1. $\underline{z}' = \mathbb{A}\underline{z}$

$$\mathbb{A} = \begin{bmatrix} 2 & -13 \\ 2 & 12 \end{bmatrix} \qquad \det(\lambda \mathbb{I} - \mathbb{A}) = 0$$

$$\begin{vmatrix} \lambda-2 & 13 \\ -2 & \lambda-12 \end{vmatrix} = 0 \qquad \lambda^2 - 14\lambda + 24 + 26 = 0$$

$$\lambda^2 - 14\lambda + 50 = 0$$

$$(\lambda-7)^2 = -1 \qquad \lambda_1 = 7+i$$

$$\lambda_2 = 7-i$$

$$\underline{h}_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad (\lambda_1 \mathbb{I} - \mathbb{A})\underline{h}_1 = 0$$

$$\begin{bmatrix} 5+i & 13 \\ -2 & -5+i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (5+i)\alpha + 13\beta = 0 \\ -2\alpha + (-5+i)\beta = 0 \end{cases} \qquad (5+i)\alpha + \beta \cdot 13 = 0$$

$$\begin{cases} \alpha = 1 \\ \beta = -\frac{5+i}{13} \end{cases}$$

Therefore

$$\underline{z} = C_1 \operatorname{Re} \begin{bmatrix} e^{(7+i)x} \\ -\frac{5+i}{13} e^{(7+i)x} \end{bmatrix} + C_2 \operatorname{Im} \begin{bmatrix} e^{(7+i)x} \\ -\frac{5+i}{13} e^{(7+i)x} \end{bmatrix}$$

$$= C_1 \operatorname{Re} \begin{bmatrix} e^{7x}(\cos x + i \sin x) \\ e^{7x}\left(-\frac{5}{13}\cos x + \frac{1}{13}\sin x + i\left(-\frac{1}{13}\cos x - \frac{5}{13}\sin x\right)\right) \end{bmatrix}$$

$$+ C_2 \operatorname{Im} \begin{bmatrix} e^{7x}(\cos x + i \sin x) \\ e^{7x}\left(-\frac{5}{13}\cos x + \frac{1}{13}\sin x + i\left(-\frac{1}{13}\cos x - \frac{5}{13}\sin x\right)\right) \end{bmatrix}$$

$$= C_1 \begin{bmatrix} e^{7x}\cos x \\ e^{7x}\left(-\frac{5}{13}\cos x + \frac{1}{13}\sin x\right) \end{bmatrix} +$$

$$+ C_2 \begin{bmatrix} e^{7x}\sin x \\ -e^{7x}\left(\frac{1}{13}\cos x + \frac{5}{13}\sin x\right) \end{bmatrix}$$

$$= \begin{bmatrix} e^{7x}\cos x & e^{7x}\sin x \\ e^{7x}\left(-\frac{5}{13}\cos x + \frac{1}{13}\sin x\right) & -e^{7x}\left(\frac{1}{13}\cos x + \frac{5}{13}\sin x\right) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

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(2) $\begin{cases} y' = \dfrac{e^{1/y}}{y-2} \\ y(x_0) = y_0 \end{cases}$ $\quad D = \{(x,y) \in \mathbb{R}^2 : y \neq 0,\ y \neq 2\}$

$$f = \frac{e^{1/y}}{y-2}$$

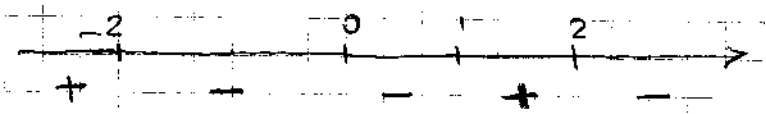
The function $f \in C^\infty(D)$ and therefore, we can conclude that $\forall (x_0, y_0) \in D$, there exists a unique solution $y = \varphi(x)$ to the Cauchy Problem and $\varphi \in C^\infty([x_0 - \delta, x_0 + \delta])$.
The global existence and uniqueness theorem cannot be applied.

