STABILIZATIONS OF THE BAUMANN-ODEN DG FORMULATION: THE 3D CASE

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ABSTRACT. We consider the Baumann-Oden Discontinuous Galerkin formulation in three dimensions in a rather general geometrical setting. Using only piecewise linear approximations (and no jump stabilizations) the method is clearly unstable. We discuss the relations of possible jump stabilizations and bubble stabilizations.

1. INTRODUCTION

Most commonly used Discontinuous Galerkin methods need the addition of suitable stabilizing terms in order to provide good convergence properties. The typical stabilizing procedure consists in the introduction of *penalty terms* that penalize the jumps of the functions (or of the *mean values* of the functions) across neighboring elements.

Sometimes, in hyperbolic or in convection dominated problems, one can also use *upwind techniques*, consisting in replacing the *average* $((u^+ + u^-)/2)$ on an internal face with the upwind value (that is, u^+ or u^-, according with the direction of the "wind"). This however, in most cases, can be seen again as a jump stabilization ([22], [20], [13]).

Another possible way of stabilizing DG methods consists in the addition of suitable terms (this time, internal to each element) of the so-called *Hughes–Franca* type: in general, the integral of the original equation (or one of the original equations), written in strong form *inside each element* in terms of the finite element unknowns (= trial functions), multiplied by a similar expression acting on the test functions. The most famous stabilization of this type, for standard Galerkin methods, is surely the SUPG stabilization of convection dominated equations [15]. A typical problem, in these cases, is the choice of the proper *stabilization coefficient* to be put in front of the stabilizing term.

In a recent paper (see [7]) we pointed out that, in DG methods, the jumps are themselves to be regarded as "equations", so that jump stabilizations (and hence *upwind*) could be regarded as Hughes–Franca stabilizations as well. And, indeed, the optimal choice of the coefficient in a jump-stabilization term is still a subject that might need a further investigation.

In standard Galerkin methods (for instance in Stokes problem or in advection-diffusion problems) one of the possible ways of stabilizing an unstable formulation is to add one or more *bubble function* per element. We recall that a bubble function is, by definition, a function whose support is contained in a single element. The bubble stabilization, in its turn, can also be seen as a Hughes–Franca stabilization

after eliminating the bubbles by static condensation. This has the effect of shifting the problem of choosing the optimal coefficient into the problem of choosing the optimal shape of the bubble (see, e.g., [6], [3], [8]). This last problem can however be solved, in some cases, with the use of Residual Free Bubbles (see [14], [21]), or Pseudo Residual Free Bubbles (see [11], [12]).

When using a discontinuous method the addition of bubble functions does not mean much, as all the basis functions already have support in a single element (hence, in a sense, they are *all*, already, bubbles). We could therefore consider that for DG methods adding bubbles is just the same as augmenting the finite element space, in an arbitrary way. For instance, in two dimensions, shifting from linear discontinuous elements to quadratic discontinuous elements could be seen as adding three bubbles per element (corresponding to x^2, y^2, and xy). The same is obviously true for any other increase of the local polynomial degree. Actually, in this paper we use the term "bubble" in a rather philosophical sense, meaning that you add these shape functions only to enhance stability, and not to enhance precision.

The problem whether the addition of bubbles could provide some additional stability for DG methods has, in this respect, a rather academic nature. However, it is intellectually tackling to check whether and when a suitable (and possibly minimal) increase in the finite element space can turn an unstable formulation into a stable one. And, possibly, any discovery in this direction can provide some additional understanding of the underlying nature of DG methods. Moreover, having seen that i) the addition of bubbles can, for other problems, (as for instance Stokes, nearly incompressible elasticity, and advection dominated transport equations) be seen as equivalent to using certain types of Hughes–Franca stabilizations. And having seen that ii) the jump stabilization is indeed a type of Hughes–Franca stabilization itself, it is natural to ask whether the jump stabilization could be obtained by adding and eliminating suitable bubbles.

In previous papers ([10], [1]) we actually proved that in two dimension, with rather general assumptions on the decomposition, the piecewise linear approximation of the Baumann-Oden DG formulation ([4], [5], [23], [24]) can indeed be stabilized adding, essentially, $k - 2$ "bubbles" in each element having k edges. The bubbles were assumed to satisfy some reasonably simple abstract properties, and we showed how to construct functions having such properties for the particular cases of meshes made of triangles and/or quadrilaterals.

Here, we discuss in more detail the three-dimensional case, and its bubble stabilization. Although, in principle, the argument used in the previous papers for two-dimensional problems generalizes almost immediately to three dimensions (implying that you can stabilize the piecewise linear discontinuous formulation by adding $k - 3$ suitably chosen bubbles per element) we concentrated our interest, for simplicity, in a stronger type of stabilization, using k bubbles in every polyhedron with k faces. In particular we show an explicit construction (in two and in three dimensions) of suitable functions that could be added in very general geometrical assumptions.

