① $\mathbb{A} = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix}$ We calculate the eigenvalues. Therefore, we need to solve

$$\det(\lambda I - \mathbb{A}) = 0$$

that is

$$\begin{vmatrix} \lambda-2 & 2 & -1 \\ 1 & \lambda-3 & 1 \\ -2 & 4 & \lambda-3 \end{vmatrix} = 0 \qquad \begin{vmatrix} \lambda-1 & \lambda-1 & 0 \\ 1 & \lambda-3 & 1 \\ -2 & 4 & \lambda-3 \end{vmatrix} = 0$$

$$(\lambda-1)\left[(\lambda-3)^2-4\right] + (\lambda-1)(-2-\lambda+3) = 0$$

$$(\lambda-1)(\lambda^2-6\lambda+5-2-\lambda+3) = 0$$

$$(\lambda-1)(\lambda^2-7\lambda+6) = 0$$

Hence

$$\lambda_1 = \lambda_2 = 1 \qquad \lambda_3 = 6$$

The general solution is given by

$$\underline{z} = e^{x}\underline{c}_1 + xe^{x}\underline{c}_2 + e^{6x}\underline{c}_3$$

Since

$$\underline{z}' = e^{x}\underline{c}_1 + (xe^{x}+e^{x})\underline{c}_2 + 6e^{6x}\underline{c}_3$$

the three unknowns satisfy

$$e^{x}\underline{c}_1 + (xe^{x}+e^{x})\underline{c}_2 + 6e^{6x}\underline{c}_3 = e^{x}\mathbb{A}\underline{c}_1 + xe^{x}\mathbb{A}\underline{c}_2 + e^{6x}\mathbb{A}\underline{c}_3$$

which yields

$$\begin{cases} \mathbb{A}\underline{c}_1 = \underline{c}_1 + \underline{c}_2 \\ \mathbb{A}\underline{c}_2 = \underline{c}_2 \\ \mathbb{A}\underline{c}_3 = 6\,\underline{c}_3 \end{cases}$$

[redacted]

If we let $\underline{c}_3 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, we have

$$\begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 6\alpha \\ 6\beta \\ 6\gamma \end{bmatrix}$$

$$\begin{cases} 2\alpha - 2\beta + \gamma = 6\alpha \\ -\alpha + 3\beta - \gamma = 6\beta \\ 2\alpha - 4\beta + 3\gamma = 6\gamma \end{cases} \qquad \begin{cases} \cancel{4\alpha + 2\beta - \gamma = 0} \\ \alpha + 5\beta + \gamma = 0 \\ 2\alpha - 4\beta - 3\gamma = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \alpha = -1 \\ \beta = 1 \\ \gamma = -2 \end{cases}$$ (up to an arbitrary multiplicative constant) $\Longrightarrow \underline{c}_3 = \begin{bmatrix} -L \\ L \\ -2L \end{bmatrix}$

If we now consider $\underline{c}_2 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, we have

$$\begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\begin{cases} 2\alpha - 2\beta + \gamma = \alpha \\ -\alpha + 3\gamma - \gamma = \beta \\ 2\alpha - 4\beta + 3\gamma = \gamma \end{cases} \qquad \begin{cases} \alpha - 2\beta + \gamma = 0 \\ \cancel{\alpha - 2\beta + \gamma = 0} \\ \cancel{\alpha - 2\beta} + \gamma = 0 \end{cases}$$

$$\underline{c}_2 = \begin{bmatrix} M \\ N \\ 2N - M \end{bmatrix}$$

We finally come to $\underline{c}_1 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$. We have

$$\begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} M \\ N \\ 2N - M \end{bmatrix}$$

$$\begin{cases} \alpha - 2\beta + \gamma = M \\ -\alpha + 2\beta - \gamma = N \\ 2\alpha - 4\beta + 2\gamma = 2N - M \end{cases} \qquad \begin{cases} \alpha - 2\beta + \gamma = M \\ \alpha - 2\beta + \gamma = -N \\ \alpha - 2\beta + \gamma = -\tfrac{1}{2}M + N \end{cases}$$

