Journal of Computational Mathematics
Vol. 00, Pages 000–000 (XXXX)

Virtual Element approximations of the Vector Potential Formulation of Magnetostatic problems

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Abstract. We consider, as a simple model problem, the application of Virtual Element Methods (VEM) to the linear Magnetostatic three-dimensional problem in the classical Vector Potential formulation. The Vector Potential is treated as a triplet of $0-forms$, approximated by nodal VEM spaces. However this is not done using three classical H^1-conforming nodal Virtual Elements, and instead we use the *Stokes Elements* introduced originally in the paper *Divergence free Virtual Elements for the Stokes problem on polygonal meshes* (ESAIM Math. Model. Numer. Anal. **51** (2017), 509–535) for the treatment of incompressible fluids.

Keywords. Virtual Element Methods, Serendipity, Magnetostatic problems, Vector Potential.

Math. classification. 65N30.

1. Introduction

In recent times, for the discretization of PDEs, there has been a considerable interest in the use of decompositions of the computational domain in polytopes. See for instance [5, 9, 26, 27, 45, 46, 51, 52, 55, 60, 62, 69, 70] and the refereences therein.

Virtual Elements were introduced a few years ago [11, 15] for the discretization of H^1-conforming spaces to be used in the numerical approximation of PDEs on very general decompositions into polygons or polyhedra, and had a wide diffusion in the last years. On one hand they were extended to the discretization of more general spaces, as H^1-nonconforming (e.g. [8]), $H(\mathrm{div})$-conforming, and $H(\mathbf{curl})$-conforming (e.g, [16, 18]). On the other hand they had other theoretical extensions through the Serendipity approach (see for instance [17]), and moreover their use has been extended to a wide variety of problems (see e.g. [1, 21, 43, 72] and the references therein). Also, the study of the possible usable decomposition and of the related interpolation errors made significant progresses in the last couple of years (see e.g. [19, 33, 35, 41, 65]).

The list of VEM contributions in the literature is nowadays quite large; we mention, e.g., [2, 6, 20, 22, 23, 36, 38, 39, 40, 49, 56, 57, 66, 67, 68, 71, 72, 73, 74] and the references therein.

Here we deal, as a simple model problem in electromagnetism, with the classical magnetostatic problem in a smooth-enough bounded domain Ω in $\mathbb{R}^3$, simply connected with a connected boundary:

given $\boldsymbol{j} \in H(\mathrm{div};\Omega)$ with $\mathrm{div}\boldsymbol{j} = 0$ in Ω, and given $\mu = \mu(\boldsymbol{x})$ with $0 < M_0 \leq \mu \leq M_1$,

$$\begin{cases} \text{find } \boldsymbol{H} \in H(\mathbf{curl};\Omega) \text{ and } \boldsymbol{B} \in H(\mathrm{div};\Omega) \text{ such that:} \\ \mathbf{curl}\boldsymbol{H} = \boldsymbol{j} \text{ and } \mathrm{div}\boldsymbol{B} = 0, \text{ with } \boldsymbol{B} = \mu\boldsymbol{H} \text{ in } \Omega, \\ \text{with the boundary conditions } \boldsymbol{B}\cdot\boldsymbol{n} = 0 \,(\text{or } \boldsymbol{H}\wedge\boldsymbol{n} = 0) \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

Clearly the formulation needs the usual adjustments if Ω is not simply connected (or does not have a simply connected boundary) in order to have uniqueness of the solution, regardless of the numerical method that one has in mind to solve it numerically. We will not deal with these issues here.

In some previous papers [12, 13] we dealt with two-dimensional and three-dimensional approximations of the above magneto-static problems using the variational formulation of Kikuchi [61]. Here, instead, we tackle the discretization of the problem in the (more classical) *Vector Potential* formulation (see e.g. [25] and the references therein). Other important contributions to the numerical approximation of Magnetostatic problems can be found, for instance, in [4, 50, 64, 25] and the references therein.

As far as we know, the vector potential formulation has not yet been tackled with Virtual Elements, and the possible benefits due to the great freedom in the element shapes have not yet been investigated in practice. Here, in particular, we also take advantage from the use of the Virtual Element spaces introduced in [20] for dealing with Stokes problems (that however are used here in a slightly different way). This choice allows the use (for test and trial functions) of vector-valued fields that have a *constant divergence* in each element. We think that, together with the generality in the element geometry, this could represent a nice feature (in particular for higher order approximations) when compared to more classical Finite Element formulations. We also point out that here the computed vector potential will have a divergence that is exactly zero.

