1) Determine the solution $u \in \mathcal{D}'(\mathbf{R})$ to the equation

$$(x^2-1)^2 u' = \delta(x) + \delta''(x-1).$$

2) Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = e^{-x^2} \text{sign} x.$$

a) Check that $f \in \mathcal{S}'(\mathbf{R})$ and study the main properties of its Fourier transform, without explicitly computing it;
b) Compute $\hat{f}$ explicitly, verifying the previous results (Suggestion: derive f and get the relation between $\hat{f}$ and $\hat{f}'$.)

3) Consider the sequence of functions $\{f_n\}_{n\in\mathbf{N}}$, $f_n : \mathbf{R} \to \mathbf{R}$, defined by

$$f_n(x) = n^\alpha \chi_{(-\frac{1}{n},\frac{1}{n})} \cos n\frac{\pi}{2}x,$$

with α a real parameter. As α ranges in $\mathbf{R}$, determine the limit of the sequence in the sense of distributions. Explain when the convergence is actually stronger.

4) Consider the differential equation in $\mathbf{R}$

(1) $$\epsilon^4 v_\epsilon^{(4)} + v_\epsilon = \delta'$$

where $\epsilon > 0$.
a) Prove that in $\mathcal{S}'(\mathbf{R})$ (1) has one and only one solution v_ϵ;
b) compute v_ϵ explicitly;
c) study the regularity of v_ϵ;
d) compute $\lim\limits_{\epsilon\to 0} v_\epsilon$ in the sense of $\mathcal{S}'(\mathbf{R})$.

5) Consider the function

$$u_\epsilon(x) = \frac{\chi_\epsilon(x)}{|x|} + 2(\log\epsilon)\,\delta(x),$$

where $\chi_\epsilon(x)$ is the characteristic function of $\mathbf{R}\backslash(-\epsilon,\epsilon)$;
a) in $\mathcal{D}'(\mathbf{R})$ determine the limit u of u_ϵ when $\epsilon \to 0$;
b) check that

$$x\,u = \text{sign } x,$$

which justifies the definition $u = \text{pf}\frac{1}{|x|}$;
c) given

$$v = \sin x(\text{pf}\frac{1}{|x|}),$$

study the main properties of its Fourier transform, without explicitly computing it;
e) compute $\hat{v}$ explicitly, verifying the previous results.

6) Compute the limit in the sense of distributions of the sequence $\{u_n\}_{n\in\mathbf{N}} \subseteq L^1_{loc}(\mathbf{R})$, where

$$u_n(x) = \frac{n^2 x}{n^2x^2+1} + \frac{1-\cos(nx)}{x} + \int_0^n \cos(tx)\,dt.$$

7) Consider the differential equation

$$\epsilon z" + xz' = 0, \qquad \epsilon > 0.$$

a) Compute $\hat{z}_\epsilon$, the Fourier transform of $z_\epsilon \in \mathcal{S}'(\mathbf{R})$, solution to the equation;
b) compute z_ϵ;

c) compute $\lim_{\epsilon\to 0^+} z_\epsilon$ in the sense of distributions;

d) compare the result found in c) with the solutions to the equation

$$xz' = 0.$$

8) Determine the tempered distribution $u \in \mathcal{S}'(\mathbf{R})$ such that

$$\hat{u}(\xi) = \ \mathrm{pv}\frac{\xi^2+2}{\xi(\xi^2+1)}.$$

9) Compute the limit in the sense of $\mathcal{D}'(\mathbf{R})$ of the sequence of functions $\{u_n\}_{n\in\mathbf{N}}$ with

$$u_n(x) = \frac{2n\,x^{2n-1}}{1+x^{4n}} + \frac{x^{2n+1}}{|x|^{2n+1}+1}.$$

10) Compute in $\mathcal{D}'(\mathbf{R})$ the limit of the sequence of functions $\{f_n\}$ defined by

$$f_n(x) = \frac{1+nx}{1+n|x|}e^{-\frac{|x|}{n}} + \frac{nx}{1+4nx^2} + n|\sin(n\pi x)|\chi_{(-\frac{1}{n},\frac{1}{n})}.$$

11) Determine the set I of the functions $u \in C^\infty(\mathbf{R}) \cap \mathcal{S}'(\mathbf{R})$ such that for any $\epsilon > 0$ and for any $x \in \mathbf{R}$ one has

$$u(x) = \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} u(t)\,dt;$$

determine $v \in \ I$ which satisfies

$$v\,\delta' = \delta.$$

12) Consider the sequence of functions $\{f_n\}$, $f_n : \mathbf{R} \to \mathbf{R}$, defined by

$$f_n(x) = \mathrm{Th}[(x-1)^{2n}] - \frac{n^2x^2}{2(1+n^2x^2)};$$

in $\mathcal{D}'(\mathbf{R})$ compute the limit of f_n and of f_n', checking whether the convergence is actually stronger or not.