A QUICK TUTORIAL ON DG METHODS FOR ELLIPTIC PROBLEMS

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Abstract. In this paper we recall a few basic definitions and results concerning the use of DG methods for elliptic problems. As examples we consider the Poisson problem and the linear elasticity problem. A hint on the nearly incompressible case is given, just to show one of the possible advantages of DG methods over continuous ones. At the end of the paper we recall some physical principles for linear elasticity problems, just to open the door towards possible new developments.

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1. Introduction. The main purpose of this paper is to present the basic features of Discontinuous Galerkin Methods for elliptic problems. We will give some hints on the basic mathematical tools typically used to study and analyze them, and on their capability to avoid some common troubles (as the discretization of nearly incompressible materials). We will state approximation properties and show how to derive a-priori estimates. We will also present some possible variants and relationships with other approaches (as mixed or hybrid methods for linear elasticity) that possibly deserve a deeper analysis.

The paper is addressed to readers with a more engineering oriented background, and an interest in continuum mechanics, with the idea to help them in getting more familiar with the basic concepts and features of DG methods, that indeed, according to the latest developments, show an interesting potential also in structural problems.

Actually, applications of DG methods to other problems, and in particular to hyperbolic problems, conservation laws and the like, started already forty years ago, and are fully developed nowadays (see e.g. [19], [36]). In this book these applications are discussed at a much higher level, (starting from the "parallel" contribution of Chi Wang Shu [37]); this is quite natural, since the interested people are (in general) already acquainted with all the basic instruments and with the applications to the more common problems.

Instead, most practitioners in structural engineering and continuum mechanics, so far, are not yet familiar with the use of DG methods, that have been pushed forward mainly by applied mathematicians and more "mathematically oriented" engineers. Hence the idea of addressing people that are less familiar with the DG machinery but are interested in trying them on their problems.

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We will not discuss issues related to a-posteriori estimates and mesh-adaptivity, a very interesting subject that however goes beyond the scope of this paper. For this we refer, for instance, to [31, 33, 32]. For the same reason, we will not discuss matters related to the solution of the final linear systems, such as the construction of efficient solvers and preconditioners (see, e.g., [9, 25]).

The paper is organized as follows. In Sect. 2 we recall some basic instruments, such as Poincaré, trace, and inverse inequalities, that will be useful in the sequel. In Sect. 3 typical tools for dealing with discontinuous functions are introduced: jumps and averages, norms and bounds for the edge contributions. Sect. 4 is devoted to the treatment of the Poisson problem; the most common DG schemes are derived and proved to be stable and consistent. Error estimates are also recalled. In Sect. 5 linear elasticity problems are treated, including the nearly incompressible case where the use of DG approximations proves to be particularly well suited for dealing with the so-called locking phenomenon. Finally, in Sect. 6 we recall some basic physical principles that are at the basis of alternative variational formulations (always for linear elasticity). The use of DG discretizations for many of these formulations is still at the beginning, and their potential is, in our opinion, still to be fully assessed.

Throughout the paper we shall follow the usual notation for Sobolev norms and seminorms, as for instance in [18]. Hence, for a geometric object $\mathcal{O}$ (as an edge, or an element, or a general domain) and a smooth-enough function v defined on $\mathcal{O}$, we will denote by

$$\|v\|^2_{0,\mathcal{O}} \equiv |v|^2_{0,\mathcal{O}} \equiv \int_{\mathcal{O}} v^2 \,\mathrm{d}\mathcal{O}$$

the (square) norm of v in $\mathbb{L}^2(\mathcal{O})$. On the other hand the notation $|v|^2_{k,\mathcal{O}}$ will indicate, for k integer ≥ 1, the square of the seminorm of v obtained summing all the squares of the $\mathbb{L}^2$ norms of all the derivatives of order k. Hence, in 2 dimensions,

$$|v|^2_{1,\mathcal{O}} \equiv \Big|\frac{\partial v}{\partial x_1}\Big|^2_{0,\mathcal{O}} + \Big|\frac{\partial v}{\partial x_2}\Big|^2_{0,\mathcal{O}}$$

$$|v|^2_{2,\mathcal{O}} \equiv \Big|\frac{\partial^2 v}{\partial x_1^2}\Big|^2_{0,\mathcal{O}} + \Big|\frac{\partial^2 v}{\partial x_1 \partial x_2}\Big|^2_{0,\mathcal{O}} + \Big|\frac{\partial^2 v}{\partial x_2^2}\Big|^2_{0,\mathcal{O}},$$

and so on.

2. Some basic mathematical instruments. We start with a few very basic inequalities. We will give a rather detailed proof in one dimension, and often only a general idea on the case of several dimensions.

2.1. Poincaré inequality. Let $v \in C^1([0,T])$ with $v = 0$ at $t_0 \in [0,T]$. Then, using the fundamental theorem of calculus we get

$$v(t) = \int_{t_0}^{t} v'(\tau)\, d\tau, \qquad \text{then, taking the absolute values,}$$

$$|v(t)| \leq \int_0^T |v'(\tau)|\, d\tau \qquad \text{then squaring both sides and using C-S}$$

$$|v(t)|^2 \leq \int_0^T |v'(\tau)|\, d\tau^2 \leq T \int_0^T |v'(\tau)|^2\, d\tau \quad \text{and integrating from 0 to } T$$

$$\int_0^T |v(t)|^2\, dt \leq T^2 \int_0^T |v'(\tau)|^2\, d\tau. \tag{2.1}$$

2.2. Trace inequalities. Let $v \in C^1([0,T])$ Then, using the fundamental theorem of calculus on the function v^2:

$$v^2(0) = v^2(t) - \int_0^t (v^2(\tau))'\, d\tau \qquad \text{and then taking the absolute value}$$

$$v^2(0) \leq v^2(t) + \int_0^T |2v(\tau)v'(\tau)|\, d\tau \quad \text{then multiplying and dividing by } \sqrt{T}$$

$$v^2(0) \leq v^2(t) + \int_0^T 2\frac{|v(\tau)|}{\sqrt{T}} \sqrt{T}|v'(\tau)|\, d\tau \qquad \text{and using } 2ab \leq a^2 + b^2$$

$$v^2(0) \leq v^2(t) + \int_0^T \Big(\frac{|v(\tau)|^2}{T} + T|v'(\tau)|^2\Big)\, d\tau; \quad \text{integrating from 0 to } T$$

$$T\, v^2(0) \leq \int_0^T v^2(t)\, dt + T \int_0^T \Big(\frac{v^2(\tau)}{T} + T(v'(\tau))^2\Big)\, d\tau; \quad \text{and dividing by } T$$

$$v^2(0) \leq \int_0^T \Big(\frac{2}{T} v^2(\tau) + T(v'(\tau))^2\Big)\, d\tau \leq \frac{2}{T}\|v\|_0^2 + T|v|_1^2. \tag{2.2}$$

2.3. Comments on the above inequalities. Note that both in (2.1) and in (2.2) the *physical dimensions* of the two terms coincide. In particular in (2.1) we have

$$\Big[\int v^2\,dt\Big] \equiv [v]^2[t] \equiv [v]^2[t]\Big[\frac{[t]^2}{[t]^2}\Big] \equiv [t]^2\Big[\frac{[v]^2[t]}{[t]^2}\Big]$$
$$\equiv [t]^2\Big[\frac{[v]}{[t]}\Big]^2[t] \equiv [t]^2\Big[\int |v'|^2\,dt\Big].$$

Similarly, considering (2.2) we easily check that

$$[v]^2 \equiv \Big[\frac{1}{t}\Big][v]^2[t] \equiv \frac{1}{[t]}\Big[\int v^2\,dt\Big]$$

and

$$[v]^2 \equiv [t]\Big[\frac{v}{t}\Big]^2[t] \equiv [t]\Big[\int |v'|^2\,dt\Big]$$

A rough interpretation of the trace inequality could be: if the value at one point is big, then either the function has a big integral (and you can use the first piece in the right-hand side of (2.2)) or it has a big derivative (and you can use the second piece in the right-hand side of (2.2)). See Figure 1.

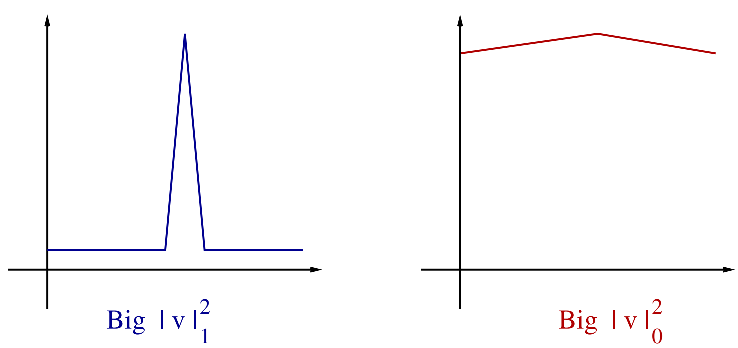


Fig. 1. *Trace inequality*

If you are big at one point (say, at zero), either you go down quickly (and have a big $|\cdot|_1$ norm), or you stay up, and have a big $|\cdot|_0$ norm. See Figure 1.

