FOURTH PART

or "Dulcis in fundo", the singular case $1 < p < 2$.

Let us now consider the singular case, namely

$$u_t - \operatorname{div}(|Du|^{p-2} Du) = 0 \qquad (4.1)$$

when $1 < p < 2$.

Remarks

* When $|Du| \to 0$, the modulus of ellipticity, $|Du|^{p-2} \to \infty$ and the Pde tends to favour its elliptic component. Otherwise stated, the diffusion process dominates over the time evolution.

* We distinguish between supercritical range, namely

$$p_* = \frac{2N}{N+1} < p < 2 \qquad (4.2)$$

and the subcritical range, that is

$$1 < p \leq \frac{2N}{N+1} = p_* \qquad (4.3)$$

As usual

$$B_\rho(x_0) = \{ x \in \mathbb{R}^N : |x - x_0| < \rho \}$$

Fix $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$ and consider cylinders of the type

$$Q_\rho(x_0,t_0) = B_\rho(x_0) \times \left\{ t_0 - \left(\frac{u(x_0,t_0)}{c^4}\right)^{2-p} \rho^p < t \le t_0 + \left(\frac{u(x_0,t_0)}{c^4}\right)^{2-p} \rho^p \right\}$$

where c is a positive parameter we will define in a moment

We have the following [Cheddar cheese Thm]

<u>Theorem 4.1</u> Let u be a non-negative solution to (4.1) for p in (4.2). There exist positive constants δ_* and c, with $\delta_*, c = \delta_*, c$ (data) such that for all $(x_0,t_0) \in E_T$ with $u(x_0,t_0) > 0$ and $Q_{8\rho}(x_0,t_0) \subset E_T$ we have

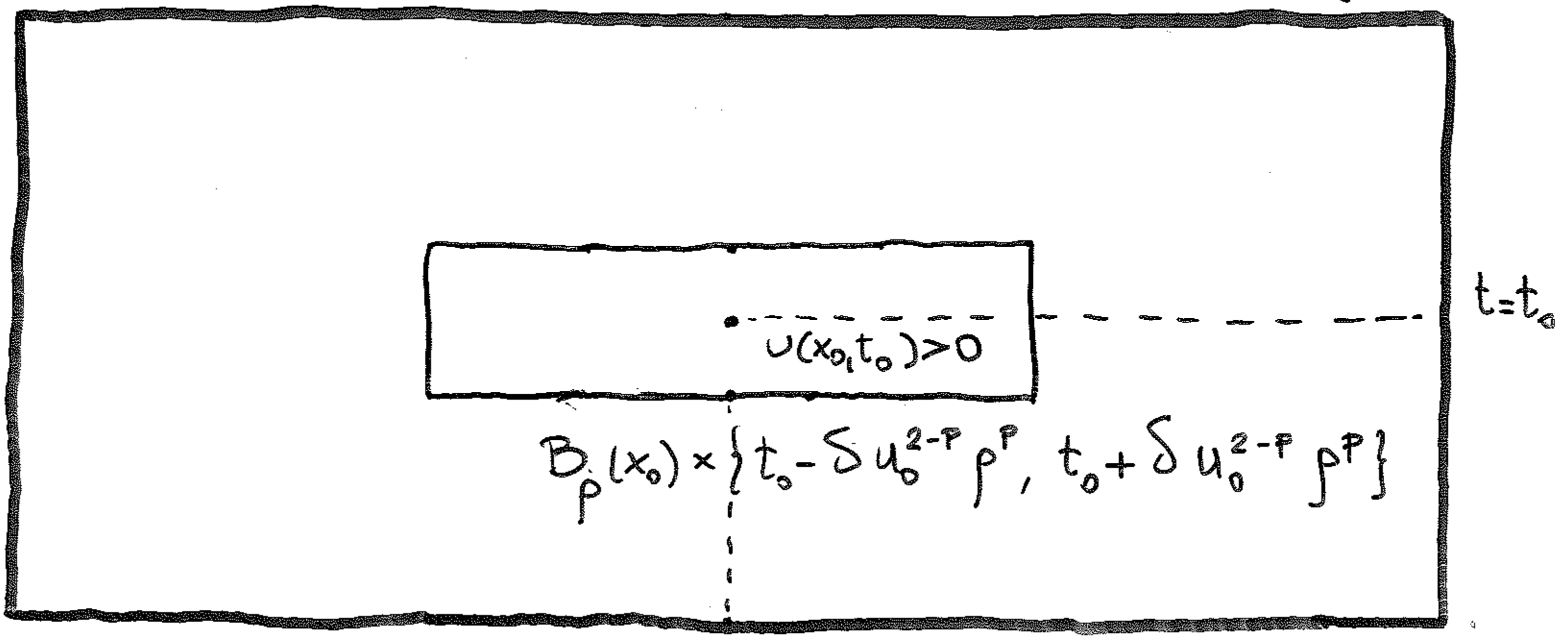
$$c\, u(x_0,t_0) \le \inf_{B_\rho(x_0)} u(\cdot, t) \qquad (4.4)$$

for all times

$$t_0 - \delta_* [u(x_0,t_0)]^{2-p} \rho^p \le t \le t_0 + \delta_* [u(x_0,t_0)]^{2-p} \rho^p$$