It has to be pointed out from the very beginning that the major interest of applying VEMs (as presented here) to the vector-potential formulation is, in practice, restricted to cases in which the solution is expected to be reasonably smooth, and hence where higher order methods could be more profitable. In particular, they cannot be applied (in the present form) to situations where the computational domain has re-entrant corner, since in that case (see e.g. [47, 30]) one cannot approximate the solution with vectors belonging to $(H^1)^3$ (as is the case for the VEMs proposed here). The same problem could occur for discontinuous coefficients (see, e.g., [48, 29]). Needless to say, it would be very interesting to extend to VEMs the tricks that have been developed for FEMs in order to use nodal elements (as for instance in [30, 32, 59, 37, 10], and the references therein). Similarly, it would also be interesting to extend to VEMs some of the ideas used in FEMs to deal with unbounded domains, as for instance in [24, 31, 63, 58]. All these issues, however, escape the aims of the present paper, where for simplicity we prove error estimates in the case where *the solution is regular*. For simplicity, we assume that Ω is a convex polyhedron and μ is constant.

A layout of the paper is as follows: in Section 2 we will introduce some basic notation, and recall some well known properties of polynomial spaces. Nothing is new there. In Section 3 we will first recall, in Subsection 3.1, the Vector Potential approach to (1.1) and its variational formulation. Then, in Subsections 3.2 and 3.3 we present the *local* two-dimensional Virtual Element spaces (of nodal type) to be used on the inter-element boundaries. Here we use a simpler (although less powerful) version of the Serendipity spaces of [17], corresponding, roughly, to the approach that is called *lazy choice* there.

Note that, instead, always with the aim of keeping the presentation as simple as possible, we do not use three-dimensional Serendipity elements to reduce the number of degrees of freedom inside the polyhedrons. Actually, as is well known, in a three-dimensional problem it is more important to reduce the number of degrees of freedom *on faces* (where static condensation is quite cumbersome to perform), than to reduce the number of degrees of freedom *internal* to polyhedrons (that can be tackled by static condensation, which is practically done in an almost automatic way by several recent direct solvers).

In Subsection 3.4 we then discuss the Virtual Element spaces to be used *inside* each polyhedron. As we said, on each face of the boundary we use a simplified version of the Serendipity elements of [17], and inside the polyhedron we use spaces inspired by [20], avoiding 3D Serendipity versions. Note that, however, from the use of a constant divergence we still have some gain in the number of internal degrees of freedom. Then in Subsection 3.5 we discuss which quantities (in our discrete spaces) are actually *computable*, out of the degrees of freedom.

In Section 4 we introduce the *global* Virtual Element spaces. We discuss their most important properties, and then we use them to define the discretised problem and to show existence and uniqueness of its solution.

In Section 5 we prove the a priori error bounds. First we bound the error between exact and approximate solutions in terms of the approximation errors (of the exact solution within the Virtual Element Spaces). Then we recall some (already classic) assumptions on the decompositions that allow to estimate the approximation errors, and we use them to derive the final error estimates. This is the part of the paper in which the regularity of the solution is used.

2. Notation and well known properties of Polynomial spaces

In two dimensions, we will denote by $\boldsymbol{x}$ the independent variable, using $\boldsymbol{x} = (x, y)$ or (more often) $\boldsymbol{x} = (x_1, x_2)$ following the circumstances. We will also use $\boldsymbol{x}^{\perp} := (-x_2, x_1)$, and in general, for a vector $\boldsymbol{v} \equiv (v_1, v_2)$: $\boldsymbol{v}^{\perp} := (-v_2, v_1)$. Moreover, for a vector $\boldsymbol{v}$ and a scalar q we will write

$$\mathrm{rot}\boldsymbol{v} := \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}, \qquad \mathbf{rot}\, q := \Big(\frac{\partial q}{\partial y}, -\frac{\partial q}{\partial x}\Big)^T. \tag{2.1}$$

In three dimensions we will denote again by $\boldsymbol{x}$ the independent variable when no confusion is likely to occur, using also $\boldsymbol{x} = (x, y, z)$ or $\boldsymbol{x} = (x_1, x_2, x_3)$, still following the circumstances.

