① $z' = \mathbb{A} z \qquad \mathbb{A} = \begin{bmatrix} 4 & -3 \\ 3 & 2 \end{bmatrix}$

$$\det(\lambda I - \mathbb{A}) = 0 \qquad \begin{vmatrix} \lambda - 4 & 3 \\ -3 & \lambda - 2 \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda + 8 + 9 = 0$$

$$(\lambda - 3)^2 + 8 = 0 \qquad \lambda = 3 \pm 2\sqrt{2}\, i$$

The eigenvalues are complex conjugate.

Let us determine the corresponding eigenvectors. We have

$$\begin{bmatrix} -1 + 2\sqrt{2}\, i & 3 \\ -3 & 1 + 2\sqrt{2}\, i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ which yields

$$h_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1 + 2\sqrt{2}\, i}{3} \\ 1 \end{bmatrix}$$

We have two linearly independent solutions, if we take

$$\underline{z}_1 = \mathrm{Re}\left(\begin{bmatrix} \frac{1 + 2\sqrt{2}\, i}{3} \\ 1 \end{bmatrix} e^{(3 + 2\sqrt{2}\, i)x} \right)$$

$$\underline{z}_2 = \mathrm{Im}\left(\begin{bmatrix} \frac{1 + 2\sqrt{2}\, i}{3} \\ 1 \end{bmatrix} e^{(3 + 2\sqrt{2}\, i)x} \right)$$

We obtain

$$z_1 = \mathrm{Re} \begin{bmatrix} \frac{1 + 2\sqrt{2}\, i}{3} \; e^{3x}(\cos 2\sqrt{2}\, x + i \sin 2\sqrt{2}\, x) \\ e^{3x}(\cos 2\sqrt{2}\, x + i \sin 2\sqrt{2}\, x) \end{bmatrix}$$

$$= \begin{bmatrix} e^{3x}\left(\frac{1}{3}\cos 2\sqrt{2}x - \frac{2\sqrt{2}}{3}\sin 2\sqrt{2}x\right) \\ e^{3x}\cos 2\sqrt{2}x \end{bmatrix}$$

$$z_2 = \begin{bmatrix} e^{3x}\left(\frac{2\sqrt{2}}{3}\cos 2\sqrt{2}x + \frac{1}{3}\sin 2\sqrt{2}x\right) \\ e^{3x}\sin 2\sqrt{2}x \end{bmatrix}$$

Therefore, the general solution of the system is

$$\underset{\sim}{z} = \begin{bmatrix} e^{3x}\left(\frac{1}{3}\cos 2\sqrt{2}x - \frac{2\sqrt{2}}{3}\sin 2\sqrt{2}x\right) & e^{3x}\left(\frac{2\sqrt{2}}{3}\cos 2\sqrt{2}x + \frac{1}{3}\sin 2\sqrt{2}x\right) \\ e^{3x}\cos 2\sqrt{2}x & e^{3x}\sin 2\sqrt{2}x \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

---

(2) $f_n = x\, e^{-n|x|}$

The functions are all odd, so it is enough to consider $[0, 10]$

Notice that

$$f_n' = e^{-nx} - nx\, e^{-nx} = e^{-nx}(1 - nx)$$

and therefore,

$$\max_{[0,10]} f_n(x) = f_n\left(\frac{1}{n}\right) = \frac{1}{n} e^{-1}$$

Since $f_n\left(\frac{1}{n}\right) \longrightarrow 0$ as $n \to \infty$, it is natural to assume that the limit should be $f = 0$.

Let us prove both in $C^0([-10, 10])$ and in $L^1(-10,10)$, endowed with their natural norms.

$$\|f_n - f\|_{C^0} = \sup_{x\in[-10,10]} |f_n(x) - f| =$$

$$= \sup_{x \in [-10,10]} |x e^{-n|x|}| = \frac{1}{n} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\| f_n - f \|_{L^1} = \int_{-10}^{10} |f_n - f| \, dx = \int_{-10}^{10} |x e^{-n|x|}| \, dx$$

$$= 2 \int_0^{10} x e^{-n|x|} \, dx = 2 \cdot \frac{1}{n} \cdot 10 \longrightarrow 0$$

as $n \to \infty$.

---

(3) $$y' = \frac{3y - 2x}{2y - 3x}$$

If we consider the local existence and uniqueness theorem, the open set which we need to consider is

$$D = \{ (x,y) \in \mathbb{R}^2 : \quad y \neq \tfrac{3}{2} x \}$$

$\forall (x_0, y_0) \in D$ there exists a unique local solution, which belongs to $C^\infty(I_{x_0})$.