[redacted]

$$r\mathbb{A} = r\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} = 1$$

$$r\mathbb{B} = r\begin{bmatrix} 1 & -2 & 1 & M \\ 1 & -2 & 1 & -N \\ 1 & -2 & 1 & -\frac{1}{2}M + N \end{bmatrix} = 2$$

unless $M = N = 0$. Therefore, we are forced to assume $M = N = 0$, which forces $\underline{c}_2 = 0$ and obtain

$$\underline{c}_1 = \begin{bmatrix} H \\ K \\ 2K - H \end{bmatrix}$$

The general solution of the system is

$$\underline{Y} = e^{6x}\begin{bmatrix} -L \\ L \\ -2L \end{bmatrix} + e^{x}\begin{bmatrix} H \\ k \\ 2k - H \end{bmatrix}$$

$$= \begin{bmatrix} -e^{6x} & e^{x} & 0 \\ e^{6x} & 0 & e^{x} \\ -2e^{6x} & -2e^{x} & 2e^{x} \end{bmatrix}\begin{bmatrix} L \\ H \\ K \end{bmatrix}$$

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(2) $\begin{cases} y' = e^{1/y}(y+2) \\ y(x_0) = y_0 \end{cases}$ $\quad D = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$

$f : D \to \mathbb{R}$. In D f has C^{∞} regularity. Therefore, $\forall (x_0, y_0) \in D$, there exists a unique local solution, which is of class C^{ω}.

The global existence and unique theorem cannot be applied. Moreover, it is apparent that $y = -2$ is a stationary solution, which cannot be crossed by any other particular solution

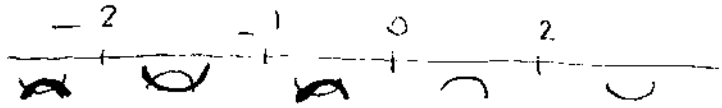
[redacted]

Moreover

$$y' \geq 0 \qquad y \geq -2$$
$$y' < 0 \qquad y < -2$$

Coming to the second order derivative, we have

$$y'' = -e^{1/y}(y+2)\frac{1}{y^2}y' + e^{1/y}y'$$
$$= e^{1/y}\left(-\frac{y+2}{y^2}e^{1/y}(y+2) + (y+2)e^{1/y}\right)$$
$$= e^{2/y}(y+2)\,\frac{y^2-y-2}{y^2}$$
$$= \frac{e^{2/y}}{y^2}(y+2)(y-2)(y+1)$$



Given an arbitrary initial condition (x_0, y_0) with $y_0 > 0$, if we separate the variables, we have