Finally we discuss the elimination of the bubbles and the resulting scheme. We show that, similarly to [16], [10], the elimination of these bubbles produces, essentially, the usual jump stabilization terms. In this sense, the paper can also be seen as a generalization to DG methods of the equivalence between bubbles and Hughes–Franca stabilizations.

For a different approach and different (but also, in our opinion, quite interesting) results concerning the stability of (these and) other DG methods *without* jump stabilizations we refer to [17], [18], [19].

The practical impact of our investigation is surely questionable, although the possibility of avoiding the jump stabilization for linear elements is surely appealing, as it leads to a more "natural" choice of the interelement fluxes. Moreover, we believe that our analysis provides a better understanding of some basic aspects and mechanisms related to DG methods, that might be of some help in designing new future methods. And as such, it might interest several curious scientists.

An outline of the paper is as follows. In the next section we recall some notation on DG methods, and the Baumann-Oden formulation for Poisson problem. Then, we introduce the discretization, using a finite element space that is made of piecewise linear functions plus suitably constructed bubbles, and we verify the stability and convergence properties of our approach. In the last section we enter more deeply into the corresponding scheme, and we analyze the effect of the bubble elimination: we prove indeed that the piecewise linear part of our solution coincides with the one that could be obtained by adding a suitable jump stabilization term.

2. The model problem and the Baumann-Oden method

Let Ω be a convex polyhedral domain, with boundary $\partial\Omega$. For every g, say, in $L^2(\Omega)$ we consider the model problem:

$$(1) \qquad -\Delta u = g \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

It is well known that problem (1) has a unique solution, that belongs to $H^2(\Omega) \cap H^1_0(\Omega)$.

Let $\{\mathcal{T}_h\}_h$ be a sequence of *compatible* decompositions of Ω into polyhedra T. Here, "compatible" means that the intersection of the closure of two different polyhedra is either empty, or a common face, or a common edge, or a common vertex. For every polyhedron T we will denote by k_T the number of its faces and by h_T its diameter. Moreover, for every face f and for every edge e we will denote by $|f|$ the area and by $|e|$ the length, respectively. We shall also assume that

(2)   There exists a constant $\rho_1 > 0$ such that for every h, for every $T \in \mathcal{T}_h$ and for every face f of T, we have

$$h_T^2 \geq |f| \geq \rho_1 h_T^2.$$

(3)   There exists a constant $\rho_2 > 0$ such that for every h, for every $T \in \mathcal{T}_h$, and for every face f of T there exists a pyramid $\mathcal{P}_f^T$ with base f, having vertex inside T and volume

$$|\mathcal{P}_f^T| \geq \rho_2 h_T^3.$$

Note that (3) implies that the number of faces of each polyhedron of the decomposition is uniformly bounded, that is

(4)   There exists an integer $k \geq 4$ such that for every h and for every $T \in \mathcal{T}_h$

$$k_T \leq k.$$

It will clearly not be restrictive to assume further that,

(5)   for every pyramid $\mathcal{P}_f^T$, the orthogonal projection of the vertex opposite to f onto the plane containing f actually falls *inside* f,

and

(6)   all the pyramids $\mathcal{P}_f^T$ in (3) have no internal points in common.

We consider now thew (infinite dimensional) space $V(\mathcal{T}_h)$ defined as

(7) $$V(\mathcal{T}_h) = \{v \in L^2(\Omega) \text{ such that } v_{|T} \in H^2(T) \ \forall T \in \mathcal{T}_h\}.$$

Elements $v \in V(\mathcal{T}_h)$ will, in general, be discontinuous when passing from one element to a neighboring one. As usual in DG methods we have therefore to introduce boundary operators as *averages* and *jumps*. As we shall deal also with vector-valued functions which are smooth in each element, but discontinuous from one element to another, we shall introduce these boundary operators for scalar and for vector-valued functions. Following [2], we set as usual:

$$\{v\} := \frac{v^+ + v^-}{2}, \qquad [\![v]\!] := v^+ \boldsymbol{n}^+ + v^- \boldsymbol{n}^-, \qquad \text{for all internal faces;}$$

$$\{\boldsymbol{\tau}\} := \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}, \qquad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}^+ \boldsymbol{n}^+ + \boldsymbol{\tau}^- \boldsymbol{n}^-, \qquad \text{for all internal faces.}$$

On the boundary faces we define $[\![v]\!] := v\boldsymbol{n}$ and $\{\boldsymbol{\tau}\} := \boldsymbol{\tau}$.

