Advanced Mathematical Methods for Engineers - February 22 2016

1. Determine the unique solution of the Cauchy Problem

$$\begin{cases} \underline{z}' = \mathbb{A}\underline{z}, \\ \underline{z}(0) = \underline{b} \end{cases}$$

where

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 0 & 2 \end{bmatrix}, \qquad \underline{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

2. Let

$$X = C^o([0,1]) \quad \text{with} \quad \|f\|_X := \sup_{x\in[0,1]} |f(x)|,$$

and consider the linear operator $A : X \to X$ given by

$$(Af)(x) = xf(x),$$

with $D_{(A)} = X$. Show that A is bounded from X to X and compute its norm $\|A\|$.

3. Consider the Cauchy Problem

$$\begin{cases} y' = \dfrac{y^2-1}{y^2+1} \\ y(x_o) = y_o, \end{cases} \qquad (x_o, y_o) \in \mathbb{R}^2.$$

Determine the main properties of its solution and draw a qualitative graph, as (x_o, y_o) ranges in $\mathbb{R}^2$.

4. Compute the Fourier transform of the tempered distribution

$$u = \mathrm{pv}\,\frac{1}{x-2} + \mathrm{pv}\,\frac{1}{x+3},$$

using

- the Fourier transform of $v = \mathrm{pv}\,\frac{1}{x}$;
- the relation between translation by a real number with respect to x, and product by a complex exponential function with respect to the conjugate variable ξ.