The global existence and uniqueness theorem cannot be applied.

Moreover, if we take $x \neq 0$, we notice that we can rewrite the equation as

$$y' = \frac{3 \frac{y}{x} - 2}{2 \frac{y}{x} - 3}$$

Therefore, we have a so-called homogeneous equation. If we set $y/x = t$, $y = xt$, $y' = xt' + t$, we obtain

$$xt' + t = \frac{3t - 2}{2t - 3} \qquad xt' = \frac{3t - 2 - 2t^2 + 3t}{2t - 3}$$

$$xt' = -\frac{2(t^2-3t+1)}{2t-3}$$ Since $t^2-3t+1=0$ has the two solutions $t = \frac{3\pm\sqrt{5}}{2}$, we conclude that the two lines

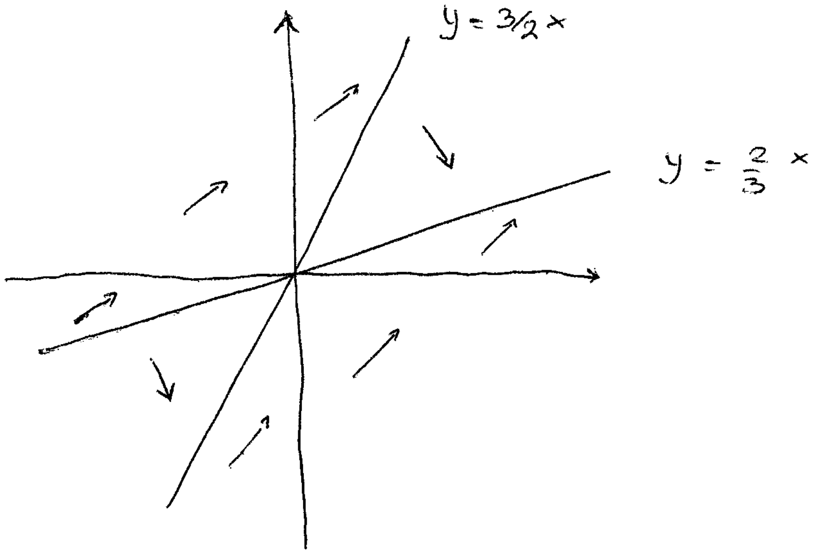
$$y = \frac{3+\sqrt{5}}{2}x, \qquad x \neq 0$$

$$y = \frac{3-\sqrt{5}}{2}x, \qquad x = 0$$

are particular solutions. Therefore, by the local existence and uniqueness theorem, any other particular solution cannot cross these two lines.

Coming to the first order derivative

$$y' \geq 0 \qquad \frac{3y-2x}{2y-3x} \geq 0$$



We can now study the second order derivative

$$y'' = \frac{(3y'-2)(2y-3x) - (2y'-3)(3y-2x)}{(2y-3x)^2}$$

$$= \frac{\cancel{6yy'} - 4y - 9xy' \cancel{+6x} - \cancel{6yy'} + 9y + 4xy' \cancel{-6x}}{(2y-3x)^2}$$