We recall some commonly used functional spaces. On a domain $\mathcal{O} \subseteq \mathbb{R}^3$ we have

$$\begin{aligned}
&H(\mathrm{div}; \mathcal{O}) = \{\boldsymbol{v} \in [L^2(\mathcal{O})]^3 \text{ with } \mathrm{div}\boldsymbol{v} \in L^2(\mathcal{O})\},\\
&H_0(\mathrm{div}; \mathcal{O}) = \{\boldsymbol{\varphi} \in H(\mathrm{div}; \mathcal{O}) \text{ with } \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \text{ on } \partial\mathcal{O}\},\\
&H(\mathbf{curl}; \mathcal{O}) = \{\boldsymbol{v} \in [L^2(\mathcal{O})]^3 \text{ with } \mathbf{curl}\boldsymbol{v} \in [L^2(\mathcal{O})]^3\},\\
&H_0(\mathbf{curl}; \mathcal{O}) = \{\boldsymbol{v} \in H(\mathbf{curl}; \mathcal{O}) \text{ with } \boldsymbol{v} \wedge \boldsymbol{n} = 0 \text{ on } \partial\mathcal{O}\},\\
&H^1(\mathcal{O}) = \{q \in L^2(\mathcal{O}) \text{ with } \mathbf{grad}\, q \in (L^2(\mathcal{O}))^3\},\\
&H^1_0(\mathcal{O}) = \{q \in H^1(\mathcal{O}) \text{ with } q = 0 \text{ on } \partial\mathcal{O}\}.
\end{aligned}$$

For an integer $s \geq -1$ we will denote by $\mathbb{P}_s$ the space of polynomials of degree $\leq s$. Following a common convention, $\mathbb{P}_{-1} \equiv \{0\}$ and $\mathbb{P}_0 \equiv \mathbb{R}$. Moreover, for $s \geq 0$

$$\mathbb{P}^h_s := \{\text{homogeneous polynomials of degree } s\}, \text{ and } \mathbb{P}^0_s(\mathcal{O}) := \{q \in \mathbb{P}_s \text{ s. t. } \int_{\mathcal{O}} q \,\mathrm{d}\mathcal{O} = 0\}. \tag{2.2}$$

For $d = 1, 2, 3$ we denote the dimension of the space $\mathbb{P}_s$ in d space dimensions by $\pi_{d,s}$:

$$\pi_{1,s} = s+1, \quad \pi_{2,s} = \frac{(s+1)(s+2)}{2}, \quad \pi_{3,s} = \frac{(s+1)(s+2)(s+3)}{6}. \tag{2.3}$$

Obviously, in d space dimensions, the (common) value of the dimension of $\mathbb{P}^0_s$ and of the space $\nabla(\mathbb{P}_s)$ will be equal to $\pi_{d,s} - 1$. The following decompositions of polynomial vector spaces are well known and will be useful in what follows.

$$(\mathbb{P}_s)^3 = \mathbf{curl}((\mathbb{P}_{s+1})^3) \oplus \boldsymbol{x}\mathbb{P}_{s-1}, \quad \text{ and } \quad (\mathbb{P}_s)^3 = \mathbf{grad}(\mathbb{P}_{s+1}) \oplus \boldsymbol{x} \wedge (\mathbb{P}_{s-1})^3. \tag{2.4}$$

Taking the **curl** of the second of (2.4) we also get :

$$\mathbf{curl}(\mathbb{P}_s)^3 = \mathbf{curl}(\boldsymbol{x} \wedge (\mathbb{P}_{s-1})^3) \tag{2.5}$$

which used in the first of (2.4) gives:

$$(\mathbb{P}_s)^3 = \mathbf{curl}(\boldsymbol{x} \wedge (\mathbb{P}_s)^3) \oplus \boldsymbol{x}\mathbb{P}_{s-1}. \tag{2.6}$$

In what follows, when dealing with the *faces* of a polyhedron (or of a polyhedral decomposition) we shall use two-dimensional differential operators that act on the restrictions to faces of scalar functions that are defined on a three-dimensional domain. Similarly, for vector valued functions we will use two-dimensional differential operators that act on the restrictions to faces of the tangential components. In many cases, no confusion will be likely to occur; however, to stay on the safe side, we will often use a superscript τ to denote the tangential components of a three-dimensional vector, and a subscript f to indicate the two-dimensional differential operator. Hence, to fix ideas, if a face has equation $x_3 = 0$ then $\boldsymbol{x}^\tau := (x_1, x_2)$ and, say, $\mathrm{div}_f \boldsymbol{v}^\tau := \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$.