We introduce now some further notation. For functions in $V(\mathcal{T}_h)$ we first introduce the elementwise gradient ∇_h, and then for u and v in $V(\mathcal{T}_h)$ we set

$$(\nabla_h u, \nabla_h v) := \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \nabla v \, \mathrm{d}x, \quad < \{\nabla_h u\}, [\![v]\!] > := \sum_{f \in \mathcal{F}_h} \int_f \{\nabla_h u\} \cdot [\![v]\!] \, \mathrm{d}s,$$

where $\mathcal{F}_h$ denotes the set of all faces of the decomposition $\mathcal{T}_h$. Setting, for u and v in $V(\mathcal{T}_h)$

(8) $$a(u, v) := (\nabla_h u, \nabla_h v) - < \{\nabla_h u\}, [\![v]\!] > + < \{\nabla_h v\}, [\![u]\!] >,$$

the Baumann-Oden "continuous" formulation of (1) is now

(9) $$\begin{cases} \text{Find } u \in V(\mathcal{T}_h) \text{ such that, } \forall v \in V(\mathcal{T}_h): \\ a(u, v) = (g, v). \end{cases}$$

In $V(\mathcal{T}_h)$ we define the jump seminorm

(10) $$\|v\|_j^2 = \sum_{f \in \mathcal{F}_h} \frac{1}{|f|^{1/2}} \int_f |[\![\overline{v}]\!]|^2 \, \mathrm{d}s,$$

where, on each face, $\overline{v}$ is *the mean value* of v on the face. We then consider the norm

(11) $$\|v\|_{V(\mathcal{T}_h)}^2 := \sum_{T \in \mathcal{T}_h} \left(|v|_{1,T}^2 + h_T^2 |v|_{2,T}^2 \right) + \|v\|_j^2,$$

We recall now the following useful result, which follows easily from a well known result of Agmon (see, e.g., [2]):

(12) $$\forall T, \ \forall f \in \partial T, \ \forall v \in H^1(T): \quad \int_f v^2 \, \mathrm{d}s \leq C_a(h_T^{-1} \|v\|_{0,T}^2 + h_T |v|_{1,T}^2),$$

with C_a only depending on the constant ρ_2 in (3). Hence, we have

$$| < \{\boldsymbol{\tau}\}, [\![\, v\,]\!] > | = \sum_{f \in \mathcal{F}_h} \int_f \{\boldsymbol{\tau}\} \cdot [\![\, v\,]\!] \, \mathrm{d}s$$

(13)

$$\leq C \left[\sum_{T \in \mathcal{T}_h} (\|\boldsymbol{\tau}\|_{0,T}^2 + h_T^2 |\boldsymbol{\tau}|_{1,T}^2) \right]^{1/2} \left[\sum_{f \in \mathcal{F}_h} \frac{1}{|f|^{1/2}} \int_f |[\![\, v\,]\!]|^2 \, \mathrm{d}s \right]^{1/2},$$

for all $\boldsymbol{\tau}$ that are componentwise in $H^1(T)$ for every T, and for all $v \in V(\mathcal{T}_h)$. From (13) and (11) we easily deduce the following proposition.

Proposition 2.1. *There exists a constant, that we denote by C_{cont}, depending only on ρ_1 and ρ_2 in* (2) *and* (3)*, such that*

(14)
$$a(u,v) \leq C_{cont} \|u\|_{V(\mathcal{T}_h)} \|v\|_{V(\mathcal{T}_h)} \qquad \forall u, v \in V(\mathcal{T}_h).$$

Proof. Inequality (14) is easily proven upon noticing that, via (12) and standard interpolation results,

$$\sum_{f \in \mathcal{F}_h} \frac{1}{|f|^{1/2}} \int_f |[\![\, v\,]\!]|^2 \, \mathrm{d}s \leq C \left[\sum_{T \in \mathcal{T}_h} \|\nabla v\|_{0,T}^2 + \sum_{f \in \mathcal{F}_h} \frac{1}{|f|^{1/2}} \int_f |[\![\, \overline{v}\,]\!]|^2 \, \mathrm{d}s \right].$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Discretization with *linear plus bubbles*

In what follows we are going to construct, for each $T \in \mathcal{T}_h$, a space $V(T)$ made of linear functions plus suitable *bubble functions*. These bubbles will not be too regular: in particular, their derivatives might exhibit jumps near the boundary, so that the bubbles will not belong to $H^2(T)$ but only to $W^{2,p}(T)$ for every $p < 2$.

At a more abstract level it will be therefore convenient to fix, once and for all, a real number p with $3/2 < p < 2$. As we have seen, we could take p, roughly speaking, *as close to* 2 *as we want, but not equal to* 2. Certain constants, indeed, will depend on $2 - p$, and degenerate as p tends to 2. This somehow justifies the choice of fixing a p once and for all.