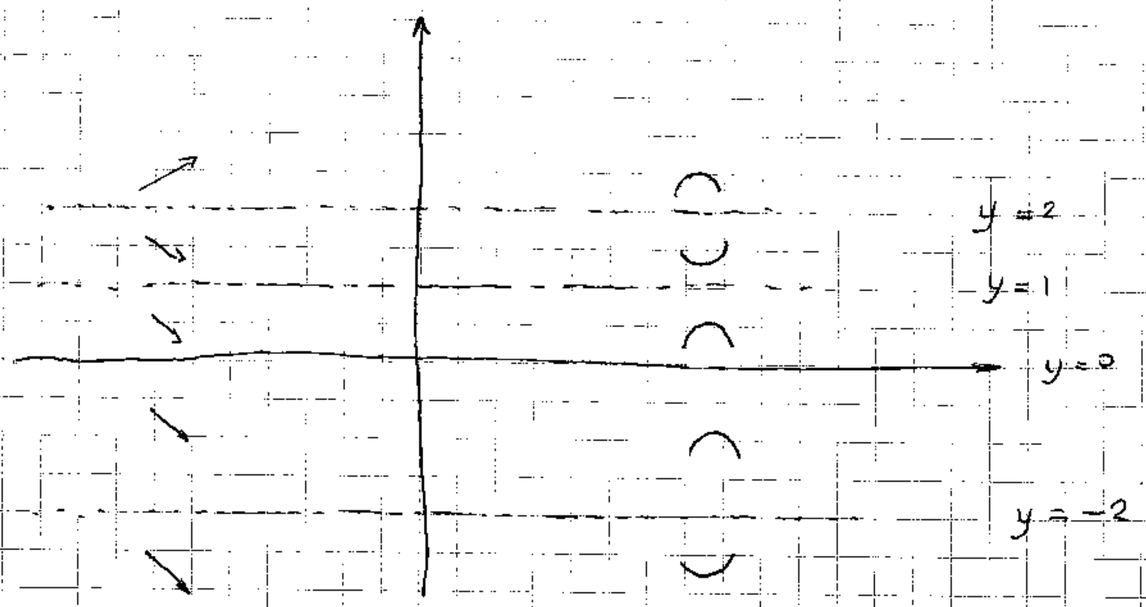
$y' \geq 0 \quad \dfrac{e^{1/y}}{y-2} \geq 0 \qquad y \geq 2$

$$y'' = \frac{-\frac{1}{y^2}(y-2)e^{1/y}y' - y'e^{1/y}}{(y-2)^2} = -y'e^{1/y}\,\frac{1 + \frac{y-2}{y^2}}{(y-2)^2}$$

$$= -\frac{e^{1/y}}{y-2}\, e^{1/y}\, \frac{y^2+y-2}{y^2(y-2)^2} = -\frac{e^{2/y}}{y^2(y-2)^2}\,\frac{(y+2)(y-1)}{y-2}$$

$y'' \geq 0$





Fix $(x_0, y_0) \in D$ with $y_0 > 2$ and separate the variables. We have

$$\int_{y_0}^{y} e^{-1/t}(t-2)\,dt = x - x_0$$

If $y \to +\infty$, the integral diverges; hence $x \longrightarrow +\infty$

If $y \to 2^+$, the integral converges; hence $x \longrightarrow L_0$ finite

Fix $(x_0, y_0) \in D$ with $0 < y < 2$ and separate the variables. We have

$$\int_{y_0}^{y} e^{-1/t}(t-2)\,dt = x - x_0$$

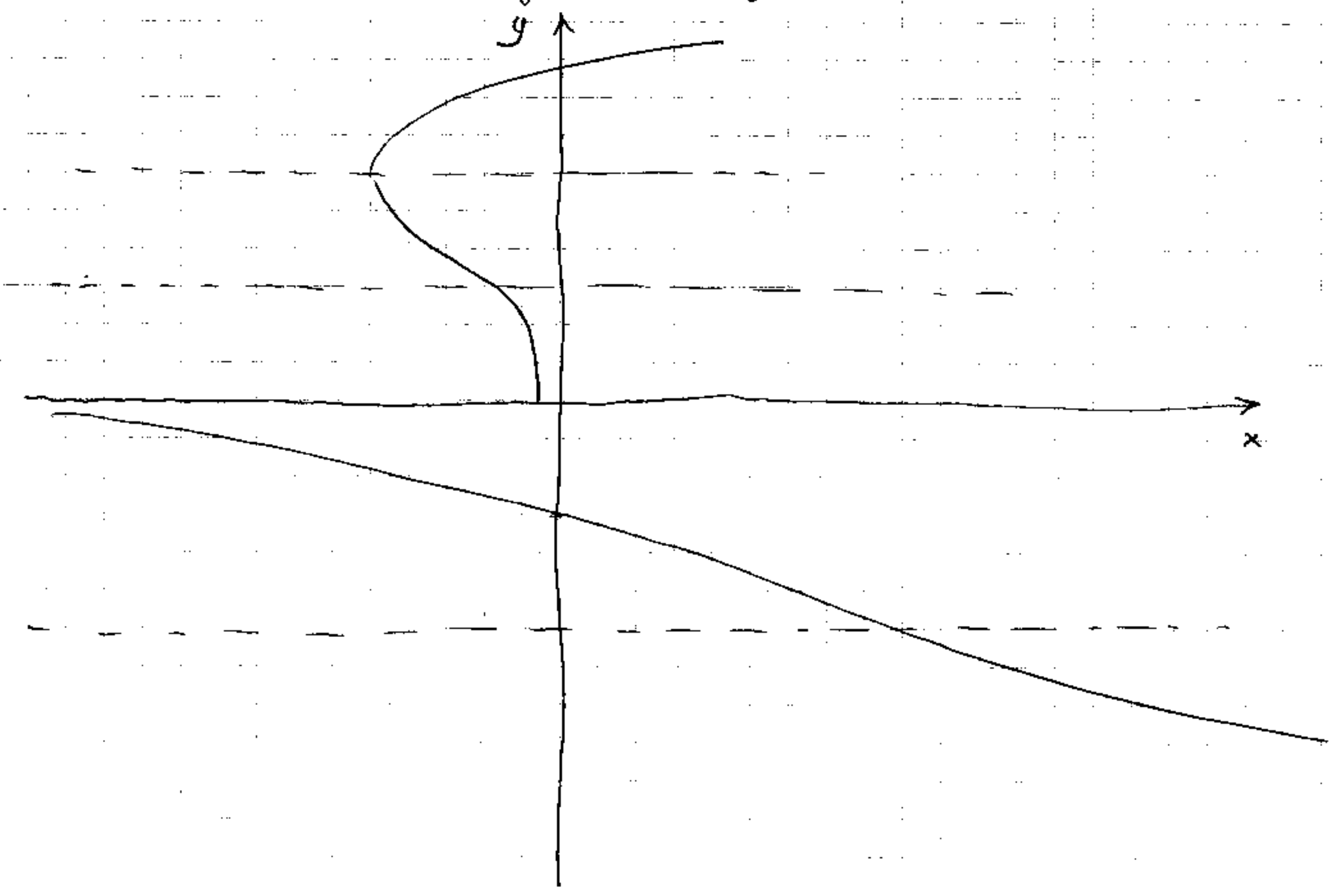
If $y \to 2^-$, the integral converges (just as before) and $x \to L_1$