3. The problem and the local spaces

3.1. The Vector Potential formulation

We recall the classical Vector Potential Formulation. The idea is to present the *magnetic induction field* $\boldsymbol{B}$ $(= \mu\boldsymbol{H})$ as the **curl** of a vector potential $\boldsymbol{A}$:

$$\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A}. \tag{3.1}$$

Then the solenoidal property $\mathrm{div}\boldsymbol{B} = 0$ will be automatically satisfied, and the Ampère law becomes

$$\mathbf{curl}\,\boldsymbol{H} \equiv \mathbf{curl}(\mu^{-1}\mathbf{curl}\,\boldsymbol{A}) = \boldsymbol{j}. \tag{3.2}$$

In turn the boundary condition $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ will be satisfied if we require that $\boldsymbol{A} \wedge \boldsymbol{n} = 0$ on $\partial\Omega$. Hence we define the space

$$\mathcal{A} := H_0(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \Omega). \tag{3.3}$$

It is easy to check that

$$\|\boldsymbol{v}\|^2_{\mathcal{A}} := \|\mu^{-1/2}\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\boldsymbol{v}\|^2_{0,\Omega} \tag{3.4}$$

is a (Hilbert) norm on $\mathcal{A}$. In our simplified assumptions (Ω convex and μ constant) we immediately have

$$c_1\|\boldsymbol{v}\|_{1,\Omega} \le \|\boldsymbol{v}\|_{\mathcal{A}} \le c_2\|\boldsymbol{v}\|_{1,\Omega} \quad \forall \boldsymbol{v} \in \mathcal{A} \tag{3.5}$$

with c_1 and c_2 depending on Ω and μ. We point out that this would hold under much milder assumptions (see e.g. [42] and, mostly, the references therein), but, as we said, we are not going to discuss regularity properties here.

We will use one of the most classical variational formulations of the vector-potential equations (see for instance [25]). We consider the problem

$$\begin{cases} \text{find } \boldsymbol{A} \in \mathcal{A} \text{ such that:} \\ a(\boldsymbol{A}, \boldsymbol{v}) := \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\Omega + \int_\Omega \mathrm{div}\boldsymbol{A}\,\mathrm{div}\boldsymbol{v}\,\mathrm{d}\Omega = \int_\Omega \boldsymbol{j} \cdot \boldsymbol{v}\,\mathrm{d}\Omega \quad \forall \boldsymbol{v} \in \mathcal{A}. \end{cases} \tag{3.6}$$

It is clear that

$$a(\boldsymbol{v}, \boldsymbol{v}) = \|\boldsymbol{v}\|^2_{\mathcal{A}} \tag{3.7}$$

so that (3.6) has a unique solution in $\mathcal{A}$. Then we check that the solution of (3.6) verifies $\mathrm{div}\boldsymbol{A} = 0$. For this we take $\varphi \in H^1_0(\Omega)$ such that $\Delta\varphi = \mathrm{div}\boldsymbol{A}$, and then we take $\boldsymbol{v} = \mathbf{grad}\varphi$ (that clearly belongs to $\mathcal{A}$). Then $\mathbf{curl}\boldsymbol{v} = 0$ and $\int_\Omega \boldsymbol{j} \cdot \boldsymbol{v}\,\mathrm{d}\Omega = 0$ as well (since $\mathrm{div}\boldsymbol{j} = 0$). Hence from (3.6) we have $\mathrm{div}\boldsymbol{A} = 0$. It also follows immediately that for $\boldsymbol{B} := \mathbf{curl}\boldsymbol{A}$ one gets $\mathrm{div}\boldsymbol{B} = 0$. Moreover from

(3.6), using $\mathrm{div}\boldsymbol{A} = 0$ and integrating by parts, we have now that $\mathbf{curl}(\mu^{-1}\mathbf{curl}\boldsymbol{A}) = \boldsymbol{j}$. Hence setting $\boldsymbol{H} := \mu^{-1}\boldsymbol{B}$ we have $\mathbf{curl}\boldsymbol{H} = \mathbf{curl}(\mu^{-1}\boldsymbol{B}) = \mathbf{curl}(\mu^{-1}\mathbf{curl}\boldsymbol{A}) = \boldsymbol{j}$. Finally, on the boundary $\partial\Omega$ we have $\boldsymbol{B}\cdot\boldsymbol{n} = \mathrm{rot}(\boldsymbol{A}\wedge\boldsymbol{n}) = 0$.

