(1) $\underline{z}' = \mathbb{A}\underline{z}$

$$\mathbb{A} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

Let us first determine the eigenvalues of $\mathbb{A}$. We need to solve

$$\det(\lambda I - \mathbb{A}) = 0 \qquad \begin{vmatrix} \lambda+1 & -2 & -2 \\ -2 & \lambda-2 & 1 \\ -2 & 1 & \lambda-2 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda+1 & 0 & -2 \\ -2 & \lambda-3 & 1 \\ -2 & 3-\lambda & \lambda-2 \end{vmatrix} = 0 \qquad \begin{vmatrix} \lambda+1 & 0 & -2 \\ -4 & 0 & \lambda-1 \\ -2 & 3-\lambda & \lambda-2 \end{vmatrix} = 0$$

$$(3-\lambda)[\lambda^2 - 1 - 8] = 0 \qquad (3-\lambda)(\lambda-3)(\lambda+3) = 0.$$

Therefore, we have $\lambda_1 = 3$ with multiplicity 2 and $\lambda_2 = -3$

The general solution is given by

$$\underline{Z} = e^{3x}\underline{c}_1 + x e^{3x}\underline{c}_2 + e^{-3x}\underline{c}_3$$

where $\underline{c}_1$, ~~and~~ $\underline{c}_2$, and $\underline{c}_3$ are proper vectors, which altogether depend on 3 arbitrary constants.

Since

$$\underline{Z}' = 3e^{3x}\underline{c}_1 + (e^{3x} + 3x e^{3x})\underline{c}_2 + (-3)e^{-3x}\underline{c}_3$$

we must have

$$3e^{3x}\underline{c}_1 + e^{3x}(1+3x)\underline{c}_2 - 3e^{-3x}\underline{c}_3 =$$
$$= e^{3x}\mathbb{A}\underline{c}_1 + x e^{3x}\mathbb{A}\underline{c}_2 + e^{-3x}\mathbb{A}\underline{c}_3$$

[redacted]

which yields

$$\begin{cases} \mathbb{A}\,\underline{c}_1 = 3\,\underline{c}_1 + \underline{c}_2 \\ \mathbb{A}\,\underline{c}_2 = 3\,\underline{c}_2 \\ \mathbb{A}\,\underline{c}_3 = -3\,\underline{c}_3 \end{cases}$$

If we denote $\underline{c}_3 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, we have

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -3\alpha \\ -3\beta \\ -3\gamma \end{bmatrix}$$

$$\begin{cases} -\alpha + 2\beta + 2\gamma = -3\alpha \\ 2\alpha + 2\beta - \gamma = -3\beta \\ 2\alpha - \beta + 2\gamma = -3\gamma \end{cases} \qquad \begin{cases} \alpha + \beta + \gamma = 0 \\ \cancel{2\alpha + 5\beta - \gamma = 0} \\ 2\alpha - \beta + 5\gamma = 0 \end{cases}$$

$$\Longrightarrow \quad \underline{c}_3 = \begin{bmatrix} -2L \\ L \\ L \end{bmatrix}$$ where $L \in \mathbb{R}$ is an arbitrary constant.

Similarly, if we let $\underline{c}_2 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, we have

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 3\alpha \\ 3\beta \\ 3\gamma \end{bmatrix}$$

$$\begin{cases} -\alpha + 2\beta + 2\gamma = 3\alpha \\ 2\alpha + 2\beta - \gamma = 3\beta \\ 2\alpha - \beta + 2\gamma = 3\gamma \end{cases} \qquad \begin{cases} \cancel{-4\alpha + 2\beta + 2\gamma = 0} \\ \cancel{2\alpha - \beta - \gamma = 0} \\ 2\alpha - \beta - \gamma = 0 \end{cases}$$

$$\Longrightarrow \quad \underline{c}_2 = \begin{bmatrix} M + N \\ 2M \\ 2N \end{bmatrix}$$ where $M, N \in \mathbb{R}$ are arbitrary constants.