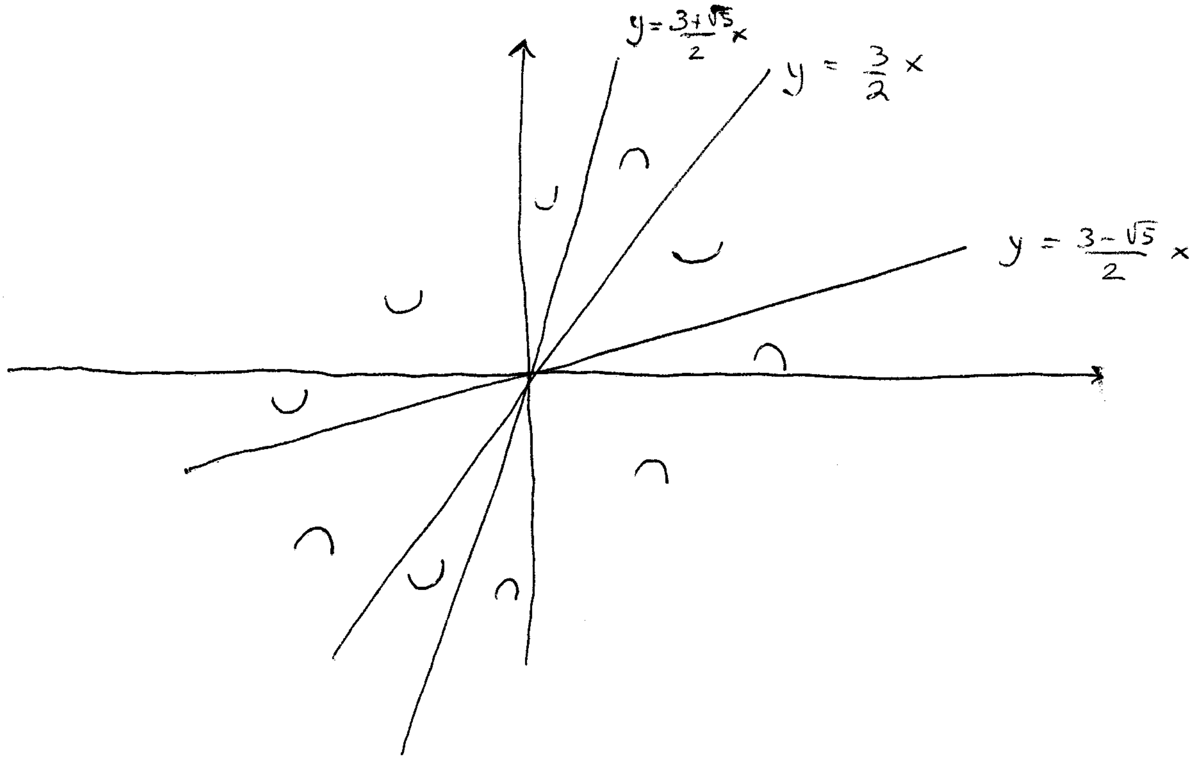
$$= \frac{5y - 5xy'}{(2y-3x)^2} = \frac{5}{(2y-3x)^2}\left(y - x \cdot \frac{3y-2x}{2y-3x}\right)$$

$$= \frac{5}{(2y-3x)^2} \; \frac{2y^2 - 3xy - 3xy + 2x^2}{2y-3x}$$

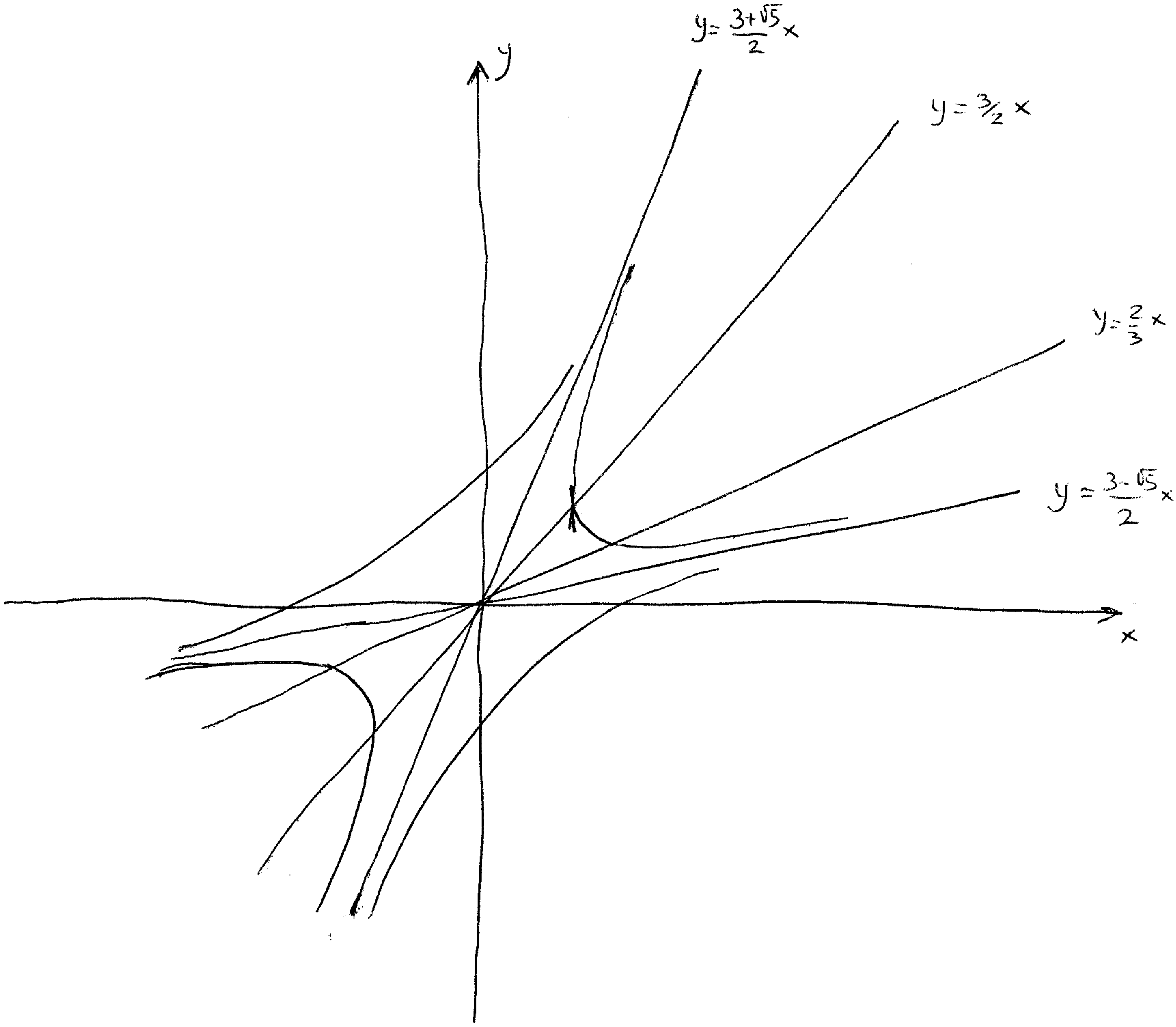
$$= \frac{10}{(2y-3x)^2} \; \frac{y^2 - 3xy + x^2}{2y-3x}$$