2.4. 2-d Versions. We summarize here the two-dimensional versions of the above inequalities, with some picture that indicates a possible proof (using the one-dimensional results) in some particular geometries.

In Figure 2 we illustrate Poincaré inequality for functions v vanishing on the edge of $\partial\Omega$ contained in the x axis.

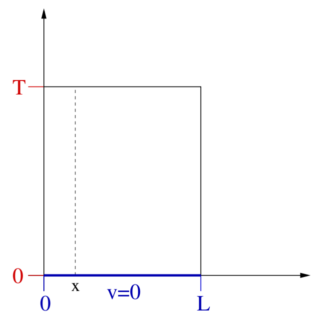


FIG. 2. *Poincaré inequality in 2 dimensions*

From the 1-d case

$$\|v(x,\cdot)\|^2_{0,]0,T[} \leq T^2 |v(x,\cdot)|^2_{1,]0,T[}$$

we deduce

$$\|v\|^2_{0,\Omega} \leq T^2 \|\frac{\partial v}{\partial t}\|^2_{0,\Omega}.$$

At a more general level, we already saw that in the estimate (2.1) the physical dimensions match. It is easy to see that, in a more general bounded domain $K \subset \mathbb{R}^d$ with characteristic length ℓ we have

$$\Big[\|v\|^2_{0,K}\Big] = [v]^2\,[\ell]^d \qquad \text{and} \qquad \Big[|v|^2_{1,K}\Big] = [v]^2\,[\ell]^{d-2}$$

so that a natural guess is

$$\|v\|^2_{0,K} \leq (d(K))^2 |v|^2_{1,K} \tag{2.3}$$

where $d(K)$ is the diameter of K, and where we have to assume, for instance, that v has zero mean value on K (or some other condition that allows to take care of constant functions).

Possibly this is a good moment to point out that the widely used definition

$$\|v\|^2_{1,K} := \|v\|^2_{0,K} + |v|^2_{1,K}$$

doesn't make any sense, unless everything has been adimensionalized: a practice rather unhealthy from the engineering point of view.

Concerning instead the trace inequality, from the one-dimensional case we have, again for the rectangle $K \equiv]0,L[\times]0,T[$

$$|v^2(x,0)| \leq \frac{2}{T}\|v(x,t)\|^2_{0,]0,T[} + T\|\frac{\partial v}{\partial t}(x,t)\|^2_{0,]0,T[} \qquad \forall x \in]0,L[.$$

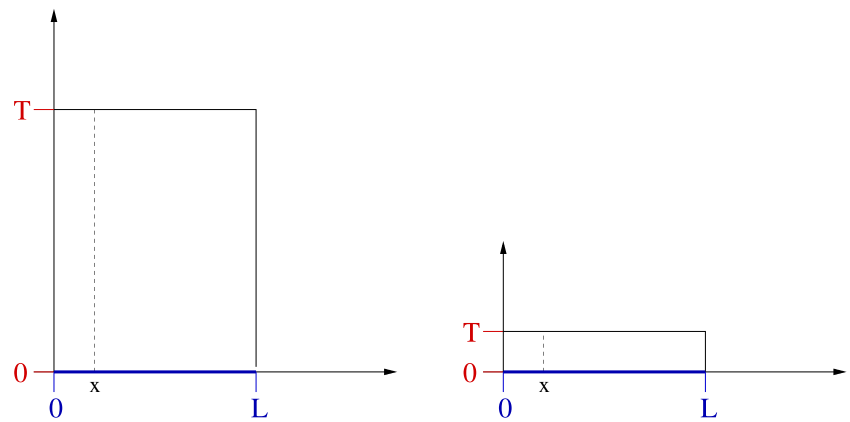


Fig. 3. *Trace inequality in 2 dimensions*

Here, and in what follows, $]a,b[$ denotes the open interval (a,b). By integrating in x from 0 to L we have:

$$\|v(\cdot,0)\|^2_{0,]0,L[} \le \frac{2}{T}\|v\|^2_{0,K} + T|\frac{\partial v}{\partial t}|^2_{0,K}$$

from which we reasonably guess the more general version

$$\|v\|^2_{0,\partial K} \le C\Big(\ell^{-1}\|v\|^2_{0,K} + \ell|v|^2_{1,K}\Big) \tag{2.4}$$

where both the constant C and the characteristic length ℓ can depend on several geometric features (see Figure 3 for a simple example where the bound on the L^2 norm of the trace on the lower edge of the rectangle depends on the height T of the rectangle), but the constant C does not depend on *the size* of K.

2.5. Inverse Inequalities. In a finite dimensional space, all norms are equivalent, in the sense that for any two norms $\|\cdot\|_{@}$ and $\|\cdot\|_{\#}$ there exist two positive constants c and C such that

$$c\,\|v\|_{@} \le \|v\|_{\#} \le C\,\|v\|_{@} \quad \text{for every } v \text{ in the space.}$$

However if the norms are, say, $\|v\|_{0,K}$ and $\|v\|_{1,K}$ the constants c and C might **depend on the size of** K. Indeed we already saw in (2.3) that in the inequality

$$\|v\|^2_{0,K} \le C|v|^2_{1,K} \tag{2.5}$$

the constant C should have physical dimension

$$\Big[C\Big] = [\ell]^2$$

and actually behave as the square of the diameter of K.

On the other hand it is natural to ask the question whether one could have (in one dimension, to start with) an inequality of the type

$$h^2|v|^2_{1,\,]0,h[} \leq C\|v\|^2_{0,\,]0,h[}$$

for some constant C. But taking

$$v = \sin(\frac{2\pi kx}{h}) \qquad k \in \mathbb{N}, \tag{2.6}$$

we have

$$\|v\|^2_{0,\,]0,h[} = \frac{h}{2} \qquad h^2|v|^2_{1,\,]0,h[} = 4\pi^2k^2\frac{h}{2}$$

and our dreams dissolve.

However....we cannot fit **all** the functions (2.6), for **all** the possible integers k, in a single *finite dimensional space*! Hence the inequality

$$h^2|v|^2_{1,\,]0,h[} \leq C\|v\|^2_{0,\,]0,h[}$$

has still some possibilities, **if** we are ready to accept a constant C that depends on the finite dimensional subspace I am using. For instance for $v = (x/h)^r$ (with r integer ≥ 1) we have

$$h^2|v|^2_{1,\,]0,h[} = \frac{h\,r^2}{2r-1} \quad \text{and} \quad \|v\|^2_{0,\,]0,h[} = \frac{h}{2r+1} \tag{2.7}$$

and we might get away with a constant C that depends on the degree of the polynomials (and actually this **is** the case). For instance in the case of (2.7) we have

$$h^2|v|^2_{1,\,]0,h[} \leq \frac{r^2(2r+1)}{2r-1}\|v\|^2_{0,\,]0,h[} \leq 3r^2\,\|v\|^2_{0,\,]0,h[}.$$

More generally, for a family of homothetic elements (see Figure 4) and an integer r there exists a $C = C(r)$ such that

$$|v|^2_{1,K} \leq\, Ch^{-2}\|v\|^2_{0,K}$$

where $h =$ diameter of K, and the inequality holds for every v polynomial of degree $\leq r$, and even more generally (see e.g. [18])

$$|v|^2_{s,K} \leq\, Ch^{-2(s-k)}\|v\|^2_{k,K} \qquad \text{for } s,\, k \text{ integers with } s \geq k \tag{2.8}$$

For a much wider and deeper review of the basic mathematical instruments for dealing with Finite Elements we refer for instance to [13].

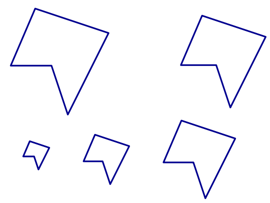
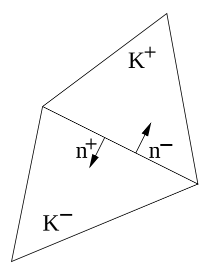


FIG. 4. *Homothetic elements*

3. Some inequalities for DG elements. As we want to deal with spaces of piecewise polynomials that can be discontinuous from one element to the neighboring one, it is natural to begin by considering the simplest case of two triangles (as in the next figure) and functions that are polynomials separately in each triangle (and possibly discontinuous from one triangle to the other).

3.1. Definition of averages and jumps. If K^+ and K^- are two elements with an edge e in common, we denote by $\mathbf{n}^+$ and $\mathbf{n}^-$ the outward unit normal at e of K^+ and K^-, respectively.