Remark 3.1. The extension of the formulation to the case where $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ only on a subset Γ of the boundary (and $\boldsymbol{H}\wedge\boldsymbol{n} = 0$ on the remaining part) is immediate by substituting the space (3.3) with

$$\mathcal{A} := \{\boldsymbol{v} \in H(\mathbf{curl};\Omega)\cap H(\mathrm{div};\Omega) \text{ with } \boldsymbol{v}\wedge\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

In the following, in order to keep the notation simpler, we stick to (3.3), the extension to the more general case being trivial.

3.2. **The local spaces on faces**

We assume that we are given a sequence of decompositions $\{\mathcal{T}_h\}_h$ of the computational domain Ω into polyhedrons P. For every polyhedron P we define

$$h_{\mathrm{P}} := \text{diameter of P} \tag{3.8}$$

and for every decomposition $\mathcal{T}_h$ we set

$$|h| := \sup_{\mathrm{P}\in\mathcal{T}_h} h_{\mathrm{P}}. \tag{3.9}$$

We also assume that each P is simply connected and convex, with all its faces also simply connected and convex. (For the treatment of non-convex faces we refer to [17]).

We note that the construction of the *local spaces* and of the whole discrete problem can be carried out in the case where μ is just *piecewise constant* (and not necessarily constant all over Ω). We will go back to the stricter assumptions in Section 5.

For the treatment of Virtual Element discretizations of problems with variable coefficients we refer, for instance, to [14] and references therein.

We will now design the Virtual Element approximation of (3.6) of order $k \geq 1$ on $\mathcal{T}_h$. We begin with the definition of the local spaces, and in particular we start by defining suitable VEM spaces on the faces. We are going to use, essentially, a particular choice of Serendipity nodal Virtual Element spaces of [17]. For this, for every integer $k \geq 1$ and for every face f we consider the Virtual Element space

$$\widetilde{V}_k(f) := \{v \in C^0(\overline{f}) \text{ such that } v_{|e} \in \mathbb{P}_k(e)\ \forall \text{ edge } e, \text{ and } \Delta_f v \in \mathbb{P}_k(f)\}. \tag{3.10}$$

In $\widetilde{V}_k(f)$ we have the natural degrees of freedom

$$\bullet \text{ value of } v(\nu), \text{ for every vertex } \nu \text{ of } f, \tag{3.11}$$

$$\bullet \text{ (for } k \geq 2) \text{ value of } \int_e v\, q_{k-2}\,\mathrm{d}e,\ \forall q_{k-2} \in \mathbb{P}_{k-2}(e), \text{ for every edge } e \text{ of } f, \tag{3.12}$$

$$\bullet \text{ value of } \int_f v\, q_k\,\mathrm{d}f,\ \forall q_k \in \mathbb{P}_k(f). \tag{3.13}$$

In $\widetilde{V}_k(f)$ we want to identify a subspace that contains all polynomials of degree $\leq k$ but uses less degrees of freedom. For this, we will use a simplified version of the Serendipity elements of [17]. We consider first the space of $\mathbb{P}_k$-bubbles on f

$$B_k(f) := \{q \in \mathbb{P}_k(f) \text{ such that } q_{|\partial f} \equiv 0\}. \tag{3.14}$$

Note that $B_k \equiv \{0\}$ for $k \leq 2$, regardless of the number of edges of f (that, obviously, will always be ≥ 3), so that the first non-trivial bubble appears on a triangular face for $k = 3$. In general, the dimension of $B_k(f)$ will always verify

$$\text{dimension of } B_k(f) =: \beta_k(f) \leq \pi_{2,k-3} \tag{3.15}$$

where $\pi_{2,k-3}$ is the number of $\mathbb{P}_k$ bubbles on a triangle. We recall that we assumed, for the sake of simplicity, that f is convex, and we define a projector $v \to \Pi_{k,f} v$ from $\widetilde{V}_k(f)$ to $\mathbb{P}_k$ as the *least squares solution* of the system:

$$\bullet \int_{\partial f} (v - \Pi_{k,f} v)\, q_k \,\mathrm{d}s = 0 \;\forall q_k \in \mathbb{P}_k(f) \tag{3.16}$$

$$\bullet \text{ (for } k \geq 3\text{)} \int_f (v - \Pi_{k,f} v)\, q_{k-3} \,\mathrm{d}f = 0 \;\forall q_{k-3} \in \mathbb{P}_{k-3}(f). \tag{3.17}$$

Proposition 3.2. *For a triangular face f the system* (3.16)-(3.17) *has a unique solution. In the other cases, the system* (3.16)-(3.17) *is over-determined (i.e. it has more equations than unknowns) but its least square solution is unique.*