$$= \frac{10}{(2y-3x)^2} \; \frac{\left(y - \frac{3+\sqrt{5}}{2}x\right)\left(y - \frac{3-\sqrt{5}}{2}x\right)}{2y-3x}$$

$$y'' \geq 0 \qquad \frac{\left(y - \frac{3+\sqrt{5}}{2}x\right)\left(y - \frac{3-\sqrt{5}}{2}x\right)}{2y-3x} \geq 0$$



If we now collect all the information we have obtained so far, we can conclude the qualitative graph is like the one given in the next figure



---

(4) $u = (2x - 3)\ \mathrm{pv}\ \dfrac{1}{3x-2}$

Since $v = \mathrm{pv}\ \dfrac{1}{x}$ is a tempered distribution and

$$\mathrm{pv}\ \frac{1}{3x-2} = \mathrm{pv}\ \frac{1}{3(x-\frac{2}{3})}$$

is a rescaling and a translation of v, we conclude that u is a tempered distribution.

Moreover

$$u = 2x\ \mathrm{pv}\ \frac{1}{3x-2} - 3\ \mathrm{pv}\ \frac{1}{3x-2}$$

$$\hat{u} = 2i \frac{d}{d\xi} \widehat{\mathrm{pv}\frac{1}{3x-2}} - 3 \widehat{\mathrm{pv}\frac{1}{3x-2}}$$

Therefore, it is enough to compute the Fourier transform of

$$\mathrm{pv}\frac{1}{3x-2} = \frac{1}{3}\,\mathrm{pv}\frac{1}{x-\frac{2}{3}}$$

Since $\widehat{\mathrm{pv}\frac{1}{x}} = \frac{\pi}{i}\,\mathrm{sign}\,\xi$, by the elementary properties of the Fourier transform, we can conclude that

$$\widehat{\mathrm{pv}\frac{1}{x-\frac{2}{3}}} = \frac{\pi}{i}\, e^{-\frac{2}{3}i\xi}\,\mathrm{sign}\,\xi$$

and

$$\hat{u} = 2i \frac{d}{d\xi}\left[\frac{1}{3}\frac{\pi}{i} e^{-\frac{2}{3}i\xi}\,\mathrm{sign}\,\xi\right] +$$

$$-3\left[\frac{1}{3}\frac{\pi}{i} e^{-\frac{2}{3}i\xi}\,\mathrm{sign}\,\xi\right].$$

$$= \frac{2}{3}\pi\left(-\frac{2}{3}i\, e^{-\frac{2}{3}i\xi}\,\mathrm{sign}\,\xi + 2\delta\right) + \pi i\, e^{-\frac{2}{3}i\xi}\,\mathrm{sign}\,\xi$$

$$= \frac{5}{9}\pi i\, e^{-\frac{2}{3}i\xi}\,\mathrm{sign}\,\xi + 2\delta.$$