Then for every pair (v^+, v^-) of smooth functions on K^+ and K^-, respectively, and for every pair $(\boldsymbol{\tau}^+, \boldsymbol{\tau}^-)$ of smooth vector valued functions on K^+ and K^-, respectively, we set

$$\{\!\{v\}\!\} := \frac{1}{2}(v^+ + v^-,) \qquad [\![v]\!] := v^+\mathbf{n}^+ + v^-\mathbf{n}^-$$

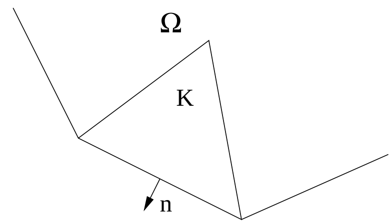
$$\{\!\{\boldsymbol{\tau}\}\!\} := \frac{1}{2}(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-), \qquad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}^+ \otimes \mathbf{n}^+ + \boldsymbol{\tau}^- \otimes \mathbf{n}^-$$

where $\mathbf{a} \otimes \mathbf{b} := \frac{1}{2}(\mathbf{a}\mathbf{b}^T + \mathbf{b}\mathbf{a}^T)$. We will also use the so called *scalar jump*: $[\![\boldsymbol{\tau}]\!]_s \equiv [\![\boldsymbol{\tau}]\!]_{nn} = \boldsymbol{\tau}^+ \cdot \mathbf{n}^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^-$

On a boundary edge, instead, for every smooth function v and for every smooth vector valued function $\boldsymbol{\tau}$ we set

$$\{\!\{v\}\!\} := v \qquad [\![v]\!] := v\mathbf{n}$$

$$\{\!\{\boldsymbol{\tau}\}\!\} := \boldsymbol{\tau}, \qquad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau} \otimes \mathbf{n}, \qquad [\![\boldsymbol{\tau}]\!]_s := \boldsymbol{\tau} \cdot \mathbf{n}$$



3.2. Piecewise integrals. Given a decomposition (that for simplicity we assume compatible) of our computational domain Ω we denote:

- the set of all elements by $\mathcal{T}_h$,
- the set of all edges by $\mathcal{E}_h$,
- the set of all *internal* edges by $\mathcal{E}_h^0$,
- the set of all *boundary* edges by $\mathcal{E}_h^\partial$.

For the sake of simplicity we also assume $\mathcal{T}_h$ to be *quasi-uniform*, meaning that there exists a positive constant γ such that

$$h_{min} \geq \gamma \, h_{max}, \tag{3.1}$$

where h_{min} and h_{max} are the minimum and maximum diameter of the elements of $\mathcal{T}_h$, respectively. This will allow us to simplify notation, and use h to denote the characteristic length of all the elements of $\mathcal{T}_h$. Moreover we set:

$$(f,g)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K f\, g \,\mathrm{d}x \qquad < f,g >_{\mathcal{E}_h} := \sum_{e\in\mathcal{E}_h} \int_e f\, g\,\mathrm{d}s$$

$$< f,g >_{\mathcal{E}_h^0} := \sum_{e\in\mathcal{E}_h^0} \int_e f\, g\,\mathrm{d}s \qquad < f,g >_{\mathcal{E}_h^\partial} := \sum_{e\in\mathcal{E}_h^\partial} \int_e f\, g\,\mathrm{d}s$$

3.3. The Magic formula. For any piecewise smooth scalar function v, and for any piecewise smooth vector valued function $\boldsymbol{\tau}$ we have now

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} v\, \boldsymbol{\tau}\cdot \mathbf{n}_K \,\mathrm{d}s = < [\![v]\!], \{\!\{\boldsymbol{\tau}\}\!\} >_{\mathcal{E}_h} + < \{\!\{v\}\!\}, [\![\boldsymbol{\tau}]\!]_s >_{\mathcal{E}_h^0}. \tag{3.2}$$

The (elementary) proof is based on the algebraic equality:

$$a_1 b_1 - a_2 b_2 = \frac{1}{2}(a_1+a_2)(b_1-b_2) + \frac{1}{2}(b_1+b_2)(a_1-a_2)$$

3.4. Continuity of edge contributions. For piecewise smooth scalar functions u and v, using on each edge

$$|\{\!\{\nabla u\}\!\}| \leq (|\nabla u^+| + |\nabla u^-|)/2, \tag{3.3}$$

and the trace inequality (2.4) we have

$$
\begin{aligned}
< [\![v]\!], \{\!\{\nabla u\}\!\} >_{\mathcal{E}_h} &\leq \sum_{e\in\mathcal{E}_h} \Big| \int_e [\![v]\!] \cdot \{\!\{\nabla u\}\!\}\, ds \Big| \\
&\leq C\Big(\sum_{e\in\mathcal{E}_h} \frac{1}{h} \|[\![v]\!]\|^2_{0,e} \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} (\|\nabla u\|^2_{0,K} + h^2 |\nabla u|^2_{1,K}) \Big)^{1/2}. \quad (3.4)
\end{aligned}
$$

If u is a piecewise polynomial of degree $\leq r$, (3.4) becomes, thanks to the inverse inequality (2.8),

$$
< [\![v]\!], \{\!\{\nabla u\}\!\} >_{\mathcal{E}_h} \leq C_r \Big(\sum_{e\in\mathcal{E}} \frac{1}{h} \|[\![v]\!]\|^2_{0,e} \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} \|\nabla u\|^2_{0,K} \Big)^{1/2}. \quad (3.5)
$$

We define now, for v piecewise smooth and $k \in \mathbb{N}$:

$$
\|v\|^2_{jump} := \sum_{e\in\mathcal{E}_h} \frac{1}{h} \|[\![v]\!]\|^2_{0,e}, \quad |\nabla v|^2_{k,h} := \sum_{K\in\mathcal{T}_h} |\nabla v|^2_{k,K}, \quad |v|^2_{k+1,h} := |\nabla v|^2_{k,h}.
$$

Often we will write $\|\cdot\|_j$ instead of $\|\cdot\|_{jump}$. We also set:

$$
\|v\|^2_{DG} := \|v\|^2_{jump} + |\nabla v|^2_{0,h} + h^2 |\nabla v|^2_{1,h} \quad (3.6)
$$

that, using (2.8), for piecewise polynomials of degree less than or equal to a given degree r is equivalent to

$$
\|v\|^2_{DG} \simeq \|v\|^2_{jump} + |\nabla v|^2_{0,h}. \quad (3.7)
$$

Then our continuity equations (3.4) and (3.5) become, respectively,

$$
< [\![v]\!], \{\!\{\nabla u\}\!\} >_{\mathcal{E}_h} \leq C \|v\|_j (|u|_{1,h} + h^2 |u|_{2,h}) \leq C \|u\|_{DG} \|v\|_{DG} \quad (3.8)
$$

and

$$
< [\![v]\!], \{\!\{\nabla u\}\!\} >_{\mathcal{E}_h} \leq C_r \|v\|_j |u|_{1,h} \leq C_r \|u\|_{DG} \|v\|_{DG}. \quad (3.9)
$$

In a quite similar way, using on each edge

$$
\{\!\{v\}\!\} \leq (|v^+| + |v^-|)/2 \qquad |[\![\nabla u]\!]| \leq (|\nabla u^+| + |\nabla u^-|)/2 \quad (3.10)
$$

one proves the inequalities

$$
< \{\!\{v\}\!\}, [\![\nabla u]\!]_s >_{\mathcal{E}_h} \leq C \|u\|_{DG} \|v\|_{DG} \quad (3.11)
$$

and

$$
< \{\!\{v\}\!\}, [\![\nabla u]\!]_s >_{\mathcal{E}_h} \leq C_r \|u\|_{DG} \|v\|_{DG}. \quad (3.12)
$$

Finally, for v piecewise smooth on a domain $\mathcal{O}$, the trace inequality gives

$$
\|v\|^2_{jump} \leq C\, (h^{-2} \|v\|^2_{0,\mathcal{O}} + |v|^2_{1,h}). \quad (3.13)
$$

4. DG for the Poisson problem. We consider now one of the simplest possible elliptic problems, in order to understand the behavior of DG methods. We will deal only with the more popular variants (SIPG, NIPG, IIPG). For a more detailed analysis of the numerous other variants of DG methods for the Poisson problem we refer for instance to [5].