If $y \to 0^+$, the integral converges and $x \to L_2$

If we take $y_0 \in (-2, 0)$, we have that

if $y \to 0^-$ the integral diverges and $x \longrightarrow -\infty$;

if $y \to -\infty$ the integral diverges and $x \longrightarrow +\infty$



(3) $u = x - \operatorname{arctg} x$

$$|u| \leq |x| + \frac{\pi}{2}$$

Since u has at most a polynomial growth, $u \in \mathcal{S}'(\mathbb{R})$.
Moreover

$$u' = 1 - \frac{1}{1+x^2}$$

$$\widehat{u'} = 2\pi\delta - \pi e^{-|\xi|}$$

$$i\xi\hat{u} = 2\pi\delta - \pi e^{-|\xi|}$$

$$\hat{u} = C\delta + 2\pi i\,\delta' + i\pi \,\mathrm{pv}\,\frac{e^{-|\xi|}}{\xi} \qquad C \text{ arbitrary constant}$$

Since u is odd and real, $\hat{u}$ is odd and purely imaginary.
Therefore $C=0$ and

$$\hat{u} = 2\pi i\,\delta' + \pi i \,\mathrm{pv}\,\frac{e^{-|\xi|}}{\xi}$$

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(4) $$\inf_{a,b\in\mathbb{R}} \int_{-1}^{1} |\cos x - (a+bx)|^2\,dx$$

We are looking for the best approximation of $f(x)=\cos x$ in terms of Legendre polynomials of degree zero and one.
We have

$$L_0 = \frac{1}{\sqrt{2}}, \qquad L_1 = \sqrt{\frac{3}{2}}\,x$$

Therefore, the best approximation is given by

$$a+bx = (\cos x, L_0)L_0 + (\cos x, L_1)L_1$$

$$(\cos x, L_0) = \int_{-1}^{1} \cos x \,\frac{1}{\sqrt{2}}\,dx = \frac{1}{\sqrt{2}} \sin x\Big|_{-1}^{1} = \sqrt{2}\sin 1$$

$$(\cos x, L_1)_{\cancel{2}} = \int_{-1}^{1} \cos x \sqrt{\frac{3}{2}}\, x \, dx = 0$$

Therefore, we conclude that

$$a + bx = \sqrt{2} \sin 1 \frac{1}{\sqrt{2}} = \sin 1$$

and

$$\inf_{(a,b)\in \mathbb{R}} \int_{-1}^{1} |\cos x - (a+bx)|^2 dx = \int_{-1}^{1} (\cos x - \sin 1)^2 dx$$

$$= \int_{-1}^{1} \left(\cos^2 x - 2\sin 1 \cos x + \sin^2 1\right) dx$$

$$= \int_{-1}^{1} \left(\frac{1+\cos 2x}{2} - 2\sin 1 \cos x + \sin^2 1\right) dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2} - 2\sin 1 \sin x + \sin^2 1 \, x\right]_{-1}^{1}$$

$$= \frac{1}{2} \cdot 2 \left(1 + \frac{\sin 2}{2} - 2\sin^2 1 + \sin^2 1\right)$$

$$= 1 + \frac{\sin 2}{2} - \sin^2 1 .$$