[redacted]

Finally, if we let $\underline{c}_1 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, we have

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 3\alpha \\ 3\beta \\ 3\gamma \end{bmatrix} + \begin{bmatrix} M+N \\ 2M \\ 2N \end{bmatrix}$$

$$\begin{cases} -4\alpha + 2\beta + 2\gamma = M+N \\ 4\alpha - 2\beta - 2\gamma = 4M \\ 4\alpha - 2\beta - 2\gamma = 4N \end{cases}$$

The coefficient matrix has $r=1$, whereas, the matrix

$$\mathbb{B} = \begin{bmatrix} -4 & 2 & 2 & M+N \\ 4 & -2 & -2 & 4M \\ 4 & -2 & -2 & 4N \end{bmatrix}$$

has $r=2$, unless we assume $M=N=0$. Therefore, by Rouché-Capelli's Theorem, in order to have solvability, we have to take $M=N=0$. Consequently $\underline{c}_2 = \underline{0}$ and

$$\underline{c}_1 = \begin{bmatrix} H+K \\ 2H \\ 2K \end{bmatrix}$$

We conclude that

$$\underline{z} = e^{3x} \begin{bmatrix} H+K \\ 2H \\ 2K \end{bmatrix} + e^{-3x} \begin{bmatrix} -2L \\ L \\ L \end{bmatrix}$$

$$= \begin{bmatrix} e^{3x} & e^{3x} & -2e^{-3x} \\ 2e^{3x} & 0 & e^{-3x} \\ 0 & 2e^{3x} & e^{-3x} \end{bmatrix} \begin{bmatrix} H \\ K \\ L \end{bmatrix}$$

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2 $\begin{cases} y' = y \sin y \\ y(x_0) = y_0 \end{cases}$

[redacted]

The function $f(x,y) = y \sin y$ belongs to $C^\infty(\mathbb{R}^2)$. Therefore, by the local existence and uniqueness Theorem, combined with the regularity Theorem, we conclude that

$$\forall (x_0, y_0) \in \mathbb{R}^2 \quad \exists! \quad y = y(x) \in C^\infty(I_{x_0}),$$

solution of the Cauchy Problem. Moreover, if we consider an arbitrary

$$S = (a,b) \times \mathbb{R} \qquad -\infty < a < b < +\infty$$

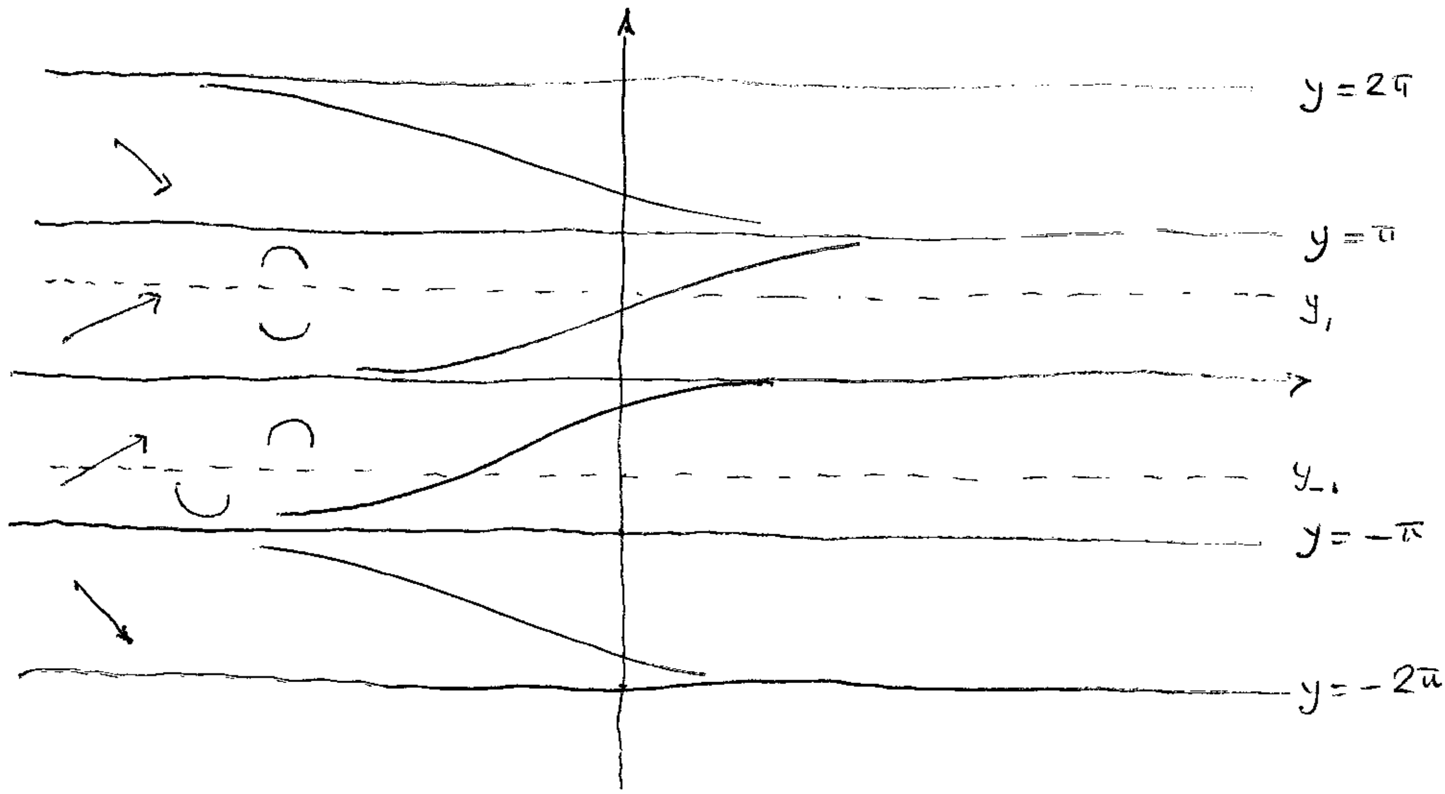
$$\forall (x,y) \in \bar{S} \qquad |f(x,y)| \leq |y|$$

