(1) Let us determine the eigenvalues of $\mathbb{A}$. We have

$$\det(\lambda \underline{I} - \mathbb{A}) = 0 \qquad \begin{vmatrix} \lambda-2 & 1 & -2 \\ -1 & \lambda-2 & 4 \\ -3 & 1 & \lambda-1 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda+1 & 0 & -(\lambda+1) \\ -1 & \lambda-2 & 4 \\ -3 & 1 & \lambda-1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 0 & -(\lambda+1) \\ +3 & \lambda-2 & 4 \\ \lambda-4 & 1 & \lambda-1 \end{vmatrix} = 0$$

$$-(\lambda+1)\left[3-(\lambda-2)(\lambda-4)\right] = 0$$

$$(\lambda+1)(\lambda^2-6\lambda+8-3) = 0 \qquad (\lambda+1)(\lambda^2-6\lambda+5) = 0$$

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = 5$$

Let us now evaluate the corresponding eigenvectors. We have

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & -1 & 4 \\ -3 & 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} -\alpha - \beta + 4\gamma = 0 \\ -3\alpha + \beta = 0 \end{cases}$$

$$\underline{h}_1 = \begin{bmatrix} L \\ 3L \\ L \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & -2 \\ -1 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} -6\alpha + 2\beta - 4\gamma = 0 \\ -\alpha - 3\beta + 4\gamma = 0 \end{cases}$$

$$\underline{h}_2 = \begin{bmatrix} M \\ -7M \\ -5M \end{bmatrix}$$

[redacted]

$$\begin{bmatrix} 3 & 1 & -2 \\ -1 & 3 & 4 \\ -3 & 1 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} 2\beta + 2\gamma = 0 \\ -\alpha + 3\beta + 4\gamma = 0 \\ -3\alpha + \beta + 4\gamma = 0 \end{cases}$$

$$\underline{h}_3 = \begin{bmatrix} -N \\ N \\ -N \end{bmatrix}$$

Therefore

$$Z = \begin{bmatrix} e^{x} & e^{-x} & -\cancel{6}e^{5x} \\ 3e^{x} & -7e^{-x} & e^{5x} \\ e^{x} & -5e^{-x} & -e^{5x} \end{bmatrix} \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

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(2) $\begin{cases} y' = \dfrac{xy}{1-xy} \\ y(x_0) = y_0 \end{cases}$ $\qquad D = \{ (x,y) \in \mathbb{R}^2 : \; xy \neq 1 \}$

D is an open set and it is apparent that $f \in C^\infty(D)$

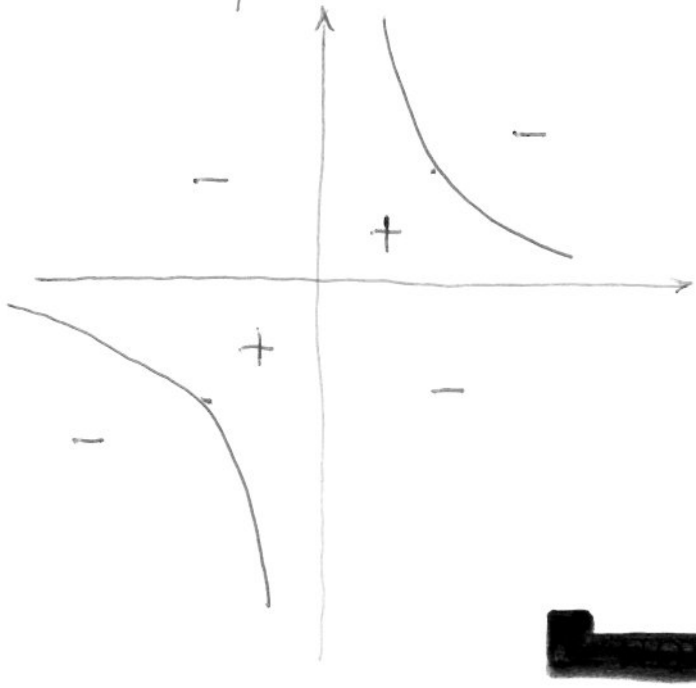
Therefore, we can apply both the local existence and uniqueness Theorem and the regularity theorem to conclude that

$$\forall (x_0, y_0) \in D \qquad \exists! \; y = y(x) \qquad y \in C^\infty(I_{x_0})$$

solution to the Cauchy problem stated above.