Given a 2-dimensional domain Ω and $f \in L^2(\Omega)$ we look for u such that

$$-\Delta\, u = f \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega \tag{4.1}$$

Given a decomposition of Ω into triangles (for simplicity) we want to use a DG method. We fix, once and for all, the degree r of the local polynomials, and *we define V_h as the space of functions v_h that are piecewise polynomials of degree $\le r$ on Ω and can be discontinuous from one triangle to another.* For a $v_h \in V_h$ we have

$$\int_\Omega -\Delta\, u\, v_h \mathrm{d}x = \sum_{T\in\mathcal{T}_h} \Big(\int_T \nabla u \cdot \nabla v_h \mathrm{d}x - \int_{\partial T} v_h\, \nabla u \cdot \mathbf{n}_T \mathrm{d}s \Big).$$

For $u =$ exact solution and $v_h \in V_h$ we have

$$\int_\Omega -\Delta\, u\, v_h \mathrm{d}x = \sum_{T\in\mathcal{T}_h} \Big(\int_T \nabla u \cdot \nabla v_h \mathrm{d}x - \int_{\partial T} v_h\, \nabla u \cdot \mathbf{n}_T \mathrm{d}s \Big)$$

that using (3.2) becomes

$$= (\nabla u, \nabla v_h)_{\mathcal{T}_h} - < \{\!\{\nabla u\}\!\}, [\![v_h]\!] >_{\mathcal{E}_h} - < [\![\nabla u]\!]_s, \{\!\{v_h\}\!\} >_{\mathcal{E}_h^0}$$

$$= (\nabla u, \nabla v_h)_{\mathcal{T}_h} - < \{\!\{\nabla u\}\!\}, [\![v_h]\!] >_{\mathcal{E}_h}. \quad \text{since } [\![\nabla u]\!] = 0$$

4.1. The three main variants. We recall that if u is the solution of problem (4.1), then for every piecewise polynomial v_h we have

$$(-\Delta\, u\,,\, v_h)_{\mathcal{T}_h} = (\nabla u, \nabla v_h)_{\mathcal{T}_h} - < \{\!\{\nabla u\}\!\}, [\![v_h]\!] >_{\mathcal{E}_h}\,.$$

For $\delta = -1, 1, 0$ (three variants) and $\alpha_{stab} > 0$ we define the discrete problem as: *Find $u_h \in V_h$ such that*

$$\begin{aligned}(f, v_h)_{\mathcal{T}_h} = (\nabla u_h, \nabla v_h)_{\mathcal{T}_h} &- < \{\!\{\nabla u\}\!\}, [\![v_h]\!] >_{\mathcal{E}_h} \\ &+ \delta < \{\!\{\nabla v_h\}\!\}, [\![u_h]\!] >_{\mathcal{E}_h} + \frac{\alpha_{stab}}{h} < [\![u_h]\!], [\![v_h]\!] >_{\mathcal{E}_h}.\end{aligned}$$

We point out that the terms in the last line are zero for $u_h = u$.

We now set

$$\begin{aligned}a_\delta(u_h, v_h) := (\nabla u_h, \nabla v_h)_{\mathcal{T}_h} &- < \{\!\{\nabla u_h\}\!\}, [\![v_h]\!] >_{\mathcal{E}_h} \\ &+ \delta < \{\!\{\nabla v_h\}\!\}, [\![u_h]\!] >_{\mathcal{E}_h} + \frac{\alpha_{stab}}{h} < [\![u_h]\!], [\![v_h]\!] >_{\mathcal{E}_h}\end{aligned} \tag{4.2}$$

so that the discrete problem becomes

$$a_\delta(u_h, v_h) = (f, v_h)_{\mathcal{T}_h} \qquad \forall v_h \in V_h.$$

We have now to check consistency and stability of all the variants, in order to prove optimal error bounds

4.2. Consistency. We note first that, for every δ and for every α_{stab}, when u is the exact solution we have, for all $v_h \in V_h$:

$$a_\delta(u, v_h) = (\nabla u, \nabla v_h)_{\mathcal{T}_h} - < \{\!\{\nabla u\}\!\}, [\![v_h]\!] >_{\mathcal{E}_h} = (f, v_h).$$

Hence, if u_h solves $a_\delta(u_h, v_h) = (f, v_h)_{\mathcal{T}_h}$ for all $v_h \in V_h$ we have the *Galerkin Orthogonality*

$$a_\delta(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h. \tag{4.3}$$

Recalling that, for piecewise smooth u and v,

$$a_\delta(u, v) := (\nabla u, \nabla v)_{\mathcal{T}_h} - < \{\!\{\nabla u\}\!\}, [\![v]\!] >_{\mathcal{E}_h} + \delta < \{\!\{\nabla v\}\!\}, [\![u]\!] >_{\mathcal{E}_h} + \frac{\alpha_{stab}}{h} < [\![u]\!], [\![v]\!] >_{\mathcal{E}_h},$$

and using the definition of the *jump-norm* together with (3.8) and (3.11) we gather easily that for all piecewise smooth u and v we have

$$a_\delta(u, v) \le C \, \|u\|_{DG} \, \|v\|_{DG}$$

with a constant C independent of the mesh-size.

4.3. Stability. We first recall that, in the subspace V_h, from the inverse inequality (2.8)

$$\|v_h\|^2_{DG} = \|v_h\|^2_j + \|\nabla v_h\|^2_{0,h} + h^2 \|\nabla v_h\|_{1,h} \simeq \|v_h\|^2_j + \|\nabla v_h\|^2_{0,h}.$$

Therefore, from the definition (4.2) we have:

$$a_\delta(v_h, v_h) := |v_h|^2_{1,h} + \alpha_{stab} \|v_h\|^2_j + (\delta - 1) < \{\!\{\nabla v_h\}\!\}, [\![v_h]\!] >_{\mathcal{E}_h}$$

so that

$$a_\delta(v_h, v_h) \ge |v_h|^2_{1,h} + \alpha_{stab} \|v_h\|^2_j - |\delta - 1| C |v_h|_{1,h} \, \|v_h\|_j \tag{4.4}$$

with a constant C independent of the mesh size. At this point it is convenient to recall that, given a quadratic form $x^2 + \alpha_s y^2 - 2\beta xy$, the associated matrix

$$\begin{pmatrix} 1 & -\beta \\ -\beta & \alpha_s \end{pmatrix}$$

is positive definite if and only if $\alpha_s > \beta^2$. In other words, for β fixed, we will always have

$$x^2 + \alpha_s y^2 - \beta xy \ge \alpha^* (x^2 + y^2)$$

for some constant $\alpha^* > 0$, whenever α_s is *big enough.* Going back to (4.4) we deduce that, for every δ,

$$\begin{aligned} a_\delta(v_h, v_h) &\ge |v_h|^2_{1,h} + \alpha_{stab} \|v_h\|^2_j - |\delta - 1| C |v_h|_{1,h} \, \|v_h\|_j \\ &\ge \alpha^* \|v_h\|^2_{DG} \quad \forall v_h \in V_h \end{aligned} \tag{4.5}$$

for some constant $\alpha^* > 0$, whenever α_{stab} is big enough.

4.4. The corresponding methods and some variants. At this point we recall that we had three choices for δ, namely $\delta = -1, 1, 0$, in the discrete bilinear form

$$a_\delta(u_h, v_h) := (\nabla u_h, \nabla v_h)_{\mathcal{T}_h} - < \{\!\{\nabla u_h\}\!\}, [\![v_h]\!] >_{\mathcal{E}_h} + \delta < \{\!\{\nabla v_h\}\!\}, [\![u_h]\!] >_{\mathcal{E}_h} + \frac{\alpha_{stab}}{h} < [\![u_h]\!], [\![v_h]\!] >_{\mathcal{E}_h} .$$

We can now comment that for all the three methods we have consistency (actually: Galerkin orthogonality) and stability (in the subspace) with a constant independent of the mesh size. We can further comment that, in particular

- for $\delta = -1$ (SIPG, [3, 39]) we have a *symmetric method*
- for $\delta = 1$ (NIPG, [10, 35]) we have stability for all α_{stab}
- for $\delta = 0$ (IIPG, [24, 38]) we have a simpler expression.

We can also consider other variants. Always for $\delta = -1, 1, 0$ we denote by Π^e_{r-1} the $L^2(e)$ projection onto the polynomials of degree $\leq r-1$ on e. We consider the following variants

$$a_\delta(u_h, v_h) := (\nabla u_h, \nabla v_h)_{\mathcal{T}_h} - < \{\!\{\nabla u_h\}\!\}, [\![v_h]\!] >_{\mathcal{E}_h} + \delta < \{\!\{\nabla v_h\}\!\}, [\![u_h]\!] >_{\mathcal{E}_h} + \frac{\alpha_{stab}}{h} < \Pi^e_{r-1}[\![u_h]\!], \Pi^e_{r-1}[\![v_h]\!] >_{\mathcal{E}_h} .$$

For $r = 1$, these variants are denoted, respectively, SIPG-0, NIPG-0, and IIPG-0. In particular, IIPG-0 has several nice features that allow an easier construction of solvers and/or pre-conditioners ([9, 8]).

Other variants include the possibility of adding, on top of the stabilizing term

$$\alpha_{stab} h^{-1} < [\![u_h]\!], [\![v_h]\!] >_{\mathcal{E}_h}, \tag{4.6}$$

an additional stabilizing term of the type

$$\beta_{stab} h < [\![\nabla u_h]\!]_s, [\![\nabla v_h]\!]_s >_{\mathcal{E}_h}$$

(see e.g.[16]).

Finally we point out that, for $\delta = 1$, and piecewise linear elements ($r = 1$) we can eliminate the jump-penalty term (4.6), and obtain stability by inserting a bubble function into the local space ([15, 1, 2, 17]).

