A second-order flux approximation for the mimetic finite difference method

Gianmarco Manzini

Istituto di Matematica Applicata e Tecnologie Informatiche (IMATI) C.N.R., Pavia, Italy

in collaboration with L. Beirao da Veiga, K. Lipnikov

9 June 2009, MAFELAP 2009
Outline:

- the model problem;
- the "low-order" MFD formulation;
- the higher-order flux approximation for:
  - piecewise-constant diffusion tensors;
  - non-constant diffusion tensors.
- final remarks.
Linear diffusion in mixed form

Linear diffusion in mixed form:
Let $\Omega$ be the domain.

- **differential form:** we solve for $p$ and $\vec{F}$:
  
  \[ \vec{F} = -K\nabla p \quad \text{in } \Omega \]
  \[ \text{div } \vec{F} = f \quad \text{in } \Omega \]

  with homogeneous Dirichletlet conditions on the boundary $\partial \Omega$.

- **variational form:** find $p \in L^2(\Omega)$ and $\vec{F} \in H(\text{div}, \Omega)$ such that:

  \[ (K^{-1}\vec{F}, \vec{G}) - (p, \text{div } \vec{G}) = 0 \quad \forall \vec{G} \in H(\text{div}, \Omega) \]
  \[ (\text{div } \vec{F}, q) = (f, q) \quad \forall q \in L^2(\Omega). \]
Mimetic degrees-of-freedom

- \{\mathcal{T}_h\}: the meshes, \( h \) is the mesh size
- \( p \approx p_h \in Q_h \), a \textbf{piecewise constant function} on \( \mathcal{T}_h \);
- \( \int_e \mathbf{n}_E^e \cdot \mathbf{F} \approx |e| F_E^e \), a \textbf{constant representation} on \( e \in \partial E \)
  (and flux conservation \( F_E^e + F_{E'}^e = 0 \)).

\( \bullet (i) \) Degrees of freedom:
  - scalar fields \( \rightarrow \) discrete scalars, \( Q_h \);
  - vector fields \( \rightarrow \) discrete vectors, \( X_h \).
Mimetic degrees-of-freedom

- $\{T_h\}$: the meshes, $h$ is the mesh size
- $p \approx p_h \in Q_h$, a **piecewise constant function** on $T_h$;
- $\int_e \mathbf{n}_E^e \cdot \mathbf{F} \approx |e| F_E^e$, a **constant representation** on $e \in \partial E$ (and flux conservation $F_E^e + F_{E'}^e = 0$).

• (i) **Degrees of freedom:**
  - scalar fields $\rightarrow$ **discrete scalars**, $Q_h$;
  - vector fields $\rightarrow$ **discrete vectors**, $X_h$. 

![Diagram](image-url)
Discrete divergence and scalar products

• (ii) Discrete divergence operator:

\[
\forall \mathbf{G} \in X_h : (\text{DIV}_h \mathbf{G})_E = \frac{1}{|E|} \sum_{e \in \partial E} |e| G^e_E,
\]

\(G^e_E\) are the degrees-of-freedom of the discrete vector \(\mathbf{G}\);

• (iii) Scalar products in \(Q_h\) and \(X_h\)

\[
[q, p]_{Q_h} = \sum_E |E| q_E p_E, \quad \text{in } Q_h
\]

\[
[F, G]_{X_h} = \sum_E [F, G]_E, \quad \text{in } X_h
\]
Discrete divergence and scalar products

(ii) Discrete divergence operator:

\[
\forall \mathbf{G} \in X_h : (DIV_h \mathbf{G})_E = \frac{1}{|E|} \sum_{e \in \partial E} |e| G^e_E,
\]

\(G^e_E\) are the degrees-of-freedom of the discrete vector \(\mathbf{G}\);

(iii) Scalar products in \(Q_h\) and \(X_h\)

\[
[q, p]_{Q_h} = \sum_E |E| q_E p_E, \quad \text{in } Q_h
\]

\[
[F, G]_{X_h} = \sum_E [F, G]_E \quad \text{in } X_h
\]
MFD formulation

The elemental scalar product $[\cdot, \cdot]_E$ satisfies

- a local *stability condition*;
- the **local** $P_0$ consistency relation:

$$
[(K_E \nabla q^1)^I, G]_E = - \int_E q^1 (\text{DIV}_h G)_E \, dV + \sum_{e \in \partial E} \int_e q^1 G^e_E \, dS;
$$

