(1) $z' = \mathbb{A} z \qquad \mathbb{A} = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$

$\det(\lambda I - \mathbb{A}) = 0 \qquad \begin{vmatrix} \lambda+3 & 0 & -2 \\ -1 & \lambda+1 & 0 \\ 2 & 1 & \lambda \end{vmatrix} = 0$

$$(\lambda+3)(\lambda^2+\lambda) - 2(-1-2(\lambda+1)) = 0$$
$$\lambda^3 + 3\lambda^2 + \lambda^2 + 3\lambda + 2 + 4\lambda + 4 = 0$$
$$\lambda^3 + 4\lambda^2 + 7\lambda + 6 = 0$$

	1	4	7	6
-2		-2	-4	-6
	1	2	3	//

Hence
$$(\lambda+2)(\lambda^2 + 2\lambda + 3) = 0$$
$$(\lambda+2)\left[(\lambda+1)^2 + (\sqrt{2})^2\right] = 0$$

$\lambda_1 = -2$

$\lambda_2 = -1 + \sqrt{2}\,i$

$\lambda_3 = \overline{\lambda_2} = -1 - \sqrt{2}\,i$

Let us compute the corresponding eigenvectors.

$$\begin{bmatrix} 1 & 0 & -2 \\ -1 & -1 & 0 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} \alpha - 2\gamma = 0 \\ \alpha + \beta = 0 \end{cases}$$

$$h_1 = \begin{bmatrix} 2L \\ -2L \\ L \end{bmatrix}$$

$$\begin{bmatrix} +2+\sqrt{2}i & 0 & -2 \\ -1 & \sqrt{2}i & 0 \\ 2 & 1 & -1+\sqrt{2}i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (2+\sqrt{2}i)\alpha \quad -2\gamma = 0 \\ -\alpha + \sqrt{2}i\beta \quad = 0 \end{cases} \qquad h_2 = \begin{bmatrix} \frac{2-\sqrt{2}i}{3} L \\ -\frac{1+\sqrt{2}i}{3} L \\ L \end{bmatrix}$$

Instead of using h_2 and $h_3 = \overline{h_2}$, we use the real and imaginary parts.

$$w_2 = \mathrm{Re} \begin{bmatrix} \frac{2-\sqrt{2}i}{3} \cancel{L} \\ -\frac{1+\sqrt{2}i}{3} \cancel{L} \\ \cancel{L} \end{bmatrix} e^{-x} (\cos\sqrt{2}x + i \sin\sqrt{2}x)$$

$$= \mathrm{Re} \begin{bmatrix} e^{-x}\left(\frac{2}{3}\cos\sqrt{2}x + \frac{\sqrt{2}}{3}\sin\sqrt{2}x\right) + i e^{-x}\left(-\frac{\sqrt{2}}{3}\cos\sqrt{2}x + \frac{2}{3}\sin\sqrt{2}x\right) \\ e^{-x}\left(-\frac{1}{3}\cos\sqrt{2}x - \frac{\sqrt{2}}{3}\sin\sqrt{2}x\right) + i e^{-x}\left(\frac{\sqrt{2}}{3}\cos\sqrt{2}x - \frac{1}{3}\sin\sqrt{2}x\right) \\ e^{-x}\cos\sqrt{2}x + i\, e^{-x}\sin\sqrt{2}x \end{bmatrix}$$

$$= \begin{bmatrix} e^{-x}(2/3 \cos\sqrt{2}x + \sqrt{2}/3 \sin\sqrt{2}x) \\ e^{-x}(-1/3 \cos\sqrt{2}x - \sqrt{2}/3 \sin\sqrt{2}x \\ e^{-x} \cos\sqrt{2}x \end{bmatrix}$$

Analogously

$$w_3 = \mathrm{Im}\,[\quad] = \begin{bmatrix} e^{-x}(-\sqrt{2}/3 \cos\sqrt{2}x + 2/3 \sin\sqrt{2}x) \\ e^{-x}(\sqrt{2}/3 \cos\sqrt{2}x - 1/3 \sin\sqrt{2}x \\ e^{-x} \sin\sqrt{2}x \end{bmatrix}$$