Proof. We note that the number of nontrivial equations in (3.16) is equal to the dimension of $\mathbb{P}_k$ minus the dimension of $B_k(f)$. For a *triangular* f, the dimension of $B_k(f)$ is equal to the dimension of $\mathbb{P}_{k-3}$, so that (3.16)-(3.17) is a square system, and it is immediate to check that the associated matrix is non-singular. For a *non triangular* face f the dimension of $B_k(f)$, according to (3.15), is *smaller* than $\pi_{2,k-3}$, and the number of equations of the system (3.16)-(3.17) is equal to $[\pi_{2,k} - \beta_k(f)] + [\pi_{2,k-3}]$, that is *bigger* than $\pi_{2,k}$ (the number of unknowns). To see that the least-squares solution is uniquely determined we have to check that for $v = 0$ the only solution is given by $\Pi_{k,f} v = 0$. For this, observe that if p, in $\mathbb{P}_k(f)$, vanishes on ∂f then either $p \equiv 0$, or p must have the form

$$p = b_\eta q^*_{k-\eta} \tag{3.18}$$

where:

- η (> 3) is the minimum number of straight lines necessary to cover ∂f,
- b_η is a polynomial in $\mathbb{P}_\eta(f)$ that vanishes on ∂f and is positive inside (remember, f is convex)
- $q^*_{k-\eta} \in \mathbb{P}_{k-\eta}(f) \subset \mathbb{P}_{k-3}(f)$.

For $v = 0$ (3.17) would then imply (taking $q_{k-3} = q^*_{k-\eta}$) that

$$0 = \int_f p\, q^*_{k-\eta} \,\mathrm{d}f = \int_f b_\eta (q^*_{k-\eta})^2 \,\mathrm{d}f, \tag{3.19}$$

and finally $p = 0$. ∎

Once the projection operator $\Pi_{k,f}$ has been defined, we can introduce for every face f the Serendipity VEM space $V_{S,k}(f)$.

Definition 3.3. The Serendipity VEM space $V_{S,k}(f)$ is defined as the subspace of $\widetilde{V}_k(f)$ made of elements v such that

$$\int_f (v - \Pi_{k,f} v)\, q^h_s \,\mathrm{d}f = 0 \quad \forall\, q^h_s \in \mathbb{P}^h_s(f),\ \forall \text{ non-negative integer } s \in [k-2, k]. \tag{3.20}$$

It is easy to see that a uni-solvent set of degrees of freedom for $V_{S,k}(f)$ is given by

$$\bullet \text{ value of } v(\nu), \text{ for every vertex } \nu \text{ of } f, \tag{3.21}$$

$$\bullet \text{ (for } k \geq 2\text{) value of } \int_e v\, q_{k-2} \,\mathrm{d}e,\ \forall q_{k-2} \in \mathbb{P}_{k-2}(e), \text{ for every edge } e \text{ of } f, \tag{3.22}$$

$$\bullet \text{ (for } k \geq 3\text{) value of } \int_f v\, q_{k-3} \,\mathrm{d}f,\ \forall q_{k-3} \in \mathbb{P}_{k-3}(f), \tag{3.23}$$

and consequently its dimension is given by $(kN_\nu(f) + \pi_{2,k-3})$ where $N_\nu(f)$ is the number of vertices of the face f and $\pi_{2,k-3} = 0$ for $k \leq 2$.

We point out that every $v \in V_{S,k}$ is still an element of $\widetilde{V}_k(f)$, and from its degrees of freedom (3.21)-(3.23) we are able to compute (through $\Pi_{k,f}$ and (3.20)) all its degrees of freedom in $\widetilde{V}_k(f)$, and in particular the quantities

$$\int_f v\, q_k \,\mathrm{d}f \quad \forall q_k \in \mathbb{P}_k(f) \text{ are computable for } v \in V_{S,k}(f). \tag{3.24}$$