$q^1$ are **linear polynomials** on $E$, (constant $K_E \approx K|_E$)

(iv) MFD method: find $p_h \in Q_h$ and $F_h \in X_h$ such that:

$$
[F_h, G]_{X_h} - [p_h, \text{DIV}_h G]_{Q_h} = 0 \quad \forall G \in X_h
$$

$$
[\text{DIV}_h F_h, q]_{Q_h} = [f^I, q]_{Q_h} \quad \forall q \in Q_h.
$$
MFD formulation

The elemental scalar product $[\cdot, \cdot]_E$ satisfies

- a local *stability condition*;
- the **local** $P_0$ consistency relation:

$$
[(K_E \nabla q^1)^I, G]_E = - \int_E q^1(\nabla_h G)_E \, dV + \sum_{e \in \partial E} \int_e q^1 G^e_E \, dS;
$$

$q^1$ are **linear polynomials** on $E$, (constant $K_E \approx K|_E$)

*(iv) MFD method:* find $p_h \in Q_h$ and $F_h \in X_h$ such that:

$$
[F_h, G]_{X_h} - [p_h, \nabla_h G]_{Q_h} = 0 \quad \forall G \in X_h
$$

$$
[\nabla_h F_h, q]_{Q_h} = [f^I, q]_{Q_h} \quad \forall q \in Q_h.
$$
**A priori** error estimates
[Brezzi-Lipnikov-Shashkov, SINUM 2005]

About the approximation of the scalar variable:

1. **first-order convergence:** if \( p \in H^2(\Omega) \) then

   \[
   \| p - p_h \|_{Q_h} \leq C h \| p \|_{H^2(\Omega)}.
   \]

2. **superconvergence:** if \( p \in H^2(\Omega) \), \( f \in H^1(\Omega) \), and \( \Omega \) is convex:

   \[
   \| p - p_h \|_{Q_h} \leq C h^2 \left( \| p \|_{H^2(\Omega)} + \| f \|_{H^1(\Omega)} \right).
   \]

**NOTE:** superconvergence of \( p \) is experimentally seen when using non-convex domains and general polygonal and polyhedral meshes.
A priori error estimates
[Brezzi-Lipnikov-Shashkov, SINUM 2005]

About the approximation of the scalar variable:

1. **first-order convergence**: if $p \in H^2(\Omega)$ then
   \[
   \| p^I - p_h \|_{Q_h} \leq C h \| p \|_{H^2(\Omega)}.
   \]

2. **superconvergence**: if $p \in H^2(\Omega)$, $f \in H^1(\Omega)$, and $\Omega$ is convex:
   \[
   \| p^I - p_h \|_{Q_h} \leq C h^2 \left( \| p \|_{H^2(\Omega)} + \| f \|_{H^1(\Omega)} \right).
   \]

**NOTE**: superconvergence of $p$ is experimentally seen when using non-convex domains and general polygonal and polyhedral meshes.
About the approximation of the flux variable:

3. **FIRST-ORDER CONVERGENCE:**
   
   if \( p \in H^2(\Omega) \) then
   
   \[ \| \vec{F}^I - F_h \|_{X_h} \leq Ch \| p \|_{H^2(\Omega)} \]

   where \( \| \cdot \|_{X_h}^2 = [\cdot, \cdot]_{X_h} \).

   - \( O(h^2) \) superconvergence on meshes of *triangles and quadrilaterals with regular refinements* (nested meshes);
   
   - but, \( O(h) \) convergence is confirmed experimentally on general polygonal and polyhedral meshes.

