Advanced Mathematical Methods for Engineers - April 5 2016

1. Consider the space $H^1(-1,1) = \{f \in L^2(-1,1) \text{ t.c. } f' \in L^2(-1,1)\}$, which is a Hilbert space endowed with the inner product

$$(f,g)_{H^1} = \int_{-1}^{1} f(x)\overline{g(x)}\,dx + \int_{-1}^{1} f'(x)\overline{g'(x)}\,dx.$$

Relying on the Gram-Schmidt method, find the best approximation of $f(x) = \cos(2x)$ in the subspace $M \subset H^1(-1,1)$, which is generated by the vectors $u_o = 1$, $u_1 = x$, $u_2 = x^2$.

2. Using the separation-of-variable method, determine the solution u of the following problem and discuss its regularity

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 & x \in]-\pi,\pi[,\ t > 0, \\ u(x,0) = x(\pi^2 - x^2) & x \in [-\pi,\pi], \\ \dfrac{\partial u}{\partial t}(x,0) = 3\cos(\dfrac{3}{2}x) & x \in [-\pi,\pi], \\ u(-\pi,t) = u(\pi,t) = 0 & t \geq 0. \end{cases}$$

3. Consider the Cauchy Problem

$$\begin{cases} y' = \ln(5 - y^2), \\ y(x_o) = y_o. \end{cases}$$

Determine the main properties of its general solution and draw a qualitative graph, as (x_o, y_o) ranges in a proper open set $D \subseteq \mathbb{R}^2$.

4. Compute the Fourier transform of the tempered distribution

$$u = x \arctan(x-1),$$

justifying all the steps.