Remark 3.4. It is easy to see that: if the face f has more than 3 edges (and $\eta > 3$, meaning that the boundary of f cannot be covered using only three straight lines), then the projection operator $\Pi_{k,f}$ could be defined using in (3.17) only the polynomials of degree $k-4$, as it would be the case in FEMs (see e.g. [3, 44]). But, in VEMs (see e.g. [17]), for a bigger and bigger η we could use fewer and fewer polynomials in (3.17). In other words: the Serendipity reduction for VEMs (as presented in [17]) becomes more and more powerful when the number of edges increases. Our present choice instead (reminiscent of what is called *the lazy choice* in [17]), ensures only a limited reduction of the number of internal degrees of freedom, but *has the advantage of working in general.* It also avoids the necessity to detect more delicate situations as, for instance, the case of a **nearly degenerate quadrilateral** with an internal angle very close to π radiants. This is a case that could be a considerable source of problems with classical Serendipity FEMs or other types of Serendipity VEMs, but that with the present choice is perfectly acceptable without any additional work, including the case of an angle **exactly** equal to π. Needless to say, if the same mesh is going to be used for many resolutions, and it contains many elements with more than 3 edges, then it would be worth to spend some additional effort on every element, and use the elements of [15] (that, in general, would be much slimmer) instead. All these choices will not affect in a significant way the theoretical treatment that follows in the present paper. □

3.3. Traces of the local spaces on ∂P

Having defined our spaces on every face f, for a given polyhedron P we can define the space of traces

$$\mathbb{B}_k(\partial\mathrm{P}) := \{\boldsymbol{v} \in (C^0(\partial\mathrm{P}))^3 \text{ such that } \boldsymbol{v}_{|f} \in (V_{S,k}(f))^3 \;\forall \text{ face } f \text{ in } \partial\mathrm{P}\}. \tag{3.25}$$

Proposition 3.5. *A unisolvent set of degrees of freedom for $\mathbb{B}_k(f)$ is given by*

- *value of $\boldsymbol{v}(\nu)$, for every vertex ν of P,* (3.26)
- *(for $k \geq 2$) value of $\int_e \boldsymbol{v}\cdot\boldsymbol{q}_{k-2}\,\mathrm{d}e$, $\forall \boldsymbol{q}_{k-2} \in (\mathbb{P}_{k-2}(e))^3$, for every edge e of P,* (3.27)
- *(for $k \geq 3$) value of $\int_f \boldsymbol{v}\cdot\boldsymbol{q}_{k-3}\,\mathrm{d}f$, $\forall \boldsymbol{q}_{k-3} \in (\mathbb{P}_{k-3}(f))^3$, for every face f of P.* (3.28)

Proof. The result follows immediately from the information on the degrees of freedom, taking into account the continuity requirements on edges and vertexes. ■

Remark 3.6. The degrees of freedom (3.12) and (3.27) could be replaced by the value of $\boldsymbol{v}$ at $k-1$ distinct points in each edge. □

3.4. Local spaces on a polyhedron

In order to define the spaces *inside* P we follow the basic ideas of [20], and we set

$$\mathcal{A}_k(\mathrm{P}) := \{\boldsymbol{v} \in (C^0(\overline{\mathrm{P}}))^3 \text{ such that } \boldsymbol{v}_{|\partial\mathrm{P}} \in \mathbb{B}_k(\partial\mathrm{P}),\ \mathbf{curl}(\Delta\boldsymbol{v}) \in (\mathbb{P}_{k-3}(\mathrm{P}))^3,\ \mathrm{div}\boldsymbol{v} \in \mathbb{P}_0(\mathrm{P})\}. \tag{3.29}$$

Following [20], we have the following properties.

Proposition 3.7. *A unisolvent set of degrees of freedom for $\mathcal{A}_k(P)$ is given by*

- *value of $\boldsymbol{v}(\nu)$, for every vertex ν of P,* (3.30)
- *(for $k \geq 2$) value of $\int_e \boldsymbol{v} \cdot \boldsymbol{q}_{k-2}\,\mathrm{d}e$, $\forall \boldsymbol{q}_{k-2} \in (\mathbb{P}_{k-2}(e))^3$, for every edge e of P,* (3.31)
- *(for $k \geq 3$) value of $\int_f \boldsymbol{v} \cdot \boldsymbol{q}_{k-3}\,\mathrm{d}f$, $\forall \boldsymbol{q}_{k-3} \in (\mathbb{P}_{k-3}(f))^3$, for every face f of P,* (3.32)
- *(for $k \geq 3$) value of $\int_P \boldsymbol{v} \cdot (\boldsymbol{x} \wedge \boldsymbol{q}_{k-3})\,\mathrm{d}P$, $\forall \boldsymbol{q}_{k-3} \in (\mathbb{P}_{k-3}(P))^3$.* (3.33)