$$\int_{y_0}^{y} \frac{e^{-1/t}}{t+2}\,dt = x - x_0$$

Relying on simple properties of improper integrals, we conclude that

$$y \to 0^+ \implies x \to L$$
$$x \to +\infty \implies y \to +\infty$$

Analogously, if $-2 < y_0 < 0$

$$x \to +\infty \implies y \to 0^-$$
$$x \to -\infty \implies y \to -2^+$$

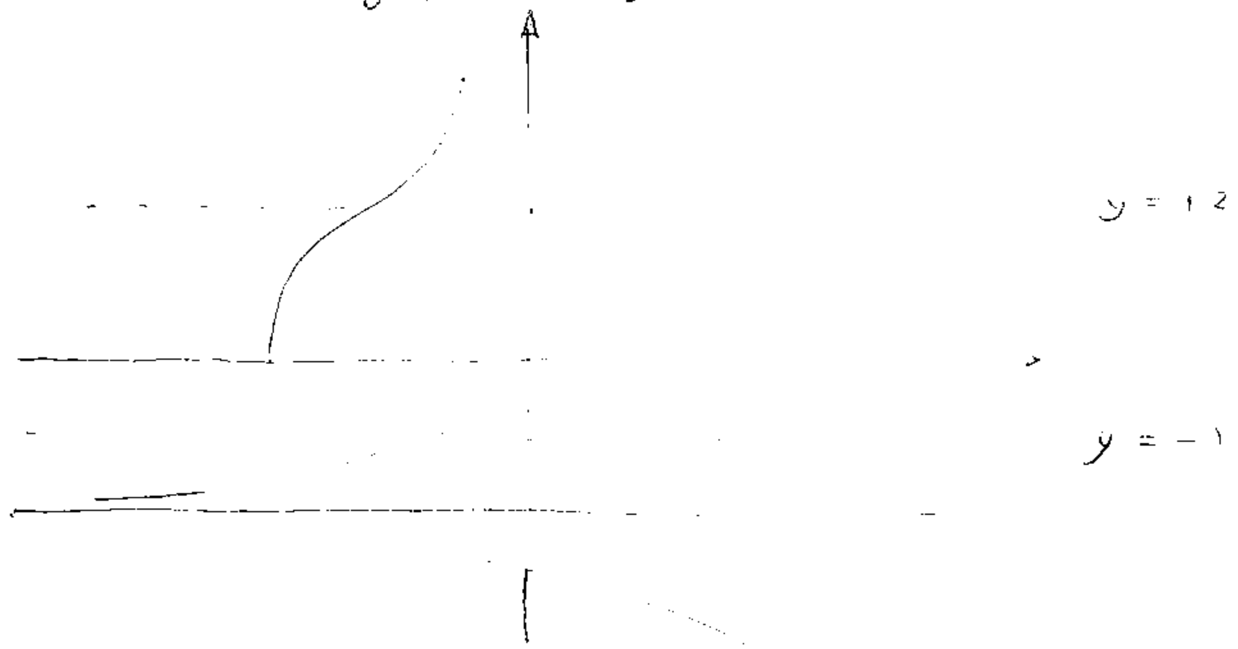
And finally, when $y_0 < -2$

$$x \to -\infty \implies y \to -2^-$$

[redacted]

$x \longrightarrow +\infty \implies y \longrightarrow -\infty$

A qualitative graph is given below



(3)

$u = \operatorname{sign} x$

$u' = (\operatorname{sign} x)' \implies u' = 2\delta$

Hence

$$\widehat{u'} = \widehat{2\delta}$$

$$i\xi \, \hat{u} = 2$$

$$\hat{u} = \frac{2}{i} \, \mathrm{pv} \, \frac{1}{\xi} + C\delta$$

Since u is an odd distribution, its Fourier transform must also be an odd distribution. Therefore,

$$\hat{u} = \frac{2}{i} \, \mathrm{pv} \, \frac{1}{\xi}$$

Finally

$$\mathcal{F}(x|x|) = \mathcal{F}(x^2 \operatorname{sign} x)$$

$$= - \frac{d}{d\xi^2} \left(\frac{2}{i} \, \mathrm{pv} \, \frac{1}{\xi} \right)$$

[redacted]

(4)

$$f(x) = \sum_{n=0}^{\infty} (f, u_n)\, u_n$$

$$= (f, u_0)\, u_0 + (f, u_1)\, u_1$$

$$(f, u_0) = \int_{\mathbb{R}} (7x - 4)\, e^{-x^2/2}\, \frac{e^{-x^2/2}}{\sqrt[4]{\pi}}\, dx$$

$$= -\frac{4}{\sqrt[4]{\pi}} \int_{\mathbb{R}} e^{-x^2}\, dx = -\frac{4}{\sqrt[4]{\pi}} \sqrt{\pi} = -4\sqrt[4]{\pi}$$

$$(f, u_1) = \int_{\mathbb{R}} (7x - 4)\, e^{-x^2/2}\, \frac{2x}{\sqrt{2\sqrt{\pi}}}\, e^{-x^2/2}\, dx$$

$$= \int_{\mathbb{R}} \frac{14x^2}{\sqrt{2\sqrt{\pi}}}\, e^{-x^2}\, dx = \frac{1}{\sqrt{2\sqrt{\pi}}} \left[14x \left(\frac{-e^{-x^2}}{2} \right) \Big|_{-\infty}^{+\infty} + 7 \int_{\mathbb{R}} e^{-x^2}\, dx \right] = \frac{1}{\sqrt{2\sqrt{\pi}}}\, 7\sqrt{\pi} = 7 \frac{\sqrt[4]{\pi}}{\sqrt{2}}$$

Finally

$$\int_{\mathbb{R}} |(7x - 4)\, e^{-x^2/2}|^2\, dx = |(f, u_0)|^2 + |(f, u_1)|^2$$

$$= 16\sqrt{\pi} + \frac{49}{2}\sqrt{\pi}.$$

[redacted]