Therefore,

$$\forall (x_0, y_0) \in \mathbb{R}^2 \quad \exists! \quad y = y(x) \in C^\infty(-\infty, \infty)$$

due to the arbitrariness of a and b.

$$y' \geq 0 \qquad y \sin y \geq 0 \qquad 2k\pi \leq y \leq (2k+1)\pi \quad k \in \mathbb{N}$$
$$-(2l+1)\pi \leq y \leq -2l\pi \quad l \in \mathbb{N}$$



[redacted]

$$y'' = y'\sin y + y y' \cos y = y \sin^2 y + y^2 \sin y \cos y$$

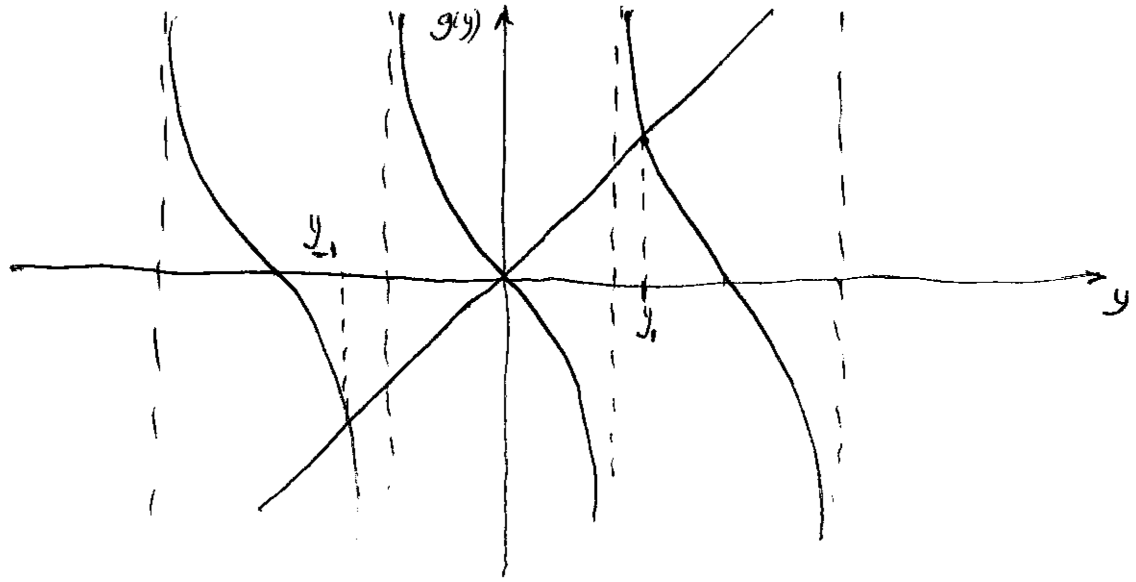
$$y'' = y \sin y \,(\sin y + y \cos y)$$

If we consider the equation

$$\sin y + y \cos y = 0$$

$$y = -\operatorname{tg} y$$

we have the situation depicted below



If we separate the variables, we have

$$\int_{y_0}^{y} \frac{dt}{t \sin t} = x - x_0$$

It is apparent that $\forall\, m \in \mathbb{Z}$ and $\forall\, y_0 \neq m\pi$

$$\lim_{y \to m\pi} \int_{y_0}^{y} \frac{dt}{t \sin t} = \infty$$

whereas $\forall\, y_0 \neq m\pi$, $\forall\, y \neq m\pi$

$$\int_{y_0}^{y} \frac{dt}{t \sin t} = \text{finite}$$

[redacted]

provided we choose

$$m\pi < y, \; y_0 < (m+1)\pi$$

Therefore the particular solutions

$$y = m\pi \qquad m \in \mathbb{Z}$$

are horizontal asymptote for all the other particular solutions.

