Advanced Mathematical Methods for Engineers - September 1 2015

1. Determine the general solution of the linear homogeneous system

$$\underline{z}' = \mathbb{A}\underline{z}, \qquad \text{where} \qquad \mathbb{A} = \begin{bmatrix} 7 & -12 & 4 \\ 4 & -9 & 4 \\ 4 & -12 & 7 \end{bmatrix}.$$

2. Consider the Cauchy Problem

$$\begin{cases} y' = y \ln y \\ y(x_o) = y_o, \end{cases} \qquad (x_o, y_o) \in \{(x,y) \in \mathbb{R}^2 : \; y > 0\}.$$

   Determine the main properties of its solution and draw a qualitative graph, as (x_o, y_o) ranges in $\{(x,y) \in \mathbb{R}^2 : \; y > 0\}$.

3. Consider

$$u = x \sin x.$$

   - Prove that $u \in \mathcal{S}'(\mathbb{R})$.
   - Compute its Fourier transform $\hat{u}$.
   - Show that the result is in accordance with the Paley-Wiener Theorem.

4. Consider the space $H^1(-\pi,\pi) = \{f \in L^2(-\pi,\pi) \;\; \text{s.t.} \;\; f' \in L^2(-\pi,\pi)\}$, which is a Hilbert space endowed with the inner product

$$(f,g)_{H^1} = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx + \int_{-\pi}^{\pi} f'(x)\overline{g'(x)}\,dx.$$

   Let $Z \subset H^1(-\pi,\pi)$ be the linear manifold generated by the vectors

$$v_1 = 1, \quad v_2 = \sin x, \quad v_3 = \cos x, \quad v_4 = \sin(2x), \quad v_5 = \cos(2x).$$

   In Z build an orthonormal system and approximate $f(x) = x$ in Z with the least mean square error.