The global existence and uniqueness theorem cannot be applied.

The figure on the right describes where the function is increasing (+) and where it is decreasing (−).



[redacted]

Notice that if (x_1, y_1) is such that $x_1 y_1 = 1$, then

$$\lim_{x \to x_1} y'(x) = \infty.$$

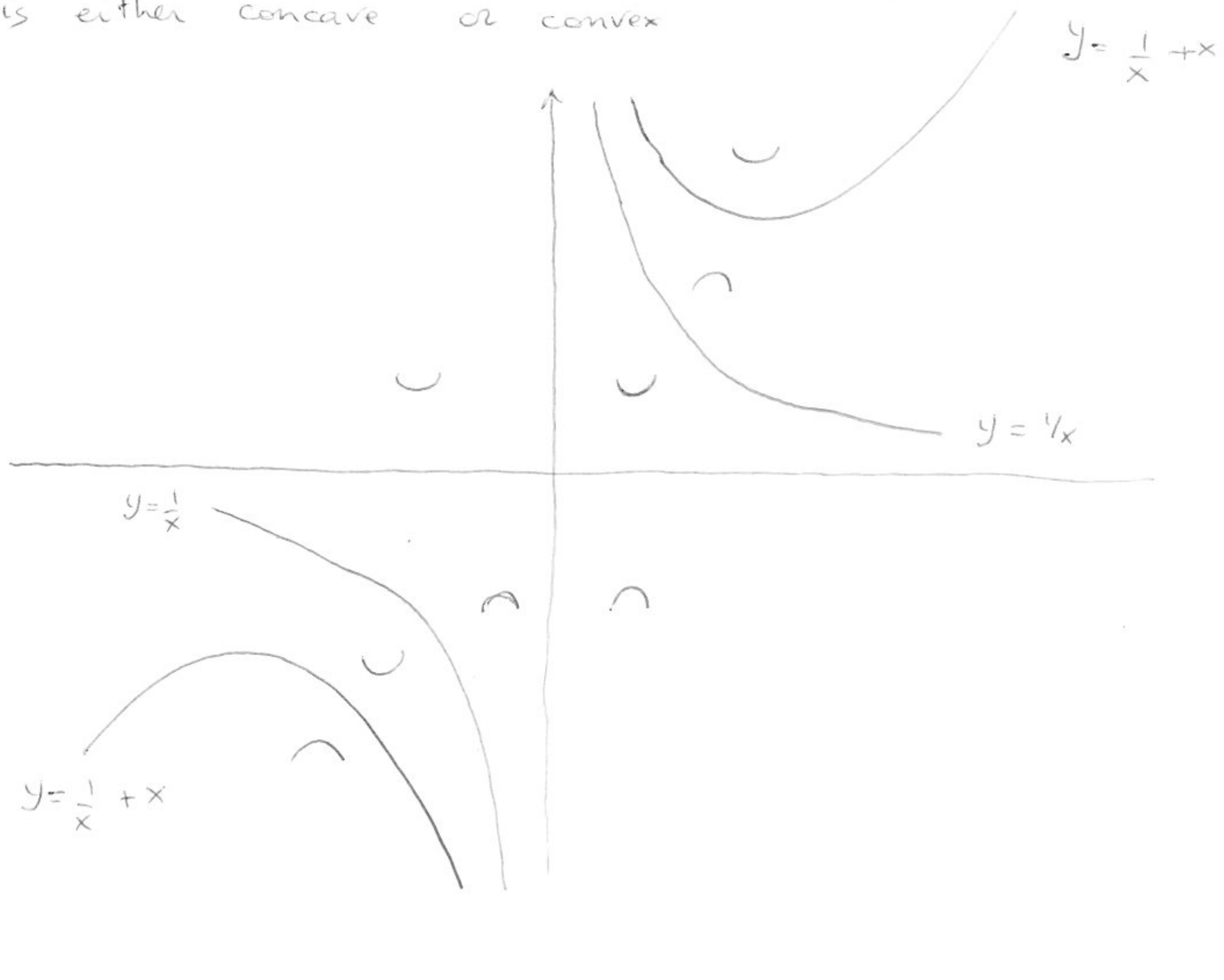
Coming to the second order derivatives, we have