4.5. Convergence. Let u_I be an approximation of u in V_h. Setting $\delta_h := u_h - u_I$ we have

$$\begin{aligned} \alpha^* \|\delta_h\|^2_{DG} &\leq a_\delta(\delta_h, \delta_h) && \text{(use the definition of } \delta_h) \\ &= a_\delta(u_h - u_I, \delta_h) && \text{(use (4.3))} \\ &= a_\delta(u - u_I, \delta_h) && \text{(use (3.8))} \\ &\leq M \, \|u - u_I\|_{DG} \, \|\delta_h\|_{DG} \end{aligned} \tag{4.7}$$

so that

$$\|u - u_h\|_{DG} \le \|u - u_I\|_{DG} + \|\delta_h\|_{DG} \le (1 + \frac{M}{\alpha_*})\|u - u_I\|_{DG}. \tag{4.8}$$

4.6. Approximation. We assume that u_I is an approximation of u in V_h with the following property: There exists an integer r (the degree of the local polynomials) and a constant C such that

$$|u_I - u|_{s,K} \le C h^{r+1-s} |u|_{r+1,K} \tag{4.9}$$

for all integers s with $0 \le s \le r$, for all h and for all element $K \in \mathcal{T}_h$.

Using (4.9) we bound first the jump norm:

$$\begin{aligned} \|u_I - u\|_j^2 &= \sum_{e\in\mathcal{E}_h} \frac{1}{h} \|[\![u_I - u]\!]\|_{0,e}^2 \le 2 \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K} \frac{1}{h} \|u_I - u\|_{0,e}^2 \\ &\le C \sum_{K\in\mathcal{T}_h} \Big(h^{-2} \|u_I - u\|_{0,K}^2 + |u_I - u|_{1,K}^2 \Big) \le C\, h^{2r} |u|_{r+1,K}^2. \end{aligned}$$

Now, always using (4.9) we bound the second part of the DG norm:

$$\begin{aligned} &\sum_{K\in\mathcal{T}_h} \Big(|\nabla(u_I - u)|_{0,K}^2 + h^2\, |\nabla(u_I - u)|_{1,K}^2 \Big) \\ &\qquad \le \sum_{K\in\mathcal{T}_h} \Big(|u_I - u|_{1,K}^2 + h^2\, |u_I - u|_{2,K}^2 \Big) \le C\, h^{2r} |u|_{r+1,K}^2. \end{aligned}$$

We conclude that under the assumption (4.9) we have

$$\begin{aligned} \|u_I - u\|_{DG}^2 &= \|[\![u_I - u]\!]\|_j^2 + \sum_{K\in\mathcal{T}_h} \Big(|\nabla(u_I - u)|_{0,K}^2 + h^2 |\nabla(u_I - u)|_{1,K}^2 \Big) \\ &\le C\, h^{2r} |u|_{r+1,K}^2. \end{aligned}$$

5. Linear Elasticity.

5.1. The problem. Given a domain Ω and a distributed load $\mathbf{f}$, we define

$$A_\mu \boldsymbol{u} := -\mathbf{div}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad A_\lambda := -\nabla \mathrm{div} \boldsymbol{u} \qquad \mathcal{A} := 2\mu A_\mu + \lambda A_\lambda$$

where μ and λ are the *Lamé* coefficients, depending on the material, and $\boldsymbol{\varepsilon}(\boldsymbol{v}) := (1/2)(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)$ is the usual symmetric gradient. Then we look for $\boldsymbol{u}$ such that

$$\mathcal{A}\boldsymbol{u} = \mathbf{f} \text{ in } \Omega \quad \text{ and } \quad \boldsymbol{u} = 0 \text{ on } \partial\Omega \tag{5.1}$$

The bilinear forms associated to the operators A_μ, A_λ, and $\mathcal{A}$ are given by

$$a_\mu(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \mathrm{d}x, \tag{5.2}$$

$$a_\lambda(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \mathrm{div}\boldsymbol{u}\,\mathrm{div}\boldsymbol{v}\mathrm{d}x, \tag{5.3}$$

and

$$a(\boldsymbol{u},\boldsymbol{v}) := 2\mu a_\mu(\boldsymbol{u},\boldsymbol{v}) + \lambda a_\lambda(\boldsymbol{u},\boldsymbol{v}), \tag{5.4}$$

respectively. Hence, setting $\mathbf{V} := (H_0^1(\Omega))^2$, the variational formulation of (5.1) reads: *find* $\boldsymbol{u} \in$ *such that*

$$a(\boldsymbol{u},\boldsymbol{v}) := 2\mu a_\mu(\boldsymbol{u},\boldsymbol{v}) + \lambda a_\lambda(\boldsymbol{u},\boldsymbol{v}) = (\mathbf{f},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{V}. \tag{5.5}$$

REMARK 5.1. *We point out that from the variational formulation* (5.5)*, taking as usual* $\boldsymbol{v} = \boldsymbol{u}$ *and using the Korn inequality*

$$C_{Korn}\|\boldsymbol{v}\|^2_{\mathbf{V}} \le a_\mu(\boldsymbol{v},\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{V} \tag{5.6}$$

we easily have

$$2\mu\, C_{Korn}\|\boldsymbol{u}\|^2_{\mathbf{V}} + \lambda\|\mathrm{div}\boldsymbol{u}\|^2_{0,\Omega} \le 2\mu a_\mu(\boldsymbol{u},\boldsymbol{u}) + \lambda\|\mathrm{div}\boldsymbol{u}\|^2_{0,\Omega} = (\mathbf{f},\boldsymbol{u}) \tag{5.7}$$

and therefore

$$\sqrt{\mu}\|\boldsymbol{u}\|_{\mathbf{V}} + \sqrt{\lambda}\|\mathrm{div}\boldsymbol{u}\|_{0,\Omega} \le C\,\|\mathbf{f}\|_{|\mathbf{V}'} \tag{5.8}$$

with a constant C *independent of* μ *and* λ*. On the other hand, we also have easily*

$$2\mu a_\mu(\boldsymbol{u},\boldsymbol{v}) + \lambda a_\lambda(\boldsymbol{u},\boldsymbol{v}) \le C\,(2\mu+\lambda)\|\boldsymbol{u}\|_{\mathbf{V}}\,\|\boldsymbol{v}\|_{\mathbf{V}} \tag{5.9}$$

with a constant C *independent of* μ *and* λ*.*

5.2. Discretization. Assume now that we have again a decomposition $\mathcal{T}_h$ of Ω into elements K. On every element K we set

$$a_K(\boldsymbol{u},\boldsymbol{v}) = 2\mu\int_K \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\mathrm{d}x + \lambda\int_K \mathrm{div}\boldsymbol{u}\,\mathrm{div}\boldsymbol{v}\mathrm{d}x.$$

and we recall the Green formula:

$$\begin{aligned} a_K(\boldsymbol{u},\boldsymbol{v}) = -2\mu\int_K (A_\mu\boldsymbol{u})\cdot\boldsymbol{v}\mathrm{d}x - \lambda\int_K (\nabla\mathrm{div}\boldsymbol{u})\cdot\boldsymbol{v}\mathrm{d}x \\ + \int_{\partial K}\Big(2\mu\mathbf{M}^\mu_{\mathbf{n}_K}(\boldsymbol{u}) + \lambda\mathbf{M}^\lambda_{\mathbf{n}_K}(\boldsymbol{u})\Big)\cdot\boldsymbol{v}\mathrm{d}s \end{aligned}$$

where $M^\mu_{\mathbf{n}_K}(\boldsymbol{u}) := \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\mathbf{n}_K$ and $M^\lambda_{\mathbf{n}_K}(\boldsymbol{u}) := (\mathrm{div}\boldsymbol{u})\mathbf{n}_K$. We also recall that the stress field $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} := 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\mathrm{div}\boldsymbol{u}\mathbb{I}\,. \qquad \text{In short} \qquad \boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),$$

where $\mathbb{I}$ is the *identity matrix.* We rewrite

$$a_K(\boldsymbol{u},\boldsymbol{v}) = (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_K, \quad \mathbf{M}_{\mathbf{n}_K}(\boldsymbol{u}) = (2\mu\mathbf{M}^{\mu}_{\mathbf{n}_K} + \lambda\mathbf{M}^{\lambda}_{\mathbf{n}_K})(\boldsymbol{u}).$$

The Green formula can then be written as

$$(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_K = (\mathcal{A}\boldsymbol{u},\boldsymbol{v})_K + < \mathbf{M}_{\mathbf{n}_K}(\boldsymbol{u}),\boldsymbol{v} >_{\partial K}\ .$$

We now introduce, in the spirit of the previous sections, the space $\mathbf{V}_h$ of piecewise polynomial (possibly discontinuous) vectors, concentrating our attention, for simplicity, on the piecewise linear case. For $\boldsymbol{u}$ and $\boldsymbol{v}$ piecewise smooth, summing over K and then applying the correspondent (for this case) of the "magic trick", we have

$$\begin{aligned}\sum_K (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})_K &= (\mathcal{A}\boldsymbol{u},\boldsymbol{v})_{\mathcal{T}_h}\\ &\quad + < \{\!\{\mathbf{M}_{\mathbf{n}_K}(\boldsymbol{u})\}\!\}, [\![\boldsymbol{v}]\!]\cdot\mathbf{n} >_{\mathcal{E}_h} + < [\![\mathbf{M}_{\mathbf{n}_K}(\boldsymbol{u})]\!]\cdot\mathbf{n}, \{\!\{\boldsymbol{v}\}\!\} >_{\mathcal{E}_h}.\end{aligned}$$