Therefore, we eventually conclude that the general solution of the linear system is given by

$$Z = \begin{bmatrix} 2e^{-2x} & e^{-x}\left(\frac{2}{3}\cos\sqrt{2}x + \frac{\sqrt{2}}{3}\sin\sqrt{2}x\right) & e^{-x}\left(-\frac{\sqrt{2}}{3}\cos\sqrt{2}x + \frac{2}{3}\sin\sqrt{2}x\right) \\ -2e^{-2x} & e^{-x}\left(-\frac{1}{3}\cos\sqrt{2}x - \frac{\sqrt{2}}{3}\sin\sqrt{2}x\right) & e^{-x}\left(\frac{\sqrt{2}}{3}\cos\sqrt{2}x - \frac{1}{3}\sin\sqrt{2}x\right) \\ e^{-2x} & e^{-x}\cos\sqrt{2}x & e^{-x}\sin\sqrt{2}x \end{bmatrix} \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

where L, M, N are real, arbitrary constants.

② $f_n = \dfrac{x}{1+nx^2}$ $\quad x \in [-10, 10]$

$$f_n' = \frac{-2nx^2 + 1 + nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

$f_n' = 0 \qquad x = \pm \dfrac{1}{\sqrt{n}}$

$$\max_{[-10,10]} |f_n(x)| = \frac{1/\sqrt{n}}{1+1} = \frac{1}{2\sqrt{n}}$$

Therefore, as $n \to \infty$ $\forall x \in [-10, 10]$ $f_n(x) \to 0$

The candidate for the limit function is $f(x) = 0$ $\forall$ $x \in [-10, 10]$

In $C^0([-10,10])$ we have

$$\sup_{x\in[-10,10]} |f_n(x) - f(x)| = \sup_{[-10,10]} |f_n(x)| = \frac{1}{2\sqrt{n}} \longrightarrow 0 \quad \text{as } n \to \infty$$

Hence, $f_n \to f$ in the sense of $C^0([-10,10])$

In $L^2(-10,10)$ we have

$$\int_{-10}^{10} |f_n - f|^2 dx = \int_{-10}^{10} |f_n|^2 dx \leq 20 \frac{1}{4n} \longrightarrow 0$$

as $n \to \infty$. Hence $f_n \to f$ also in the sense of $L^2(-10,10)$.

③ $\begin{cases} y' = xy\, e^{-y^2} \\ y(0) = y_0 \end{cases}$ $\quad f = xy\, e^{-y^2} \quad f \in C^\infty(\mathbb{R}^2)$

Therefore, $\forall y_0 \in \mathbb{R}$, due the local existence and uniqueness theorem, we can conclude there exists a unique solution.

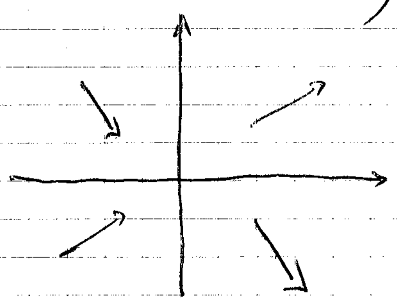
Consider $S = [-a, a] \times \mathbb{R}$. In S we have

$$|f(x,y)| = |xy\, e^{-y^2}| = |x|\,|y|\, e^{-y^2}$$
$$\leq a|y|$$

Therefore, the global existence and uniqueness theorem is satisfied by taking $L_1 = 0$, $L_2 = a$ and we can conclude that for any $a > 0$, in $[-a, a]$ there exists a unique solution to our Cauchy Problem and such a solution is of class C^∞.

Moreover, it is immediate to check that $y = 0$ is a solution.

$y' \geq 0 \qquad xy\, e^{-y^2} \geq 0 \qquad xy \geq 0$



$$y'' = y e^{-y^2} + x y' e^{-y^2} - 2x y^2 y' e^{-y^2}$$
$$= y e^{-y^2} + x^2 y\, e^{-2y^2} - 2x^2 y^3 e^{-2y^2}$$

$$= y e^{-y^2} \left(1 + x^2 (1 - 2y^2) e^{-y^2} \right)$$

$y'' \geq 0 \qquad y e^{-y^2} \left(1 + x^2 (1-2y^2) e^{-y^2} \right) \geq 0$

$\qquad\qquad y \cancel{e^{-y^2}} \left(1 + x^2 (1-2y^2) e^{-y^2} \right) \geq 0$