$$y'' = \frac{(y+xy')(1-xy) + xy(y+xy')}{(1-xy)^2} = \frac{y+xy'}{(1-xy)^2}$$

$$= \frac{y + x\,\frac{xy}{1-xy}}{(1-xy)^2} = \frac{y - xy^2 + x^2y}{(1-xy)^3}$$

$$= \frac{y\left[1 + x^2 - xy\right]}{(1-xy)^2(1-xy)}$$

Therefore, the figure below gives the regions where y is either concave or convex



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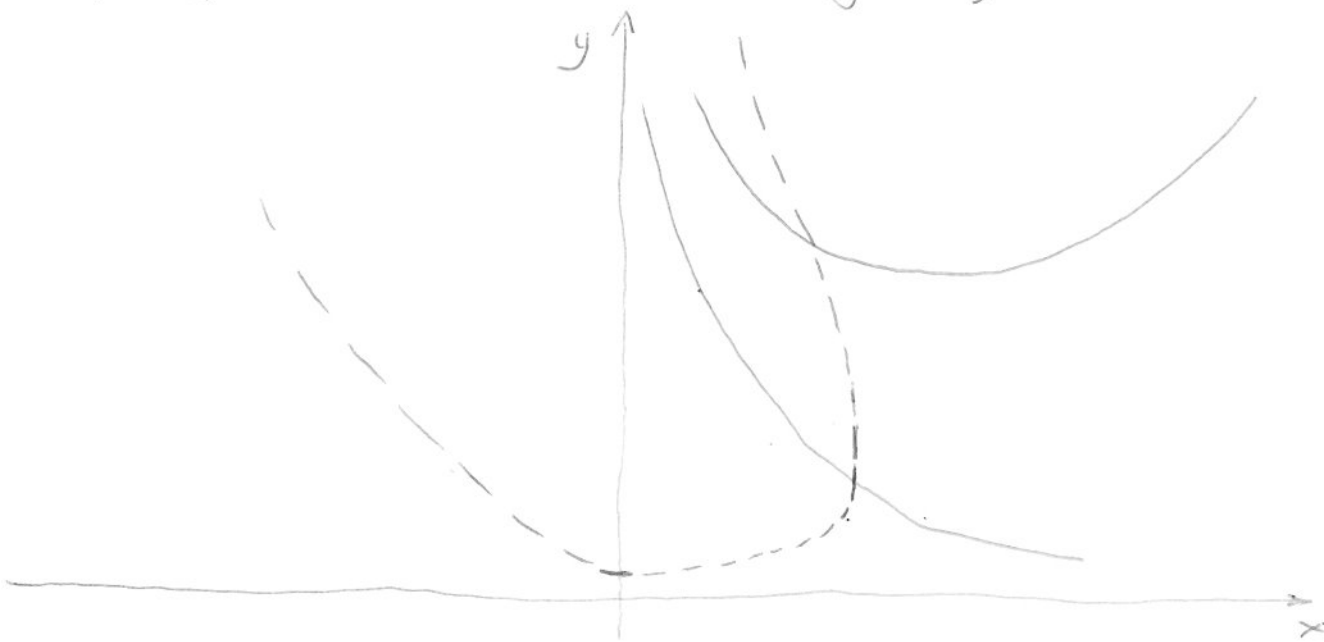
Taking into account all the previous information, a qualitative graph has the following behavior (for the sake of simplicity we restrict ourselves to $y > 0$)



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(3) $u \in \mathcal{S}'(\mathbb{R})$ since it is a regular function whose growth at infinity is controlled by a polynomial of arbitrary degree.