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(3)

$$u_n = \frac{2n\, x^{2n-1}}{1 + x^{4n}} + n \cos nx$$

Let us define

$$v_n = \frac{2n\, x^{2n-1}}{1 + x^{4n}}$$

$$w_n = n \cos nx$$

and compute the two limits separately.

We have

$$n \cos nx = (\sin nx)' = \left(- \frac{\cos nx}{n}\right)''$$

$\forall\, \varphi \in \mathcal{D}(\mathbb{R})$

$$\left| \int_{\mathbb{R}} - \frac{\cos nx}{n}\, \varphi(x)\, dx \right| \leq \frac{1}{n} \int_{\mathbb{R}} |\varphi(x)|\, dx \xrightarrow[n \to \infty]{} 0$$

Therefore

$$- \frac{\cos nx}{n} \xrightarrow[\mathcal{D}'(\mathbb{R})]{n \to \infty} 0$$

For the continuity of the derivation in the space of distributions, we conclude that

$$n \cos nx \xrightarrow[\mathcal{D}'(\mathbb{R})]{n \to \infty} 0$$

As for the other sequence, we remark that

$$\frac{2n\, x^{2n-1}}{1 + x^{4n}} = \left(\operatorname{arctg} x^{2n}\right)'$$

[redacted]

Therefore, as before, if we prove that

$$\bar{v}_n \xrightarrow[\mathcal{D}']{n\to\infty} \bar{v}$$

where $\bar{v}_n = \operatorname{arctg} x^{2n}$, then $v_n \xrightarrow[\mathcal{D}']{n\to\infty} v'$

Notice that

$$\operatorname{arctg} x^{2n} \longrightarrow \begin{cases} 0 & \text{if } |x| < 1 \\ \pi/4 & \text{if } |x| = 1 \\ \frac{\pi}{2} & \text{if } |x| > 1 \end{cases}$$

We want to show that $\forall R > 1$

$$\operatorname{arctg} x^{2n} \xrightarrow[L^1(-R,R)]{n\to\infty} \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)$$

We have to show that

$$\lim_{n\to\infty} \left\| \operatorname{arctg} x^{2n} - \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)\right\|_{L^1(-R,R)} =$$

$$= \lim_{n\to\infty} \int_{-R}^{R} \left|\operatorname{arctg} x^{2n} - \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)\right| dx = 0.$$

We have

$$\int_{-R}^{R} \left|\operatorname{arctg} x^{2n} - \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)\right| dx = 2\int_0^1 \operatorname{arctg} x^{2n}\, dx +$$

$$+ 2\int_1^R \left(\frac{\pi}{2} - \operatorname{arctg} x^{2n}\right) dx \le 2\int_0^1 x^{2n}\, dx +$$

$$+ 2\int_1^R \operatorname{arctg} \frac{1}{x^{2n}}\, dx \le 2\, \frac{x^{2n+1}}{2n+1}\Big|_0^1 +$$

$$+ 2\int_1^R \frac{1}{x^{2n}}\, dx = \frac{2}{2n+1} - \frac{2}{2n+1}\, \frac{1}{x^{2n+1}}\Big|_1^R$$

$$= \frac{4}{2n+1} - \frac{2}{(2n+1)\, R^{2n+1}} \xrightarrow[n\to\infty]{} 0.$$

Much in the same way, we can prove that for any $-\infty < a < b < +\infty$

$$\operatorname{arctg} x^{2n} \xrightarrow[L^1(a,b)]{n \to \infty} \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)$$

Let us now consider $\varphi \in \mathcal{D}(\mathbb{R})$. We want to show that

$$\int_{\mathbb{R}} \operatorname{arctg} x^{2n} \varphi(x) \xrightarrow[n \to \infty]{} \int_{\mathbb{R}} \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right) \varphi(x)$$

It is enough to show that $\int_{\mathbb{R}} \left[\operatorname{arctg} x^{2n} - \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)\right] \varphi(x)$ $\xrightarrow[n \to \infty]{} 0$. Indeed

$$\left| \int_{\mathbb{R}} \left[\operatorname{arctg} x^{2n} - \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)\right] \varphi(x)\, dx \right| \leq$$

$$\leq \int_{[a,b]} \left|\operatorname{arctg} x^{2n} - \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)\right| |\varphi(x)|\, dx$$

$$\leq \operatorname{Max}_{[a,b]} |\varphi| \int_{[a,b]} \left|\operatorname{arctg} x^{2n} - \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)\right| dx$$

$\xrightarrow[n \to \infty]{} 0$ where $[a,b]$ is the support of φ.