Notice that if $|y| \leq \frac{1}{\sqrt{2}}$, we have that

$y > 0 \implies y'' > 0$

$y < 0 \implies y'' < 0$

When $|y| > 1/\sqrt{2}$

$y'' \geq 0 \qquad y \left(1 - x^2 (2y^2 - 1) e^{-y^2} \right) \geq 0$

$$\begin{cases} y \geq 0 \\ |x| \leq \dfrac{e^{y^2}}{2y^2 - 1} \end{cases}$$

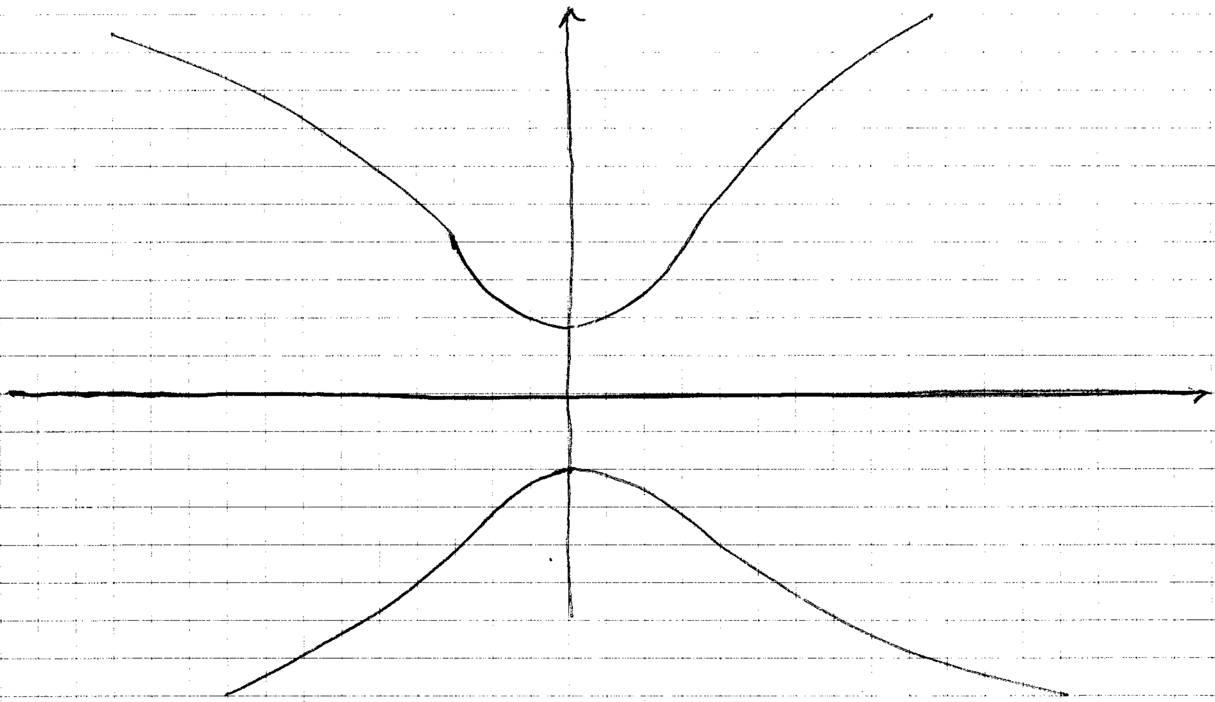
or

$$\begin{cases} y \leq 0 \\ |x| \geq \dfrac{e^{y^2}}{2y^2 - 1} \end{cases}$$

Even though it is not easy to draw a qualitative graph of the function

$$x = \frac{e^{y^2}}{2y^2 - 1} \qquad |y| > 1/\sqrt{2},$$

nevertheless, we have enough information to conclude



④ $u = e^{-x^2} \left(\mathrm{pv} \frac{1}{x} \right)$

Since $\mathrm{pv} \frac{1}{x}$ is a tempered distibution and e^{-x^2} is a bounded, C^∞ function, we conclude that u is a tempered distibution.

Coming to the Fourier transform

$$x u = e^{-x^2}$$

$$\widehat{x u} = \widehat{e^{-x^2}}$$

$$i \frac{d}{d\xi} \hat{u} = \sqrt{\pi}\, e^{-\xi^2/4}$$

$$\hat{u} = -\sqrt{\pi}\, i \int_0^\xi e^{-t^2/4}\, dt.$$