Hence, we assume that for every element $T \in \mathcal{T}_h$ we are given a finite dimensional space $V(T) \subset W^{2,p}(T)$, and we consider the (finite element) space

$$V_h := \prod_{T \in \mathcal{T}_h} V(T),$$

and the corresponding space of gradients

$$\Sigma_h := \nabla_h(V_h).$$

We point out that, by usual Sobolev embedding theorems, $p > 3/2$, in three dimensions, implies that $W^{2,p}(T) \subset H^s(T)$ for some $s > 3/2$. This in turn implies that for an element $w \in W^{2,p}(T)$ we can define both the trace $w_{|\partial T}$ and the trace of the normal derivative $\partial w/\partial n$ on ∂T, and both belong to $L^2(\partial T)$. Hence we are allowed to consider the discrete problem:

(15)
$$\begin{cases} \text{Find } u_h \in V_h \text{ such that, } \forall v \in V_h : \\ (\nabla_h u_h, \nabla_h v) - < \{\nabla_h u_h\}, [\![\, v\,]\!] > + < \{\nabla_h v\}, [\![\, u_h\,]\!] > = (g, v), \end{cases}$$

that, using (8), can also be written

$$(16) \qquad\qquad a(u, v) = (g, v) \qquad \forall v \in V_h.$$

In the finite element space V_h we introduce the usual DG norm

$$(17) \qquad\qquad \|v\|^2 := |v|_{1,h}^2 + \|v\|_j^2,$$

where $| \cdot |_{1,h}$ is the H^1–broken seminorm. It will be convenient (and not really restrictive) to assume that in each $V(T)$ (as well as for its gradients $\Sigma(T) := \nabla(V(T))$) we have an inverse inequality of the form

(18) $\exists \rho_3 > 0$ such that for every h and for every $T \in \mathcal{T}_h$ we have

$$i) \quad h_T \, |v|_{1,T} \le \rho_3 \|v\|_{0,T} \qquad \text{for every } v \text{ in } V(T)$$
$$ii) \quad h_T \, |\boldsymbol{\tau} \cdot \boldsymbol{n}|_{0,\partial T}^2 \le \rho_3 |\boldsymbol{\tau}|_{0,T}^2 \qquad \text{for every } \tau \text{ in } \Sigma(T)$$

We observe that for the latter inequality we could not use (12) as $\boldsymbol{\tau}$ does not belong to $(H^1(T))^2$ but only to $(W^{1,p}(T)^2$.

We note immediately that, using our assumptions on the decomposition (2)-(3) together with (18), then *on V_h the DG norm (17) is equivalent to the norm (11) originally introduced in $V(\mathcal{T}_h)$*. In particular we have

$$(19) \qquad\qquad \|v_h\|_{V(\mathcal{T}_h)} \le C_{inv} \|v_h\| \le C_{inv} \|v_h\|_{V(\mathcal{T}_h)} \qquad \forall v_h \in V_h,$$

where C_{inv} depends only on ρ_1, ρ_2, and ρ_3. In a similar way (13) could be simplified to

$$(20) \qquad\qquad | < \{\boldsymbol{\tau}\}, [\![\, v \,]\!] > | \le C_s \|\boldsymbol{\tau}\|_0 \|v\|_j \qquad \forall \boldsymbol{\tau} \in \Sigma_h, \ \forall v \in V_h.$$

Hence, we immediately have the following result.

Proposition 3.1. *There exists a constant, that we denote again by C_{cont}, depending only on ρ_1, ρ_2, and ρ_3 such that*

$$(21) \qquad a(u_h, v_h) \le C_{cont} \|u_h\| \, \|v_h\| \qquad \forall u_h, v_h \in V_h. \qquad\qquad \square$$

Our first task should be to prove *stability* of the bilinear form $a(u, v)$ in the DG norm (17). For this, we make some further assumptions on the choice of the spaces $V(T)$. More precisely, we assume that each $V(T)$ is constructed as the union of the space P_1 of polynomials of degree less than or equal to one, plus k_T "bubbles", one for each face f of T, having the following properties:

(22)   • for each face f the corresponding b_f^T has support contained in $\mathcal{P}_f^T$;

(23)   • each b_f^T belongs to $H_0^1(T) \cap W^{2,p}(T) \quad \forall p < 2$;

(24)   • on the boundary of T each b_f^T verifies

$$\frac{\partial b_f^T}{\partial \mathbf{n}^T}_{|\partial T} = \chi_f, \quad \chi_f = \text{ characteristic function of } f,$$

$\mathbf{n}^T$ being the outward normal unit vector to ∂T;

(25)   • $\exists \rho_4 > 0$ such that for each T and for each f of T we have

$$\|b_f^T\|_{H^1(T)}^2 \le \rho_4 \, h_T^3;$$

(26)   • the inverse inequality (18) holds.

It is not difficult to check that the spaces $V(T)$ will then verify the abstract assumptions of [1]. In particular, the most crucial of the properties required in [1] is proved in the following proposition.