Remarks

* The geometrical setting



$Q_{\delta\rho}(x_o, t_o) \subset E_T$

* (4.4) is simultaneously a

1) Forward in time H.I.

$$(4.5) \qquad c\, u(x_o, t_o) \le \inf_{B_\rho(x_o)} u(\cdot, t_o + \vartheta \rho^p)$$

$$\vartheta = \delta [u(x_o, t_o)]^{2-p}$$

and a

2) Backward in time H.I.

$$(4.6) \qquad c\, u(x_o, t_o) \le \inf_{B_\rho(x_o)} u(\cdot, t_o - \vartheta \rho^p)$$

As a consequence, we have

3) Elliptic H.I.

$$c\, u(x_0,t_0) \le \inf_{B_\rho(x_0)} u(\cdot\,, t_0) \tag{4.7}$$

* c and δ_* tend to zero as $p \longrightarrow 2$ and as $p \longrightarrow p_*$, but they can be stabilized as $p \longrightarrow 2$ if we limit ourselves to the forward in time case. No elliptic and/or backward H.I. can hold for $p=2$
* Later comments on the subcritical range

The forward Harnack Inequality

We have already seen why an intrinsic waiting time is needed in the degenerate case. What about here?

<u>Notice</u> Non-negative $\underbrace{\text{solutions}}_{\text{in a bounded domain}}$ with homogeneous Dirichlet data on the boundary and non-negative initial data become extinct in finite time, that is $\exists\, T = T(\text{data}, u_0)$ such that

$$u(x,t) > 0 \qquad \text{for} \quad t < T$$
$$u(x,T) = 0 \qquad \text{for} \quad t \geq T$$

$\forall x \in E$.

<u>Consequence</u> A Harnack inequality with waiting time independent of u cannot hold.

The elliptic Harnack Inequality

* It makes precise the previous heuristic argument about diffusion vs time evolution

* The parabolic component is not lost because u is required to exist for a sufficiently large time interval about t_o.

* It was originally proved for the porous media equation in [7]

The backward Harnack Inequality

* It reflects the tendency of the solution to become extinct in finite time, as discussed before
* Despite its form, in the inequality the time is not reversed: indeed the solution must exist in a large time-interval about t_0.
* It's a new fact, at least for local solution where the boundary data play no role

Question What about the subcritical case $1 < p \leq \frac{2N}{N+1}$?

Answer The range $\frac{2N}{N+1} < p < 2$ is optimal for an intrinsic Harnack inequality to hold.

Case I: No forward in time H. I. for p subcritical (strictly subcritical)

Indeed solution of the Cauchy Problem for non-negative initial data $u_0 \in L^1(\mathbb{R}^N)$ became extinct after a finite time T.

Fix $(x_0, t_0) \in \mathbb{R}^N \times (0, T)$ where t_0 is so close to T to satisfy

$$T - t_0 < \frac{\delta c^{4(2-p)}}{8^p} t_0$$

where δ and c are the constants in (4.4). Now choose $\rho > 0$ so large that

$$\delta\,[u(x_0,t_0)]^{2-p}\rho^p = T - t_0$$

For such a choice $Q_{\delta\rho}(x_0,t_0) \subset \mathbb{R}^N \times (0,+\infty)$ however the intrinsic forward H.I. fails $\left[\inf_{B_\rho(x_0)} u(\cdot,T) = 0,\ u(x_0,t_0) > 0\right]$

<u>Case II</u> No elliptic H.I. for p strictly subcritical, $1 < p < \frac{2N}{N+1}$.

Indeed

$$u(x,t) = C(N,p)\ \frac{(T-t)_+^{\frac{1}{2-p}}}{|x|^{\frac{p}{2-p}}}$$

is a non-negative solution, unbounded near $x=0$ for all $t<T$ and finite otherwise $\Longrightarrow$ No elliptic H.I.

<u>Notice</u> For $1 < p \le p_*$ solutions need not be bounded, as it is the case here, but the lack of an elliptic H.I. is not due to the possible unboundedness.