When $\boldsymbol{u}$ is the exact solution and $\boldsymbol{v} = \boldsymbol{v}_h$ is an element of $\mathbf{V}_h$ we obviously have $[\![\mathbf{M}_{\mathbf{n}_K}(\boldsymbol{u})]\!]\cdot\mathbf{n} = 0$. Hence

$$(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}_h)_{\mathcal{T}_h} - < \{\!\{\mathbf{M}_{\mathbf{n}_K}(\boldsymbol{u})\}\!\}, [\![\boldsymbol{v}_h]\!]\cdot\mathbf{n} >_{\mathcal{E}_h} = (\mathbf{f},\boldsymbol{v}_h)_{\mathcal{T}_h}. \tag{5.10}$$

5.3. The discretized problem. As before, from (5.10) we take inspiration in order to write the discretized problem. Taking again into account that the regularity of the exact solution implies $[\![\mathbf{M}_{\mathbf{n}_K}(\boldsymbol{u})]\!]\cdot\mathbf{n} = 0$ as well as $[\![\boldsymbol{u}]\!] = 0$, we introduce the bilinear form

$$\begin{aligned}B_h(\boldsymbol{u},\boldsymbol{v}) :=& (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})_{\mathcal{T}_h} - < \{\!\{\mathbf{M}_{\mathbf{n}_K}(\boldsymbol{u})\}\!\}, [\![\boldsymbol{v}]\!]\cdot\mathbf{n} >_{\mathcal{E}_h}\\ &+ \delta < \{\!\{\mathbf{M}_{\mathbf{n}_K}(\boldsymbol{v})\}\!\}, [\![\boldsymbol{u}]\!]\cdot\mathbf{n} >_{\mathcal{E}_h} + \frac{\alpha_{stab}}{h} < [\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!] >_{\mathcal{E}_h},\end{aligned} \tag{5.11}$$

where again we can take $\delta = -1, 1, 0$ (three methods) and $\alpha_{stab} > 0$ is a stabilization parameter. We consider then the discretized problem

$$\textit{Find } \boldsymbol{u}_h \in \mathbf{V}_h \textit{ such that: } \quad B_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = (\mathbf{f},\boldsymbol{v}_h)_{\mathcal{T}_h} \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h. \tag{5.12}$$

It is immediate to see, from (5.10) and (5.12), that *Galerkin orthogonality* holds:

$$B_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = 0 \qquad \forall \boldsymbol{v}_h \in \mathbf{V}_h. \tag{5.13}$$

Moreover, defining, as in (3.6),

$$\|\mathbf{v}\|^2_{DG} := |\boldsymbol{v}|^2_{1,h} + \sum_K h_K^2 |\boldsymbol{v}|^2_{2,K} + \sum_e \frac{1}{h_e} \|[\![\boldsymbol{v}]\!]\|^2_{0,e},$$

we have, with arguments quite similar to the ones of the previous section and using the DG version of (5.6) (see [12]), that for α_{stab} big enough we have *stability*:

$$\exists \kappa_s > 0 \text{ such that } \kappa_s \mu \|\boldsymbol{v}_h\|^2_{DG} \le B_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \mathbf{V}_h, \tag{5.14}$$

with a κ_s independent of μ, λ and h. Similarly, using (5.9) in every element and following again the same arguments used for Poisson problem in the previous section, we can also prove *continuity*

$$\exists M > 0 \text{ s. t. } B_h(\boldsymbol{u}, \boldsymbol{v}) \le M\,(\mu + \lambda)\,\|\boldsymbol{u}\|_{DG}\|\boldsymbol{v}\|_{DG} \quad \forall \boldsymbol{u},\, \boldsymbol{v} \in H^2(\mathcal{T}_h), \tag{5.15}$$

with an M independent of μ, λ and h.

5.4. The nearly incompressible case. As we saw, for every λ and μ positive we have stability (see(5.14)) and continuity (see(5.15)). However, for $\lambda >> \mu$ we have a mismatch between the stability and the continuity constant.

Let us see the effects of this on the classical error estimate. Let $\boldsymbol{u}_I$ be an approximation of the solution $\boldsymbol{u}$ in $\mathbf{V}_h$. Setting $\boldsymbol{\eta}_h := \boldsymbol{u}_h - \boldsymbol{u}_I$ we have, as in (4.7) and (4.8),

$$\begin{aligned} \kappa_s \mu \|\boldsymbol{\eta}_h\|^2_{DG} &\le B_h(\boldsymbol{\eta}_h, \boldsymbol{\eta}_h) = B_h(\boldsymbol{u}_h - \boldsymbol{u}_I, \boldsymbol{\eta}_h) \\ &= B_h(\boldsymbol{u} - \boldsymbol{u}_I, \boldsymbol{\eta}_h) \le M(\mu + \lambda)\,\|\boldsymbol{u} - \boldsymbol{u}_I\|_{DG}\,\|\boldsymbol{\eta}_h\|_{DG} \end{aligned} \tag{5.16}$$

so that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{DG} \le \|\boldsymbol{u} - \boldsymbol{u}_I\|_{DG} + \|\boldsymbol{\eta}_h\|_{DG} \le (1 + \frac{M\,(\mu + \lambda)}{\mu \kappa_s})\|\boldsymbol{u} - \boldsymbol{u}_I\|_{DG}$$

and for $\lambda >> \mu$ we are in deep trouble. Actually, if instead of DG methods we were using traditional H^1-conforming methods we would face the so-called *locking phenomenon*, and the solution $\boldsymbol{u}_h$ of our discretized problem would be bounded, but would not converge to the exact solution $\boldsymbol{u}$.

With DG methods, instead, we have good results: let us see why. As a first step, we recall the so-called *inf-sup* condition for the continuous problem (5.5)

$$\exists\, \beta > 0 \text{ such that } \qquad \inf_{q \in Q} \sup_{\boldsymbol{v} \in \mathbf{V}} \frac{(\mathrm{div}\boldsymbol{v}, q)}{\|q\|_Q\,\|\boldsymbol{v}\|_{\mathbf{V}}} \ge \beta > 0, \tag{5.17}$$

where $Q := L^2(\Omega)/\mathbb{R}$ is the subspace of $L^2(\Omega)$ made of functions with zero mean value.

5.5. Solving Troubles with DG . We can now start proving error bounds for the discretized problem (5.12). We restart as in (5.16), setting now $\boldsymbol{\delta}_h := \boldsymbol{u}_h - \boldsymbol{u}_I$, and stop at

$$\kappa_s \mu \|\boldsymbol{\delta}_h\|^2_{DG} \le B_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) = B_h(\boldsymbol{u}_h - \boldsymbol{u}_I, \boldsymbol{\delta}_h) = B_h(\boldsymbol{u} - \boldsymbol{u}_I, \boldsymbol{\delta}_h). \tag{5.18}$$

Instead of bounding brutally the last term, we now observe that

$$B_h(\boldsymbol{u}-\boldsymbol{u}_I,\boldsymbol{\delta}_h) \le 2\mu\, C\|\boldsymbol{u}-\boldsymbol{u}_I\|_{DG}\,\|\boldsymbol{\delta}_h\|_{DG}+\frac{\alpha_{stab}}{h}\Big|<[\![\boldsymbol{u}-\boldsymbol{u}_I]\!]_s,[\![\boldsymbol{\delta}_h]\!]_s>_{\mathcal{E}_h}\Big|$$
$$+\,\lambda\Big|(\mathrm{div}(\boldsymbol{u}-\boldsymbol{u}_I,\mathrm{div}\boldsymbol{\delta}_h)_{\mathcal{T}_h}+<\{\mathrm{div}(\boldsymbol{u}-\boldsymbol{u}_I)\},[\![\boldsymbol{\delta}_h]\!]\cdot\mathbf{n}>_{\mathcal{E}_h}$$
$$+\,\delta<\{\mathrm{div}\boldsymbol{\delta}_h\},[\![\boldsymbol{u}-\boldsymbol{u}_I]\!]\cdot\mathbf{n}>_{\mathcal{E}_h}\Big|. \quad (5.19)$$

The only way to bound this with a constant that does not depend on λ would be to find a $\boldsymbol{u}_I$, in the subspace $\mathbf{V}_h$, such that:

a) $\displaystyle\int_K \mathrm{div}(\boldsymbol{u}-\boldsymbol{u}_I)\mathrm{d}x = 0 \qquad \forall K;$

b) $\displaystyle\int_e (\boldsymbol{u}-\boldsymbol{u}_I)\cdot\mathbf{n}\mathrm{d}s = 0 \qquad \forall e;$

c) $\|\boldsymbol{u}-\boldsymbol{u}_I\|_{r,K} \le C\,h^{2-r}\|\boldsymbol{u}\|_2 \qquad \forall K, \quad r=0,1.$

Property **a)** would cancel the term $(\mathrm{div}(\boldsymbol{u}-\boldsymbol{u}_I,\mathrm{div}\boldsymbol{\delta}_h)_{\mathcal{T}_h}$, since $\mathrm{div}\boldsymbol{\delta}_h$ (for our piecewise linear elements) is constant in each element. Moreover, (since $\mathrm{div}\boldsymbol{u}_I$ is piecewise constant) it will also imply

$$\|\lambda\mathrm{div}(\boldsymbol{u}-\boldsymbol{u}_I)\|_{0,K} \le C\,h_K\|\lambda\mathrm{div}\boldsymbol{u}\|_{1,K}$$

on every element K.