Proposition 3.2. *There exists a constant $\rho_5 > 0$ (depending only on $\rho_1,..., \rho_4$) such that: for all h, for all $T \in \mathcal{T}_h$, and for all function $\varphi \in L^2(\partial T)$, constant on each face of ∂T, there exists $v(\varphi) \in V(T)$ such that*

$$\|v(\varphi)\|_{1,T}^2 \le \rho_5 \, h_T^{-1} \|\varphi\|_{0,\partial T}^2, \tag{27}$$

$$\frac{\partial v(\varphi)}{\partial \mathbf{n}_f^T} = h_T^{-1} \varphi \quad on \; \partial T \qquad \forall T \in \mathcal{T}_h. \tag{28}$$

Proof. It is enough, for $T \in \mathcal{T}_h$ and for each face f of T, to take

$$v(\varphi) = (\varphi_{|f}) h_T^{-1} b_f^T \qquad in \; \mathcal{P}_f^T. \tag{29}$$

From (24) we easily have (28). From (25) , (29), and (2) we then have, for each face,

$$\|v(\varphi)\|_{1,\mathcal{P}_f^T}^2 = h_T^{-2} (\varphi_{|f})^2 \|b_f^T\|_{1,\mathcal{P}_f^T}^2 \le \rho_4 \, h_T (\varphi_{|f})^2 \tag{30}$$

$$= \rho_4 \frac{h_T}{|f|} \|\varphi\|_{0,f}^2 \le \frac{\rho_4}{\rho_1} h_T^{-1} \|\varphi\|_{0,f}^2.$$

$\square$

We point out that the assumption (made in [1]) that the *degree* of the bubbles is uniformly bounded is not needed here (and is somehow replaced by the inverse inequality (18)).

The stability of the problem (16) will then follow easily, by the same procedure as in [1]. In particular we have the following result.

Theorem 3.3. *There exists a constant K, depending only on ρ_1–ρ_4, such that: for every $u_h \in V_h$ there exists a v_h in V_h, different from zero, such that*

$$a(u_h, v_h) \ge K \|u_h\| \, \|v_h\|. \tag{31}$$

The above stability result, together with the continuity property of Proposition 3.1, and natural consistency properties, give then the following convergence result, with classical instruments.

Theorem 3.4. *In the above assumptions, for every $g \in L^2(\Omega)$ the discrete problem (15) has a unique solution u_h. Moreover, the distance between u_h and the solution u of (1) can be estimated as*

$$\|u - u_h\|_{V(\mathcal{T}_h)} \le C \, h \, |u|_{2,\Omega}. \tag{32}$$

where C is a constant depending only on ρ_1–ρ_4.

4. CHOICE AND ELIMINATION OF THE BUBBLES

Here we would like first to describe how bubble-functions satisfying (22)-(26) could be constructed. Then, we want to analyze the nature of the resulting scheme and the effect of the elimination of such bubbles.

Let us start from the first task. For each $T \in \mathcal{T}_h$, and for each face f of T, we assume that the plane containing f is the (x, y) plane, that the vertex of $\mathcal{P}_f^T$ is in

the half-space $z > 0$, and that $z = d(x, y)$ is the (piecewise linear) function whose graph is $\{\partial \mathcal{P}_f^T\} \setminus \{f\}$. On the interval $]0, 1[$ we consider the function

$$(33) \qquad t :\longrightarrow \psi(t) := t^3 - 2t^2 + t.$$

For each $\delta > 0$ we consider then the scaled function ψ_δ

$$(34) \qquad t :\longrightarrow \psi_\delta(t) := \delta\psi(t/\delta).$$

Finally, in $\mathcal{P}_f^T$ we set

$$(35) \qquad b_f^T(x, y, z) := \psi_{d(x,y)}(z),$$

that is (writing d in place of $d(x, y)$):

$$(36) \qquad b_f^T(x, y, z) := d\Big(\big(\tfrac{z}{d}\big)^3 - 2\big(\tfrac{z}{d}\big)^2 + \big(\tfrac{z}{d}\big)\Big) = \frac{z^3}{d^2} - 2\frac{z^2}{d} + z.$$

and we set $b_f^T = 0$ elsewhere. It is not too difficult to see that all the above assumptions (22)-(26) are verified. Note that, in particular, $z \leq d$ in the whole pyramid $\mathcal{P}_f^T$, so that the function b_f^T is everywhere bounded by $C\,h_T$ for some constant C depending only on $\rho_1 - \rho_4$. Similarly, as the gradient of $d(x, y)$ is bounded by a constant independent of h, it follows easily from (36) that the gradient of b_f^T is everywhere bounded by a constant independent of h, always using the fact that $z \leq d$.