We now have

$$u' = \frac{2x}{1+x^2}$$

If we transform both sides and take into account the fundamental properties of the Fourier transform, we have

$$i\xi\hat{u} = 2i\frac{d}{d\xi}\pi e^{-|\xi|}$$

$$\xi\hat{u} = -2\pi e^{-|\xi|}\frac{|\xi|}{\xi}$$

$$\xi\hat{u} = -2\pi e^{-|\xi|}\frac{\xi}{|\xi|}$$

If we proceed formally, simplifying both sides, we can conclude that

$$\hat{u} = -2\pi \,\mathrm{pf}\, \frac{e^{-|\xi|}}{|\xi|}$$

[redacted]

where

$$\mathrm{pf}\ \frac{e^{-|\xi|}}{|\xi|} = \lim_{\epsilon \to 0} \frac{e^{-|\xi|}}{|\xi|}\, \chi_{(-\infty,-\epsilon)\cup(\epsilon,+\infty)} + 2 \ln \epsilon\, \delta$$

where the limit is taken in the sense of $\mathscr{S}'$. Let us show that such a limit exists. By definition, $\forall\, \varphi \in \mathscr{S}$ we have

$$< \mathrm{pf}\ \frac{e^{-|\xi|}}{|\xi|}, \varphi > = \lim_{\epsilon \to 0} -\int_{-\infty}^{-\epsilon} \frac{e^{\xi}}{\xi} \varphi(\xi) + \int_{\epsilon}^{+\infty} \frac{e^{-\xi}}{\xi} \varphi(\xi) + 2 \ln \epsilon\, \varphi(0)$$

We consider

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{+\infty} \frac{e^{-\xi}}{\xi} \varphi(\xi) + \ln \epsilon\, \varphi(0)$$

since the other term is analogous. We can rewrite

$$\varphi(\xi) = \varphi(0) + \xi\, \psi(\xi)$$

where $\psi \in \mathscr{S}$. Therefore

$$\int_{\epsilon}^{+\infty} \frac{e^{-\xi}}{\xi} \varphi(\xi) + \ln \epsilon\, \varphi(0) = \int_{\epsilon}^{\infty} \frac{e^{-\xi}}{\xi} \left(\varphi(0) + \xi \psi(\xi) \right) +$$

$$+ \ln \epsilon\, \varphi(0) = \int_{\epsilon}^{\infty} \frac{e^{-\xi}}{\xi} \varphi(0) + \ln \epsilon\, \varphi(0) + \int_{\epsilon}^{\infty} e^{-\xi} \psi(\xi)$$

When $\epsilon \to 0$, the last term gives no problem. Let us concentrate on the first two ones. If we integrate by parts, we have

$$\int_{\epsilon}^{\infty} \frac{e^{-\xi}}{\xi} \varphi(0) + \ln \epsilon\, \varphi(0) = e^{-\xi} \ln \xi \Big|_{\epsilon}^{\infty} \varphi(0) + \int_{\epsilon}^{+\infty} e^{-\xi} \ln \xi\, \varphi(0) + \ln \epsilon\, \varphi(0) = - e^{-\epsilon} \ln \epsilon\, \varphi(0) + \ln \epsilon\, \varphi(0) + \varphi(0) \int_{\epsilon}^{\infty} e^{-\xi} \ln \xi$$

$$= \epsilon\, \varphi(0) \ln \epsilon\, \frac{1 - e^{-\epsilon}}{\epsilon} + \varphi(0) \int_{\epsilon}^{\infty} e^{-\xi} \ln \xi$$

[redacted]

When $\epsilon \to 0$ the first term vanishes, whereas the second one has finite limit.