We have therefore proved that

$$\operatorname{arctg} x^{2n} \xrightarrow[\mathcal{D}'(\mathbb{R})]{n \to \infty} \frac{\pi}{2}\left(1 - \chi_{(-1,1)}\right)$$

Therefore

$$2n \frac{x^{2n-1}}{1 + x^{4n}} \xrightarrow[\mathcal{D}'(\mathbb{R})]{n \to \infty} \frac{\pi}{2}\left(-\delta_{-1} + \delta_1\right)$$

and

$$u_n \longrightarrow \frac{\pi}{2}\left(\delta_1 - \delta_{-1}\right)$$

[redacted]

④ We need to build an orthonormal system for Z.
We use Gram-Schmidt's method

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{3}{\left[\int_{-1}^{1} 3^2\,dx + \int_{-1}^{1} 0^2\,dx\right]^{1/2}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$u_2 = \frac{v_2 - (v_2, u_1)u_1}{\|v_2 - (v_2, u_1)u_1\|}$$

We have

$$v_2 - (v_2, u_1)\,u_1 = 2x - \left[\int_{-1}^{1} 2x \frac{1}{\sqrt{2}}\,dx + \int_{-1}^{1} 2\cdot 0\,dx\right]\frac{1}{\sqrt{2}}$$

$$= 2x - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\, 0 = 2x$$

$$\|v_2 - (v_2, u_1)u_1\| = \left[\int_{-1}^{1} (2x)^2\,dx + \int_{-1}^{1} 2^2\,dx\right]^{1/2}$$

$$= \left[4\frac{x^3}{3}\Big|_{-1}^{1} + 4x\Big|_{-1}^{1}\right]^{1/2} =$$

$$= \left(2\,\frac{4}{3} + 2\cdot 4\right)^{1/2} = \left(8\cdot\frac{4}{3}\right)^{1/2} = 4\sqrt{\frac{2}{3}}$$

Hence

$$u_2 = \frac{2x}{4\sqrt{2/3}} = \sqrt{\frac{3}{2}}\;\frac{x}{2} = \sqrt{\frac{3}{8}}\,x$$

Finally

$$u_3 = \frac{v_3 - (v_3, u_1)u_1 - (v_3, u_2)u_2}{\|v_3 - (v_3, u_1)u_1 - (v_3, u_2)u_2\|}$$

We have

$$v_3 - (v_3, u_1)u_1 - (v_3, u_2)u_2 = 4x^2 - \left[\int_{-1}^{1} 4x^2\cdot\frac{1}{\sqrt{2}}\,dx + \int_{-1}^{1} 8x\cdot 0\,dx\right]\frac{1}{\sqrt{2}}$$

$$- \left[\int_{-1}^{1} 4x^2\cdot\sqrt{\frac{3}{8}}\,x\,dx + \int_{-1}^{1} 8x\cdot\sqrt{\frac{3}{8}}\,dx\right] =$$

[redacted]

$$= 4x^2 - \frac{1}{2}\,\frac{4}{3}x^3\Big|_{-1}^{1} = 4x^2 - \frac{4}{3} = \frac{4}{3}(3x^2-1)$$

$$\|V_3 - (V_3,U_1)U_1 - (V_3,U_2)U_2\| = \left[\int_{-1}^{1}\left(\frac{4}{3}(3x^2-1)\right)^2 dx + \right.$$

$$\left. + \int_{-1}^{1}\left(\frac{4}{3}6x\right)^2 dx\right]^{1/2} = \left[\frac{16}{9}\cdot 2\int_0^1 (9x^4-6x^2+1)\,dx + 128\int_0^1 x^2\,dx\right]^{1/2}$$

$$= \left[\frac{32}{9}\left[\frac{9}{5} - \frac{6}{3} + 1\right] + \frac{128}{3}\right]^{1/2} = 4\sqrt{\frac{128}{45}} = \frac{2\cdot 2\cdot 8\sqrt{2}}{3\sqrt{5}} = \frac{2\,16\sqrt{2}}{3\sqrt{5}}$$

Hence

$$U_3 = \frac{\frac{4}{3}(3x^2-1)}{\sqrt{2}\;16\cdot 2}\cdot 3\sqrt{5} = \frac{1}{2}\sqrt{\frac{5}{32}}\,(3x^2-1) = \sqrt{\frac{5}{128}}(3x^2-1)$$

The approximation w of e^x with the least mean square error is given by

$$w = (e^x, U_1)U_1 + (e^x, U_2)U_2 + (e^x, U_3)U_3$$

We omit the final calculations.

[redacted]