We would like now to have a closer look on the resulting scheme. For this, we split (as usual when dealing with bubble stabilizations) the discrete unknown u_h as

$$(37) \qquad u_h := u_L + u_B,$$

where u_L, in each element T, takes the linear part (that is the part in P_1) and u_B takes the bubble part. We number the internal faces from 1 to $\mathcal{N}_i$, and then the boundary faces from 1 to $\mathcal{N}_b$. We observe that we are going to have *two bubbles* for each internal faces (corresponding to the two elements sharing that face) and just *one* bubble for each boundary face, for a total of $2\mathcal{N}_i + \mathcal{N}_b$ bubbles. Before numbering the bubbles, for each internal face we choose an orientation $\mathbf{n}_f$ (once and for all), and then we use it to number the bubbles as follows. For each internal face k, $k = 1, ..., \mathcal{N}_i$, we denote by b_{2k-1} (and by b_{2k}, resp.) the bubble corresponding to the face number k and to the element from which $\mathbf{n}_f$ exits (enters, resp.). For $j = 2\mathcal{N}_i + 1, ..., 2\mathcal{N}_i + \mathcal{N}_b$, we take instead b_j as the bubble associated to the boundary face numbered by $j - 2\mathcal{N}_i$. Conversely, given a bubble-index j between 1 and $2\mathcal{N}_i + \mathcal{N}_b$ we can associate to it, in a unique way, a *face* $f(j)$ and an *element* $T(j)$ (that sometimes, for brevity, will be indicated just by f and T, when no confusion can occur) as the face and the element associated with the bubble b_j in the obvious way.

At this point we can write

$$(38) \qquad u_B(x, y, z) = \sum_{j=1}^{2\mathcal{N}_i + \mathcal{N}_b} \beta_j\, b_j(x, y, z).$$

We note now that, having assumed that the pyramids $\mathcal{P}_f^T$ have disjoint interiors, we obviously have

$$(39) \qquad \int_\Omega \nabla b_i \cdot \nabla b_j\, \mathrm{d}x = 0, \qquad \text{for } i \neq j.$$

Moreover, as each bubble is in H_0^1 of the corresponding element, we clearly have

$$(40) \qquad [\![\, b_j \,]\!] = 0 \qquad \forall j,$$

so that, setting

$$(41) \qquad D_j := \int_\Omega |\nabla b_j|^2 \, \mathrm{d}x, \qquad \forall j,$$

we easily get

$$(42) \qquad a(b_i, b_j) = D_j \delta_{i,j},$$

where $\delta_{i,j}$ is the usual Kronecker symbol. On the other hand, for each discontinuous piecewise linear v_L we have

$$(43) \quad \int_\Omega \nabla_h v_L \cdot \nabla b_j \, \mathrm{d}x = \int_{T(j)} \nabla v_L \cdot \nabla b_j \, \mathrm{d}x$$

$$= -\int_{T(j)} \Delta v_L \, b_j \, \mathrm{d}x + \int_{\partial T(j)} \frac{\partial v_L}{\partial \mathbf{n}^T} b_j \, \mathrm{d}s = 0,$$

and

$$(44) \qquad < [\![\, b_j \,]\!], \{\nabla v_L\} >\, = 0,$$

while for each T and for each face f of T we have, thanks to (24),

$$(45) \qquad < [\![\, v_L \,]\!], \{\nabla b_f^T\} >\, = \begin{cases} \dfrac{1}{2} \displaystyle\int_f [\![\, v_L \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s & \forall \text{ internal face } f \\[2ex] \displaystyle\int_f [\![\, v_L \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s & \forall \text{ boundary face } f, \end{cases}$$

that we write concisely as

$$(46) \qquad < [\![\, v_L \,]\!], \{\nabla b_f^T\} >\, = c_f \int_f [\![\, v_L \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s$$

with

$$c_f = 1/2 \text{ on internal faces}, \ c_f = 1 \text{ on boundary faces.}$$

Using again the expression (8) we have now, for every piecewise linear function v_L and for every bubble b_j

$$(47) \qquad a(v_L, b_j) =\, < [\![\, v_L \,]\!], \{\nabla b_j\} >\, = c_f \int_{f(j)} [\![\, v_L \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s,$$

and

$$(48) \qquad a(b_j, v_L) = -\, < [\![\, v_L \,]\!], \{\nabla b_j\} >\, = -c_f \int_{f(j)} [\![\, v_L \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s,$$

where we recall that $T = T(j)$ and $f = f(j)$ are the element and the face corresponding to the index j. Collecting all the above properties, we then have, for every $j = 1, ..., 2\mathcal{N}_i + \mathcal{N}_b$

$$(49) \qquad a(u_L + u_B, b_j) = a(u_L, b_j) + a(u_B, b_j) = c_f \int_{f(j)} [\![\, u_L \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s + \beta_j \, D_j.$$

Using (16) with $v = b_j$ gives then

$$(50) \qquad c_f \int_{f(j)} [\![\, u_L \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s + \beta_j D_j = \int_{\mathcal{P}_f^T} g \, b_j \, \mathrm{d}x,$$

that we write as

$$(51) \quad \beta_j = (G_j - c_f \int_{f(j)} [\![\, u_L \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s)/D_j \equiv (G_j - c_f \int_{f(j)} [\![\, \overline{u_L} \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s)/D_j,$$

where we have set

$$G_j := \int_{\mathcal{P}_f^T} g \, b_j \, \mathrm{d}x.$$

Having computed u_B as a function of u_L we can go back to our equation (16), this time with $v = v_L$, piecewise linear test function. We have

$$a(u_L + u_B, v_L) = a(u_L, v_L) + a(u_B, v_L) = a(u_L, v_L) - c_f \sum_{j=1}^{2\mathcal{N}_i + \mathcal{N}_b} \beta_j \int_{f(j)} [\![\, v_L \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s.$$