Can we do better than that?
Linear representation of the numerical flux field

\( X_h \), degrees of freedom for flux fields

We consider the **linear** representation of the numerical flux:

\[
G = \{ G^e \} \in X_h \quad \Rightarrow \quad G^e(\xi) = G_0^e + \frac{\xi - \xi_e}{h_e} \cdot \vec{G}_1^e;
\]

- \( G_0^e \) is the low order flux unknown;
- \( \vec{G}_1^e \) is the slope on the edge (or face) \( e \);

How can we build a scalar product for this \( X_h \)?
An MFD scheme with second-order fluxes
Gyrya-Lipnikov (JCP, 2008), Beirao-M. (SISC, 2008)

IDEA: use the same construction as for the low-order scheme with:

- **the discrete divergence is:**

\[
(DIV_h G)_E |E| = \sum_{e \in \partial E} \int_e G^e_\xi \, dS = \sum_{e \in \partial E} \int_e \left( G^e_0 + \frac{\nabla G^e_1 \cdot (\xi - \xi_e)}{h_e} \right) dS
\]

\[
= \sum_{e \in \partial E} |e| G^e_0 \quad \text{is the same of the low-order MFD!}
\]

- the local scalar products satisfy the \( P_1 \) consistency relation:

\[
[(K_E \nabla q^2)^I, G]_E = -\int_E q^2 (DIV_h G)_E \, dV + \sum_{e \in \partial E} \int_e q^2 G^e_0 \, dS.
\]

for any \( q^2 \in P_2(E) \) and taking \( K_E := K|_E = \text{constant} \).

Be aware that \( K|_E \) must be constant on \( E \)!
IDEA: use the same construction as for the low-order scheme with:

- **the discrete divergence is:**

\[
(DIV_h G)_E \mid E \mid = \sum_{e \in \partial E} \int_e G^e_E(\xi) \, dS = \sum_{e \in \partial E} \int_e \left( G^e_0 + \mathbf{G}^e_1 \cdot \frac{(\xi - \xi^e)}{h_e} \right) \, dS
\]

\[= \sum_{e \in \partial E} |e| \, G^e_0 \quad \text{is the same of the low-order MFD!}

- the local scalar products satisfy the \( P_1 \) consistency relation:

\[\left[(K_E \nabla q^2)^\tau, G\right]_E = - \int_E q^2 (DIV_h G)_E \, dV + \sum_{e \in \partial E} \int_e q^2 G^e_E \, dS.\]

for any \( q^2 \in P_2(E) \) and taking \( K_E := K\mid_E = \text{constant.} \)

**Be aware that \( K\mid_E \) must be constant on \( E! \)
Sequence of “randomized” meshes
Beirao-M. (SISC, 2008)

- the nodes of a regular mesh are randomly displaced;
- refined meshes are not nested into coarser meshes.
Pressure relative error, $Q_h$-norm

Beirao-M. (SISC, 2008)

- dashed lines → low-order scheme;
- continuous lines → high-order scheme;
- circles (isotropic):
  \[
  K = \begin{pmatrix}
  1 & 0.25 \\
  0.25 & 1
  \end{pmatrix};
  \]
- squares (anisotropic):
  \[
  K = \begin{pmatrix}
  1 & 0.25 \\
  0.25 & 0.1
  \end{pmatrix}.
  \]

Pressure approximation is superconvergent!
Flux relative error, $X_h$-norm
Beirao-M. (SISC, 2008)

- dashed lines → low-order scheme;
- continuous lines → high-order scheme;
- circles (isotropic):
  \[ K = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 1 \end{pmatrix}; \]
- squares (anisotropic):
  \[ K = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 0.1 \end{pmatrix}. \]

High-order flux approximation is second order!
CPU costs versus flux relative error in $X_h$-norm

Beirao-M. (SISC, 2008)

- dashed lines → low-order scheme;
- continuous lines → high-order scheme;
- circles (isotropic):
  \[ K = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 1 \end{pmatrix}; \]
- squares (anisotropic):
  \[ K = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 0.1 \end{pmatrix}. \]

High-order flux approximation is not expensive!

A High-order MFD Method
9 June 2009, MAFELAP 2009
Non-constant $K$, pressure relative error, $Q_h$-norm

- black curve (circles): low-order scheme;
- red curve (squares): high-order scheme;
- diffusion tensor:
  $$
  \begin{pmatrix}
  (x + 1)^2 + y^2 & -xy \\
  -xy & (x + 1)^2
  \end{pmatrix};
  $$
- exact solution:
  $$
  p(x, y) = x^2 + y^2 + xy + \sin(2\pi x) \sin(2\pi y).
  $$

Pressure approximation is still superconvergent!
Non-constant $K$, flux relative error, $X_h$-norm

