(x_0,t_0) > 0$.

<u>Notice</u>: We do not have a similar result for the p-laplacian.

Remarks

* It means that non-negative solutions are locally analytic in the space variables and at least Lipschitz continuous in time

* By a straightforward approximation procedure, Theorem 4.3 continues to hold for points $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) = 0$. (see [7])

* It continues to hold for equations with the full quasi-linear structure, provided $\mathbb{A}$ and B are locally analytic in $E_T \times \mathbb{R}^{N+1}$.

Main technical component in the proof

Proposition 4.2 ($L^1_{loc}-L^\infty_{loc}$ Harnack-type estimates)

Let u be a non-negative, weak solution to (4.1) - (4.2). There exists a positive constant γ depending only upon the data, such that for all cylinders

$$K_{2\rho}(y)\times[s-(t-s),\, s+(t-s)]\subset E_T$$

we have

$$\sup_{K_\rho(y)\times[s,t]} u \le \frac{\gamma}{(t-s)^{N/\lambda}}\left(\inf_{2s-t<\tau<t}\int_{K_{2\rho}(y)} u(x,\tau)\,dx\right)^{p/\lambda}$$

$$+\gamma\left(\frac{t-s}{\rho^p}\right)^{\frac{1}{2-p}}$$

where, as usual, $\lambda = N(p-2)+p$

Notice $\quad \frac{2N}{N+1}<p<2 \iff \lambda>0$

Starting from $K_\rho(y)$, u is required to exist in a larger neighborhood $K_{2\rho}(y)$ and for a significant interval about s

A final remark for the singular case

Hölder continuity: it holds for bounded solutions in the whole range $1 < p < 2$.

Harnack inequality: it holds in the forward, backward and elliptic norm only for $\frac{2N}{N+1} < p < 2$ and implies the Hölder continuity.

In the subcritical range $1 < p \leq \frac{2N}{N+1}$ the diffusion is so fast, the decay in the space variables can be so strong, that the solutions, although Hölder continuous, do not have a controlled growth, as provided by the Harnack inequality.