Substituting the value of β_j given by (51) we have then

$$(52) \quad a(u_L + u_B, v_L) = a(u_L, v_L) - \sum_{j=1}^{2\mathcal{N}_i + \mathcal{N}_b} c_f \frac{G_j}{D_j} \int_{f(j)} [\![\, \overline{v_L} \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s$$

$$+ \sum_{j=1}^{2\mathcal{N}_i + \mathcal{N}_b} c_f^2 \frac{1}{D_j} \int_{f(j)} [\![\, \overline{u_L} \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s \int_{f(j)} [\![\, \overline{v_L} \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s.$$

Thus, after elimination of the bubbles, the scheme becomes

$$(53) \quad \begin{cases} \text{Find } u_L \text{ p.w. linear such that } \forall v_L \text{ p.w. linear :} \\[2mm] a(u_L, v_L) + \displaystyle\sum_{j=1}^{2\mathcal{N}_i + \mathcal{N}_b} \frac{c_f^2 \, |f(j)|}{D_j} \int_{f(j)} [\![\, \overline{u_L} \,]\!] \cdot [\![\, \overline{v_L} \,]\!] \, \mathrm{d}s \\[4mm] = (g, v_L) + \displaystyle\sum_{j=1}^{2\mathcal{N}_i + \mathcal{N}_b} c_f \frac{G_j}{D_j} \int_{f(j)} [\![\, \overline{v_L} \,]\!] \cdot \mathbf{n}_f^T \, \mathrm{d}s. \end{cases}$$

The similarities with the so-called *NIPG method* (or *jump-stabilized Baumann-Oden*) are clear. We recall, in particular, that the lowest order NIPG method [24] reads

$$(54) \quad \begin{cases} \text{Find } u_L \text{ piecewise linear linear such that :} \\[2mm] a(u_L, v_L) + \displaystyle\sum_f \frac{c_f}{h_f} \int_f [\![\, \overline{u_L} \,]\!] \cdot [\![\, \overline{v_L} \,]\!] \, \mathrm{d}s = (g, v_L) \qquad \forall v_L \text{ piecewise linear,} \end{cases}$$

where: the sum ranges over all the faces; h_f is a characteristic length attached to the face f (and of the order of $|f|^{1/2}$); c_f is a coefficient attached to the face f (in most cases all the c_f's are taken equal to 1), and often (but not necessarily) the jumps of the averages $[\![\, \overline{u_L} \,]\!]$ and $[\![\, \overline{v_L} \,]\!]$ are replaced by the true jumps $[\![\, u_L \,]\!]$ and $[\![\, v_L \,]\!]$.

To compare (53) and (54), note first that, on a regular mesh and for a smooth right-hand side g the terms D_j and G_j coming from two different elements sharing the same face will be very similar to each other, so that the change in orientation on $\mathbf{n}_f^T$ will make the sum of the two contribution negligible. On the other hand, we could have taken a different function instead of ψ in (33). For instance we could have taken the function

$$(55) \qquad\qquad \psi(t) = (t^3 - 2t^2 + t)(1 - 5t/2),$$

that has zero mean value on $]0,1[$ (and shares all the other necessary properties of the choice (33)). Then for a piecewise constant g all the G_j would be zero. We can summarize the above discussion in the following theorem.

Theorem 4.1. *Let us start with the formulation* (16)*, with a finite element space made of piecewise linear (discontinuous) polynomials. Assume that we add, for each element, a number of bubbles equal to the number of faces, and assume that the bubbles are constructed as in* (35)*, with ψ given as in* (55)*. Assume finally that the right-hand side g is constant in each element. Then, the augmented problem has a unique solution, that converges to the solution u of* (1) *with an error of order $O(h)$ in the DG norm* (17)*. Moreover, the linear part of the solution coincides with the solution of the stabilized Baumann-Oden formulation (also known as NIPG), in which the bilinear form* (8) *has been stabilized by adding the term*

$$(56) \qquad \sum_{j=1}^{2\mathcal{N}_i+\mathcal{N}_b} \frac{c_f^2|f(j)|}{D_j} \int_{f(j)} [\![\, \overline{u_L}\,]\!] \cdot [\![\, \overline{v_L}\,]\!],$$

where $f(j)$ is the face corresponding to the index j, and each D_j is given by (41) *(and, according with* (25)*, is locally of the order of h_T^3).*

Note that, as c_f is either equal to $1/2$ or equal to one, and $|f|$ is clearly of the order of h_T^2, then the coefficient in front of the integral in (56) is of the order of h_T^{-1}, exactly as in (54).