Property **b)** would cancel the term $<\{\mathrm{div}\boldsymbol{\delta}_h\},[\![\boldsymbol{u}-\boldsymbol{u}_I]\!]\cdot\mathbf{n}>_{\mathcal{E}_h}$ since, again, $\mathrm{div}\boldsymbol{\delta}_h$ is constant in each element (and therefore its trace is constant on each edge).

Property **c)** takes care of the the jump terms. Indeed, combined with (3.13) it will provide

$$\|\boldsymbol{u}-\boldsymbol{u}_I\|_j^2 \le C\,(h^{-2}\|\boldsymbol{u}-\boldsymbol{u}_I\|_{0,\mathcal{O}}^2+|\boldsymbol{u}-\boldsymbol{u}_I|_{1,h}^2) \le C\,h^2\|\boldsymbol{u}\|_{2,\mathcal{O}}^2.$$

Recalling the $H(\mathrm{div})$-conforming Finite Elements (as for instance the BDM_1 spaces [14]), we see that such a $\boldsymbol{u}_I$ can be easily constructed, and our work is concluded.

We note that the BDM_1 is not a subspace of $\mathbf{V}$, so that the above construction could not be used to prove convergence for traditional continuous Galerkin approximations.

Remark 5.2. *The use of $\boldsymbol{u}_I$ in the above construction was instrumental to derive error bounds for fully discontinuous approximations. The idea, however, can be used to construct semi-discontinuous approximations, that is, with $\mathbf{V}_h \subset H(\mathrm{div})$ only, thus guaranteeing continuity of the normal component but not of the tangential component. This approach was used, for instance, in [23] for the Stokes problem.*

6. Alternative formulations. In this Section we recall some basic physical principles that are the basis for several numerical methods. For convenience and simplicity we restrict our attention to linear elasticity problems, although the range of applications (of the physical principles and of the related numerical methods) is much wider.

6.1. Minimum potential energy. The primal formulation of the linear elasticity problem (say, with homogeneous Dirichlet boundary conditions all over $\partial\Omega$) that we saw already in the previous section is based on the *minimum potential energy* principle:

$$\frac{1}{2}(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})) - (\mathbf{f},\boldsymbol{v}) = \text{ minimum,} \tag{6.1}$$

that is equivalent to our variational equation (5.5), that we repeat here for convenience of the reader

$$a(\boldsymbol{u},\boldsymbol{v}) \equiv (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})) = (\mathbf{f},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{V} = (H^1_0(\Omega))^d.$$

6.2. Complementary Energy. We introduce the following notation

$$\boldsymbol{\Sigma} := (L^2(\Omega))^{d\times d}_{sym}$$

$$\forall \mathbf{g} \quad \text{we set} \quad \boldsymbol{\Sigma}_{\mathbf{g}} := \{\boldsymbol{\tau} \in \boldsymbol{\Sigma} \ \text{ with } \ \mathbf{div}\,\boldsymbol{\tau} + \mathbf{g} = 0\}$$

that we are going to use mainly for $\mathbf{g} = \mathbf{f}$ or $\mathbf{g} = \mathbf{0}$. The dual formulation of elasticity problems is based on the *complementary energy* principle:

$$\frac{1}{2}(\mathbb{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\sigma}) = \text{ minimum over } \boldsymbol{\Sigma}_{\mathbf{f}}, \tag{6.2}$$

giving rise to the variational equation

$$\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_{\mathbf{f}} \quad \text{and} \quad (\mathbb{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_{\mathbf{0}}.$$

6.3. The Hellinger-Reissner principle. The Hellinger - Reissner principle is at the basis of the two more common mixed formulations. The principle reads:

$$\frac{1}{2}(\mathbb{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\sigma}) - (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\sigma}) + (\mathbf{f},\boldsymbol{u}) = \text{ stationary.} \tag{6.3}$$

The (Euler-Lagrange) equations of (6.3) are:

$$\begin{cases} (\mathbb{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\tau}) - (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\tau}) = 0 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma} \\ (\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\sigma}) = (\mathbf{f},\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{V} \end{cases} \tag{6.4}$$

This is the *primal mixed* formulation for elasticity.

On the other hand, the Euler-Lagrange equations (6.4) become, upon integration by parts,

$$\begin{cases} (\mathbb{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\tau}) + (\boldsymbol{u},\mathbf{div}\,\boldsymbol{\tau}) = 0 & \forall\boldsymbol{\tau}\in\boldsymbol{\Sigma} \text{ with } \mathbf{div}\boldsymbol{\tau}\in(L^2(\Omega))^d \\ (\boldsymbol{v},\mathbf{div}\,\boldsymbol{\sigma}) = -(\mathbf{f},\boldsymbol{v}) & \forall\boldsymbol{v}\in(L^2(\Omega))^d \end{cases} \tag{6.5}$$

This is the *dual mixed* formulation for elasticity.

6.4. Discontinuous approximations. In the discretization of (6.3) one clearly chooses either $-(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\tau})$ or $(\boldsymbol{u},\mathbf{div}\,\boldsymbol{\tau})$ depending on whether one takes *continuous displacements* or *continuous (normal) stresses*, and this, as we have seen, corresponds to using *primal mixed* or *dual mixed* methods, respectively.

Clearly, if **both** displacements ($\boldsymbol{v}$) **and** stresses ($\boldsymbol{\tau}$) are approximated by **discontinuous** piecewise polynomials, the two above terms are no longer equal. Indeed one has

$$(\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\tau})_h + (\boldsymbol{v},\mathbf{div}\,\boldsymbol{\tau})_h = <\{\!\{\boldsymbol{\tau}\}\!\},[\![\boldsymbol{v}]\!]> + <[\![\boldsymbol{\tau}]\!],\{\!\{\boldsymbol{v}\}\!\}> \tag{6.6}$$

in the best tradition of DG methods. Here, and in what follows, for functions v, w piecewise smooth, $(v,w)_h$ will indicate the scalar product:

$$(v,w)_h = \sum_{K\in\mathcal{T}_h}(v,w)_{0,K}.$$

6.5. Towards Hybrid Methods. Formula (6.6) opens the door towards Hybrid methods. Assume that your discretization allows you to know the displacements only at the interelement boundaries (to fix the ideas, because the displacement trial and test functions are defined, inside each element, to be the solution of some PDE). On the other hand, in this case you can reasonably take them (that is, the displacements) to be *single valued* on the skeleton, so that $[\![\boldsymbol{v}]\!]=0$ in (6.6). Then, if $\mathbf{div}_h\,\boldsymbol{\tau}=0$ in each element K, (6.6) becomes

$$(\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\tau})_h = <[\![\boldsymbol{\tau}]\!],\{\!\{\boldsymbol{v}\}\!\}>, \tag{6.7}$$

so that in (6.4) you can write $<[\![\boldsymbol{\tau}]\!],\{\!\{\boldsymbol{v}\}\!\}>$ instead of $(\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\tau})_h$. Similarly, when $\mathbf{div}_h\,\boldsymbol{\sigma}+\mathbf{f}=0$ equation (6.6) gives

$$(\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\sigma})_h - (\mathbf{f},\boldsymbol{v}) = <[\![\boldsymbol{\sigma}]\!],\{\!\{\boldsymbol{v}\}\!\}>, \tag{6.8}$$

that can be used in the second equation of (6.4).