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(4) From v_0 we build u_0.
We have

$$u_0 = \frac{v_0}{\|v_0\|_{H_1}} = \frac{1}{\left[\int_{-1}^{1} |v_0|^2 dx + \int_{-1}^{1} |v_0'|^2 dx \right]^{1/2}} = \frac{1}{\sqrt{2}}$$

Let us now build u_1

$$u_1 = \frac{v_1 - (v_1, u_0) u_0}{\|v_1 - (v_1, u_0) u_0\|_{H^1}}$$

$$(v_1, u_0) = \int_{-1}^{1} v_1 u_0 \, dx + \int_{-1}^{1} v_1' u_0' \, dx = \int_{-1}^{1} \frac{1}{\sqrt{2}} x \, dx$$

$$= \frac{1}{\sqrt{2}} \left. \frac{x^2}{2} \right|_{-1}^{1} = \frac{1}{\sqrt{8}} \cdot 0 = 0$$

$$v_1 - (v_1, u_0) u_0 = x$$

$$\|v_1 - (v_1, u_0) u_0\|_{H^1}^2 = \int_{-1}^{1} x^2 dx + \int_{-1}^{1} 1^2 dx$$

$$= 2 \frac{1}{3} + 2 = \frac{8}{3}$$

Quindi

$$u_1 = \frac{x}{\sqrt{\frac{8}{3}}} = \sqrt{\frac{3}{8}} \, x$$

Infine

$$u_2 = \frac{v_2 - (v_2, u_0) u_0 - (v_2, u_1) u_1}{\| v_2 - (v_2, u_0) u_0 - (v_2, u_1) u_1 \|}$$

We have

$$(v_2, u_0) = \int_{-1}^{1} x^2 \frac{1}{\sqrt{2}} + \int_{-1}^{1} 2x \cdot 0 = \frac{2}{\sqrt{2}} \frac{1}{3} = \frac{\sqrt{2}}{3}$$

[redacted]

$$(v_2, u_1) = \int_{-1}^{1} x^2 \cdot \sqrt{\frac{3}{8}}\, x + \int_{-1}^{1} 2x \cdot \sqrt{\frac{3}{8}} = 0.$$

$$v_2 - (v_2, u_0)u_0 - (v_2, u_1)u_1 = x^2 - \frac{\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = \frac{3x^2 - 1}{3}$$

$$\| v_2 - (v_2, u_0)u_0 - (v_2, u_1)u_1 \|^2 = \int_{-1}^{1} \left(\frac{3x^2 - 1}{3} \right)^2 dx + \int_{-1}^{1} (2x)^2\, dx$$

$$= \frac{1}{9} \cdot 2 \cdot \int_0^1 (9x^4 - 6x^2 + 1)\, dx + 2 \cdot 4 \left[\frac{x^3}{3} \right]_0^1 =$$

$$= \frac{2}{9} \left(9 \cdot \frac{1}{5} - 2 + 1 \right) + \frac{8}{3} = \frac{2}{9} \left(\frac{18 - 20 + 10}{10} \right) + \frac{8}{3}$$

$$= \frac{2 \cdot 8}{90} + \frac{240}{90} = \frac{256}{90} = \frac{128}{45}$$

Finally

$$u_2 = \frac{3x^2 - 1}{3} \Big/ \sqrt{\frac{128}{45}} = \frac{3x^2 - 1}{3} \, \frac{3\sqrt{5}}{\sqrt{128}}$$

$$= \sqrt{\frac{5}{128}} \, (3x^2 - 1).$$

[redacted]