# Particular Solutions to Linear Equations

Both for constant-coefficient and Euler linear equations, in some instances the search of a particular solution $\varphi(x)$ can be simplified and the method of variation of constants can be avoided; $\varphi(x)$ can be calculated relying on the method we briefly outline below. The coefficients, which appear in the expression of $\varphi(x)$ are to be determined, requiring $\varphi(x)$ to identically satisfy the differential equation.

## 1 Linear Constant-Coefficient Equations

$$(*) \qquad a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = F(x)$$

1. $F(x) = P_m(x)$ polynomial in $x$ of degree $m \geq 0$:
   $\varphi(x) = P_q(x)$ polynomial in $x$ (coefficients to be determined) of degree $q$, where $q = m + r$, and $r$ is the lowest derivation order for the variable $y$ on the left-hand side of (*) (Notice that $y$ is considered as a derivative of order zero, and therefore $a_n \neq 0 \;\Rightarrow\; r = 0$).

2. $F(x) = he^{kx}$ ($h$ e $k$ given constants):
   - If $k$ is not a root of the characteristic equation, then $\varphi(x) = Ae^{kx}$, $A$ constant to be determined;
   - If $k$ is a root of the characteristic equation with multiplicity $r$, then $\varphi(x) = Ax^r e^{kx}$, $A$ constant to be determined.

3. $F(x) = P_m(x)e^{kx}$, $P_m(x)$ polynomial in $x$ of degree $m > 0$:
   $\varphi(x) = P_q(x)e^{kx}$, $P_q(x)$ polynomial in $x$ (coefficients to be determined) of degree $q$, where $q = m$ if $k$ is not a root of the characteristic equation, and $q = m + r$ if $k$ is a root of the characteristic equation with multiplicity $r$.

4. $F(x) = h\sin(kx)$ or $F(x) = h\cos(kx)$ ($h$ e $k$ given constants):
   - If $\pm ik$ are not roots of the characteristic equation, then $\varphi(x) = A\sin(kx) + B\cos(kx)$, $A,\ B$ constants to be determined;
   - If $\pm ik$ are root of the characteristic equation with multiplicity $r$, then $\varphi(x) = x^r(A\sin(kx) + B\cos(kx))$, $A,\ B$ constants to be determined.

5. $F(x) = h\sinh(kx)$ or $F(x) = h\cosh(kx)$ ($h$ e $k$ given constants):
   - If $\pm k$ are not roots of the characteristic equation, then $\varphi(x) = A\sinh(kx) + B\cosh(kx)$, constants to be determined;
   - If $+k$ o $-k$ are root of the characteristic equation, then recall that

     $$\sinh(kx) = \frac{e^{kx} - e^{-kx}}{2}, \qquad \cosh(kx) = \frac{e^{kx} + e^{-kx}}{2}$$

     and rely on cases 2) and 7).

6. $F(x) = e^{px}\sin(qx)$ or $F(x) = e^{px}\cos(qx)$ ($p$ e $q$ given constants):
    - If $p \pm iq$ are not roots of the characteristic equation, then $\varphi(x) = e^{px}(A\sin(qx) + B\cos(qx))$, $A,\ B$ constants to be determined;
    - If $p \pm iq$ are root of the characteristic equation with multiplicity $r$, then $\varphi(x) = x^r e^{px}(A\sin(qx) + B\cos(qx))$, $A,\ B$ constants to be determined.
7. If $F(x)$ is the sum of functions considered in the previous examples, let $\varphi(x)$ be the sum of the corresponding particular solutions.

**Remark 1.1** *If in $\varphi(x)$ there are terms, which are particular solutions to the homogeneous linear equation associated to (*), they can be discarded, since they are already taken into account in the general solution to the homogeneous equation (the only thing that might possibly change are the arbitrary, multiplicative constants).*

# 2 Euler Equations

(**) $$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy' + a_ny = F(x)$$

1. $F(x) = hx^k$ ($h$ e $k$ given constants):
    - If $k$ is not a root of the characteristic equation, then $\varphi(x) = Ax^k$, $A$ constant to be determined;
    - If $k$ is a root of the characteristic equation with multiplicity $r$, then $\varphi(x) = Ax^k\log^r x$, $A$ constant to be determined.
2. $F(x) = P_m(\log x)$ polynomial in $\log x$ of degree $m > 0$:
   $\varphi(x) = P_q(\log x)$ polynomial in $\log x$ (coefficients to be determined) of degree $q$, where $q = m + r$, and $r$ is the multiplicity (if any) of the root $\lambda = 0$ of the characteristic equation.
3. If $F(x)$ is the sum of functions considered in the previous examples, let $\varphi(x)$ be the sum of the corresponding particular solutions.

**Remark 2.1** *If in $\varphi(x)$ there are terms, which are particular solutions to the homogeneous linear equation associated to (*), they can be discarded, since they are already taken into account in the general solution to the homogeneous equation (the only thing that might possibly change are the arbitrary, multiplicative constants).*