Using both (6.7) and (6.8) (always for $\mathbf{div}_h\boldsymbol{\tau}=0$ and $\mathbf{div}_h\boldsymbol{\sigma}+\mathbf{f}=0$, respectively) in (6.4), we have then

$$\begin{cases} (\mathbb{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\tau}) - <[\![\boldsymbol{\tau}]\!],\{\!\{\boldsymbol{u}\}\!\}> = 0 & \forall\boldsymbol{\tau}\in\boldsymbol{\Sigma_0} \\ <[\![\boldsymbol{\sigma}]\!],\{\!\{\boldsymbol{v}\}\!\}> = 0 & \forall\boldsymbol{v}\in\mathbf{V} \end{cases} \tag{6.9}$$

6.6. Dual Hybrid Methods. The general strategy for constructing a dual hybrid method is as follows.

Pick up a *particular solution* $\boldsymbol{\sigma}_f$ (such that $\mathbf{div}_h\boldsymbol{\sigma}+\mathbf{f}=0$), and write $\boldsymbol{\sigma}=\boldsymbol{\sigma}_f+\boldsymbol{\sigma}_0$ with $\boldsymbol{\sigma}_0$ to be found. Then look for $\boldsymbol{\sigma}_0$ and $\boldsymbol{u}$ such that

$$\begin{cases} (\mathbb{C}^{-1}(\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_f),\boldsymbol{\tau}_0) - < [\![\boldsymbol{\tau}_0]\!], \{\!\{\boldsymbol{u}\}\!\} > = 0 \quad \forall \boldsymbol{\tau}_0 \in \boldsymbol{\Sigma}_0 \\ < [\![\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_f]\!], \{\!\{\boldsymbol{v}\}\!\} > = 0 \quad \forall \boldsymbol{v} \in \mathbf{V}. \end{cases} \tag{6.10}$$

Note that the values of $\boldsymbol{u}$ and $\boldsymbol{v}$ are used *only at the interelement boundaries*. Separating $\boldsymbol{\sigma}_f$, and considering $\boldsymbol{\sigma}_0$ as the *true stress unknown*, we have then the final formulation: *Find $\boldsymbol{\sigma}_0 \in \boldsymbol{\Sigma_0}$ and $\boldsymbol{u}$ on the skeleton such that*

$$\begin{cases} (\mathbb{C}^{-1}\boldsymbol{\sigma}_0,\boldsymbol{\tau}_0) - < [\![\boldsymbol{\tau}_0]\!], \{\!\{\boldsymbol{u}\}\!\} > = -(\mathbb{C}^{-1}\boldsymbol{\sigma}_f,\boldsymbol{\tau}_0) \quad \forall \boldsymbol{\tau}_0 \in \boldsymbol{\Sigma_0} \\ < [\![\boldsymbol{\sigma}_0]\!], \{\!\{\boldsymbol{v}\}\!\} > = - < [\![\boldsymbol{\sigma}_f]\!], \{\!\{\boldsymbol{v}\}\!\} > \quad \forall \boldsymbol{v} \in \mathbf{V}. \end{cases} \tag{6.11}$$

Note: When you discretize (6.11) you will need *sufficiently many* $\boldsymbol{\tau}_0$ to control $\{\!\{\boldsymbol{u}\}\!\}$....

6.7. Primal Hybrid. Assume now that, in the primal formulation (6.1), we start with *discontinuous* $\boldsymbol{u}$ and $\boldsymbol{v}$. One possibility to do this would be to proceed as in the previous section. Another possibility, however, is to consider that we are actually dealing with a minimization problem, and to consider the interelement continuity (here $[\![\boldsymbol{u}]\!]=0$) as a *constraint*. Then we could introduce a Lagrange multiplier (that will turn out to be the normal component of the stress field $\boldsymbol{\sigma}$ at the interelement boundaries), obtaining the two equations

$$\begin{cases} (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_h + < \{\!\{\boldsymbol{\sigma}\}\!\}, [\![\boldsymbol{v}]\!] > = (\mathbf{f},\boldsymbol{v}) \quad \forall \boldsymbol{v} \\ < \{\!\{\boldsymbol{\tau}\}\!\}, [\![\boldsymbol{u}]\!] > = 0 \quad \forall \boldsymbol{\tau}. \end{cases} \tag{6.12}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are (a priori) discontinuous from one element to the other while $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are defined *only at the interelement boundaries.*

Equations (6.12) are the basis for the **Primal Hybrid Methods**. Note that, in this case, you will need *sufficiently many* $\boldsymbol{v}$'s to control $\{\!\{\boldsymbol{\sigma}\}\!\}$.

6.8. Nonconforming methods. In discretizing (6.12) you will restrict yourself to consider displacement fields $\boldsymbol{u}$ and $\boldsymbol{v}$ in some subspace (of discontinuous p.w. polynomials) $\mathbf{V}_h$. To fix the ideas, assume that the elements of $\mathbf{V}_h$ are, piecewise, polynomials of degree k for some $k \geq 1$. In a similar way, you will assume that you have, at the interelement boundaries, a space $\boldsymbol{\Sigma}_h$ to discretize the normal components of the stress field, made of piecewise (actually: "*egdewise*") polynomials of degree m. We can assume, for the sake of simplicity, that $m < k$ (otherwise, in general, the *inf-sup* condition would fail, since you will not have *sufficiently many* $\boldsymbol{v}$'s to control $\{\!\{\boldsymbol{\sigma}\}\!\}$). At this point you might restrict your attention to displacements that belong to the space V_{nc} defined by

$$V_{nc} := \{\boldsymbol{v} \in \mathbf{V}_h \ \text{ such that } \ < \{\!\{\boldsymbol{\tau}\}\!\}, [\![\boldsymbol{v}]\!] >= 0 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\}.$$

Then you will just look for $\boldsymbol{u} \in V_{nc}$ such that

$$(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_h = (\mathbf{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V_{nc}.$$

This could obviously be seen as using a *Nonconforming Finite Element Method*.

In a quite similar way you could instead start from (6.11), and introduce a discretized space $\boldsymbol{\Sigma}_h$ (made of piecewise polynomial symmetric tensors) and a discretized space $\mathbf{V}$ made of edgewise polynomial vectors on the skeleton. Then you could think of using an $H(\mathbf{div})$-nonconforming space of the form

$$\boldsymbol{\Sigma}_{nc} := \{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \ \text{ such that } \ < \{\!\{\boldsymbol{v}\}\!\}, [\![\boldsymbol{\tau}]\!] >= 0 \quad \forall\, \boldsymbol{v} \in \mathbf{V}_h\}.$$

6.9. Hybridizing dual mixed methods. Let us recall the Hellinger -Reissner principle for *dual mixed* elements (6.5), that uses "continuous stresses" (i.e. "$H(\mathbf{div})$-conforming") and discontinuous displacements. Assume now that you want to use, a priori, *discontinuous stresses* $\boldsymbol{\sigma}$, and enforce back their continuity by means of a Lagrange multiplier.

Then you will consider spaces $\mathbf{V}_h$ and $\boldsymbol{\Sigma}_h$ made of discontinuous piecewise polynomials, and a space of edge-wise polynomials $\mathbb{M}_h$, and look for $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$, $\boldsymbol{u} \in \mathbf{V}_h$, and $\mathcal{U} \in \mathbb{M}_h$ such that

$$(\mathbb{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\boldsymbol{u}, \mathbf{div}\, \boldsymbol{\tau})_h - < \{\!\{\mathcal{U}\}\!\}, [\![\boldsymbol{\tau}]\!] >= 0 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \tag{6.13}$$

$$-(\boldsymbol{v}, \mathbf{div}\, \boldsymbol{\sigma})_h = (\mathbf{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{V}_h \tag{6.14}$$

$$< \{\!\{\mathcal{V}\}\!\}, [\![\boldsymbol{\sigma}]\!] >= 0 \quad \forall\, \mathcal{V} \in \mathbb{M} \tag{6.15}$$

Noting that in (6.13)-(6.15) $\mathbf{V}_h$ and $\boldsymbol{\Sigma}_h$ are "bubbles-spaces" (meaning that you can easily have a basis made of vectors and tensors (respectively) having support in a single element), we can eliminate $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ by *static condensation*, and end up with a system of the type

$$\Lambda(\{\!\{\mathcal{U}\}\!\}, \{\!\{\mathcal{V}\}\!\}) =< F, \{\!\{\mathcal{V}\}\!\} > \quad \forall\, \mathcal{V}$$

whose matrix is, in general, symmetric and positive definite. Remember however that you will still need some sort of *inf-sup* condition. Indeed, recalling the hybridized formulation (6.13)-(6.15), if you are interested only in the $\mathcal{U}$ variable (eliminating the others by static condensation), you cannot avoid an *inf-sup* condition: you need *sufficiently many* $\boldsymbol{\tau}$*'s to control* $\{\!\{\mathcal{U}\}\!\}$ (that appears only in the first equation (6.13)).

This procedure, originally introduced by Fraeijs de Veubeke ([26]) has been first analyzed for Poisson problem in [4] and is used in a rather systematic way when dealing with mixed finite element methods for scalar elliptic problems. Apart from the paramount advantage of going back to a single elliptic problem, the procedure has many additional advantages:

- The Lagrange multiplier $\mathcal{U}$ is a good approximation of $\boldsymbol{u}$ at the interfaces. You can postprocess $\mathcal{U}$ and get an approximation of $\boldsymbol{u}$ *one order better* than the original one coming from the mixed formulation (see e.g.[4]).
- In many cases, $\mathcal{U}$ can be computed directly using suitable *nonconforming* discretizations of the *primal formulation* (see e.g.[34])
- In many problems, $\mathcal{U}$ can be identified with the *flux variable* of Finite Volumes and DG Methods, with many interesting features to be exploited (see, e.g., [20]-[21]).

However, the application to linear elasticity problems is less spectacular, since the combined need to work with symmetric stress fields, to have an *inf-sup* condition and to have $H(\mathrm{div})$ compatibility is a considerable source of troubles. See for instance [6], [11],[22], [29] for some recent attempts using reduced symmetry (and the references therein for earlier attempts), and see as well [7] and [28] for an attempt to use nonconforming elements.

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