- black curve (circles): low-order scheme;
- red curve (squares): high-order scheme;
- diffusion tensor:
  \[
  \begin{pmatrix}
  (x + 1)^2 + y^2 & -xy \\
  -xy & (x + 1)^2
  \end{pmatrix}
  \]
- exact solution:
  \[
  p(x, y) = x^2 + y^2 + xy \
  \sin(2\pi x) \sin(2\pi y).
  \]

Higher order flux approximation is NOT second order!
A new scalar product suitable to non-constant $K$

**IDEA:** introduce the **modified local** $P_1$ consistency relation:

$$\left[(\mathcal{P}^1_E(K_E \nabla q^2))^I, G\right] = - \int_E q^2 (\text{DIV}_h G)_E \, dV + \sum_{e \in \partial E} \int_e q^2 G^e \, dS$$

$\mathcal{P}_E^1$ is the orthogonal projector from $(H^2(E))^d$ onto $(P_1(E))^d$ vectors.

**Convergence analysis** is possible assuming that:

$p \in H^3(\Omega)$; $K$ is such that $K|_E := K_E \in W^{2,\infty}(E)$:

$$\| \hat{F}^I - F_h \|_{X_h} = \mathcal{O}(h^2)$$

(which also includes the case of (piecewise) constant $K$).

We can also prove the $\mathcal{O}(h^2)$ superconvergence of the scalar unknowns and of the post-processed (quadratic) solution.
A new scalar product suitable to non-constant $K$


**IDEA:** introduce the **modified local $P_1$ consistency relation:**

$$
\left[ (P^1_E(K_E \nabla q^2))^I, \mathbf{G} \right]_E = - \int_E q^2 (\nabla \cdot (h \mathbf{G}))_E \, dV + \sum_{e \in \partial E} \int_{e} q^2 G^e E \, dS
$$

$P^1_E$ is the orthogonal projector from $(H^2(E))^d$ onto $(P_1(E))^d$ vectors.

**Convergence analysis** is possible assuming that:

$p \in H^3(\Omega)$; $K$ is such that $K|_E := K_E \in W^{2,\infty}(E)$:

$$
\left\| \mathbf{F}^I - \mathbf{F}_h \right\|_{X_h} = \mathcal{O}(h^2)
$$

(which also includes the case of (piecewise) constant $K$).

We can also prove the $\mathcal{O}(h^2)$ superconvergence of the scalar unknowns and of the post-processed (quadratic) solution.
Non-constant \( K \), flux relative error, \( X_h \)-norm

\[ p(x, y) = x^2 + y^2 + xy + \sin(2\pi x) \sin(2\pi y) \]

- low-order scheme;
- high-order scheme;
- \( \cdots \);
- diffusion tensor:

\[
\begin{pmatrix}
(x + 1)^2 + y^2 & -xy \\
-xy & (x + 1)^2
\end{pmatrix};
\]
Non-constant $K$, flux relative error, $X_h$-norm

- low-order scheme;
- high-order scheme;
- high-order scheme + projection;
- diffusion tensor:
  \[
  \begin{pmatrix}
  (x + 1)^2 + y^2 & -xy \\
  -xy & (x + 1)^2
  \end{pmatrix};
  \]
- $p(x, y) = x^2 + y^2 + xy + \sin(2\pi x) \sin(2\pi y)$

We recover the second-order flux approximation!
Non-constant $K$, pressure relative error, $Q_h$-norm

- low-order scheme;
- high-order scheme;
- high-order scheme + projection;
- diffusion tensor:
  \[
  \begin{pmatrix}
  (x + 1)^2 + y^2 & -xy \\
  -xy & (x + 1)^2 
  \end{pmatrix};
  \]
- $p(x, y) = x^2 + y^2 + xy + \sin(2\pi x) \sin(2\pi y)$

Pressure approximation does not change!
Summary

We developed a new MFD method for second-order elliptic problems that

- shows a second-order convergence rate in the flux approximation both for constant and non-constant tensor fields;
- still has the good properties of the mimetic schemes:
  - it mimics properties of continuous operators; e.g. $\text{DIV}_h$, satisfy a discrete Green-like formula;
  - it works on element of very general shape;
  - a theoretical foundation can be established along the guidelines of the original MFD method that proves convergence and provides a priori estimates of the approximation error.