References

[1] P. F. Antonietti, F. Brezzi and L.D. Marini, Bubble Stabilizations of DG methods, submitted to *Comput. Methods Appl. Mech. Engrg.*

[2] D.N. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.* (2002), **39**, 1749–1779.

[3] C. Baiocchi, F. Brezzi, and L.P. Franca, Virtual bubbles and GaLS. *Comput. Methods Appl. Mech. Engrg.* (1993), **105**, 125–141.

[4] C. E. Baumann and J. T. Oden, A discontinuous hp finite element method for the Navier-Stokes equations, 10th. International Conference on Finite Element in Fluids, 1998.

[5] C. E. Baumann and J. T. Oden, A discontinuous hp finite element method for convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.* (1999), **175**, 311–341.

[6] F. Brezzi, M.O. Bristeau, L.P. Franca, M. Mallet, and G. Rogé, A relationship between stabilized finite element methods and the Galerkin method with bubble functions. *Comput. Methods Appl. Mech. Engrg.* (1992), **96**, 117–129.

[7] F. Brezzi, B. Cockburn, L.D. Marini, and E. Süli, Stabilization mechanisms in discontinuous Galerkin finite element methods.*Comput. Methods Appl. Mech. Engrg.* (2006), **195**, No.25-28, 3293–3310.

[8] F. Brezzi, L. P. Franca, T. J. R. Hughes, and A. Russo, $b = \int g$. *Comput. Methods Appl. Mech. Engrg.* (1997), **142**, 353–360.

[9] F. Brezzi, L.D. Marini, Error estimates for the three-field formulation with bubble stabilization. *Math. Comp.*, (2001) **7**0, 911–934.

[10] F. Brezzi, L.D. Marini, Bubble stabilization of discontinuous Galerkin methods, in *W. Fitzgibbon, R. Hoppe, J. Periaux, O. Pironneau, Y. Vassilevski (Eds.): Proceedings of International Conference on the occasion of the 60th birthday of Y.A. Kuznetsov* Institute of Numerical Mathematics of the Russian Academy of Sciences. Moskow, (2006) 25–36.

[11] F. Brezzi, L.D. Marini, and A. Russo, Applications of Pseudo Residual-Free Bubbles to the Stabilization of Convection-Diffusion Problems. *Comput. Methods Appl. Mech. Engrg.* (1998), **166**, 51–63.

[12] F. Brezzi, L.D. Marini, and A. Russo, On the choice of a Stabilizing Subgrid for Convection-Diffusion Problems. *Comput. Methods Appl. Mech. Engrg.* (2005), **194**, 127–148.

[13] F. Brezzi, L.D. Marini, and E. Süli, Discontinuous Galerkin methods for first-order hyperbolic problems. *Math. Models Methods Appl. Sci.* (2004), **14**, No.12, 1893–1903.

[14] F. Brezzi and A. Russo, Choosing bubbles for advection-diffusion problems. *Math. Models Meth. Appl. Sci.* (1994), **4**, 571–587.

[15] A. N. Brooks and T. J. R. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.* (1982), **32**, 199–259.

[16] A. Buffa, Error estimate for a stabilised domain decomposition method with nonmatching grids, *Numer. Math.*, (2002) **9**0, 617–640.

[17] E. Burman and B. Stamm, Symmetric and non-symmetric discontinuous Galerkin methods stabilized using bubble enrichment, *C. R. Acad. Sci. Paris, Sr. I* (2008), **346**, No. 1-2, 103–106.

[18] E. Burman and B. Stamm, Low order discontinuous Galerkin methods for second order elliptic problems, EPFL-IACS report 04.2007, submitted to *SIAM J. Numer. Anal.*.

[19] E. Burman and B. Stamm, Minimal stabilization of discontinuous Galerkin finite element methods for hyperbolic problems, *J. Sci. Comput.* (2007), **33**, No.2, 183–208.

[20] B. Cockburn, Discontinuous Galerkin methods, *Zeitschirift fur Angewandte Mathematik und Mechanick* (2003), **83**, No.11, 731–754.

[21] L. P. Franca and A. Russo, Deriving upwinding, mass lumping and selective reduced integration by RFB. *Appl. Math. Letters* (1996), **9**, 83–88.

[22] P. Lesaint, and P.-A. Raviart, On a finite element method for solving the neutron transport equation. In: *Mathematical Aspects of Finite Elements in Partial Differential Equations* (Ed. C.A. deBoor), Academic Press, New York, 1974, 89–123.

[23] J. T. Oden, I. Babuška, and C. E. Baumann, A discontinuous *hp* finite element method for diffusion problems, *J. Comput. Phys.*, (1998), **146**, 491–519.

[24] B. Rivière, M. F. Wheeler, and V. Girault, Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. *Comp. Geo.* (1999), **3**, 337–360.

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