Advanced Mathematical Methods for Engineers - September 15 2016

1. Determine the solution of the Cauchy Problem

$$\begin{cases} y' = y - y^3, \\ y(0) = \dfrac{1}{2}. \end{cases}$$

2. Consider the space $H^1(-\pi, \pi) = \{f \in L^2(-\pi, \pi) \ \text{ s.t. } \ f' \in L^2(-\pi, \pi)\}$, which is a Hilbert space endowed with the inner product

$$(f, g)_{H^1} = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx + \int_{-\pi}^{\pi} f'(x)\overline{g'(x)}\, dx.$$

   Let $Z \subset H^1(-\pi, \pi)$ be the linear manifold generated by the vectors

$$v_1 = 1, \quad v_2 = \sin x, \quad v_3 = \cos x, \quad v_4 = \sin(2x), \quad v_5 = \cos(2x).$$

   In Z build an orthonormal system and approximate $f(x) = |x|$ in Z with the least mean square error.

3. Consider the Cauchy Problem

$$\begin{cases} y' = 4y - y^3, \\ y(x_o) = y_o. \end{cases}$$

   Determine the main properties of its general solution and draw a qualitative graph, as (x_o, y_o) ranges in $\mathbb{R}^2$.

4. Compute the Fourier Transform of the tempered distribution $u = \operatorname{sign} x$, taking into account that in the sense of distributions $(\operatorname{sign} x)' = 2\delta$.

   Then, relying on the previous result, and on the fundamental properties of the Fourier transform, compute

$$\mathcal{F}((\sin x) \operatorname{sign} x).$$