Proof. First we recall that the values (3.30)-(3.32) determine uniquely the *boundary values* of a $\boldsymbol{v}$ in $\mathcal{A}_k(\mathrm{P})$. Consequently, the (constant) value of *the divergence* of $\boldsymbol{v}$ is also determined uniquely, using the mean value of $\boldsymbol{v} \cdot \boldsymbol{n}$ on $\partial\mathrm{P}$. Hence we just need to show that adding the degrees of freedom (3.33) we can determine uniquely $\boldsymbol{v} \in \mathcal{A}_k(\mathrm{P})$. For that it would be enough to restrict our attention to the elements of $\mathcal{A}_k(\mathrm{P})$ that belong to the subspace

$$\mathrm{AUX} := \{\boldsymbol{v} \in (H^1_0(\mathrm{P}))^3 \text{ such that } \mathrm{div}\boldsymbol{v} = 0\} \tag{3.34}$$

(meaning that their values in (3.30)-(3.32) are all zero), and show that the values of (3.33) would determine uniquely a $\boldsymbol{v}$ among them.
For this we check first that the number of conditions in (3.33) matches the dimension of $\mathcal{A}_k(\mathrm{P})\cap\mathrm{AUX}$. We observe that an element $\boldsymbol{v}$ of AUX belongs to $\mathcal{A}_k(\mathrm{P})$ *if and only if* $\mathbf{curl}\Delta\boldsymbol{v}$ is in $(\mathbb{P}_{k-3})^3$, and this amounts to $3\pi_{3,k-3} - \pi_{3,k-4}$ conditions: indeed, remember that a vector valued polynomial $\boldsymbol{q}$ of degree $k-3$, in order to be a **curl**, must have a zero divergence, which amounts to $\pi_{3,k-4}$ conditions. On the other hand, (3.33) amounts to $3\pi_{3,k-3} - \pi_{3,k-4}$ conditions as well, since for all vectors $\boldsymbol{q}_{k-3}$ of the form $\boldsymbol{q}_{k-3} = \boldsymbol{x}q_{k-4}$ (with $q_{k-4} \in \mathbb{P}_{k-4}$) the product $\boldsymbol{x} \wedge \boldsymbol{q}_{k-3} \equiv \boldsymbol{x} \wedge \boldsymbol{x}q_{k-4}$ is identically zero.
Hence, we are reduced to prove that **if** $\boldsymbol{v} \in \mathcal{A}_K\cap\mathrm{AUX}$ has the values (3.33) all equal to zero **then** we must have $\boldsymbol{v} = 0$. We observe that $\mathbf{curl}(\Delta\boldsymbol{v})$ is in $(\mathbb{P}_{k-3}(\mathrm{P}))^3$, and, being a **curl**, has zero divergence; we deduce that $\mathbf{curl}(\Delta\boldsymbol{v})$ is equal to the **curl** of some polynomial vector in $(\mathbb{P}_{k-2}(\mathrm{P}))^3$. Using then (2.5) we have that there exists a $\boldsymbol{q}^*_{k-3} \in (\mathbb{P}_{k-3}(\mathrm{P}))^3$ such that

$$\mathbf{curl}(\Delta\boldsymbol{v}) = \mathbf{curl}(\boldsymbol{x} \wedge \boldsymbol{q}^*_{k-3}), \tag{3.35}$$

implying, since P is simply connected, that

$$\Delta\boldsymbol{v} = \boldsymbol{x} \wedge \boldsymbol{q}^*_{k-3} + \nabla s \tag{3.36}$$

for some $s \in H^1(\mathrm{P})$. Next, we note that for $\boldsymbol{v} \in AUX$ we have $\boldsymbol{v} = 0$ on $\partial\mathrm{P}$ and $\mathrm{div}\boldsymbol{v} = 0$ in P. Integrating by parts and using (3.36) and (3.33) we have then

$$\int_\mathrm{P} |\nabla\boldsymbol{v}|^2\,\mathrm{dP} = -\int_\mathrm{P} \boldsymbol{v} \cdot \Delta\boldsymbol{v}\,\mathrm{dP} = -\int_\mathrm{P} \boldsymbol{v} \cdot (\nabla s + \boldsymbol{x} \wedge \boldsymbol{q}^*_{k-3})\,\mathrm{dP} = 0 + 0 = 0 \tag{3.37}$$

and the proof is completed. ■

Remark 3.8. Clearly, another (conceptually simpler) option would be to take as $\mathcal{A}_k$ the space of triplets of C^0 VEMs as in [14], similarly to what is done for these problems when using FEMs. The advantage with the present choice is in the use of a **constant divergence**, that will allow to have a truly divergence-free solution, as well as a reduction of the number of degrees of freedom in P (that has nothing to do with the possible use of 3D Serendipity elements). □

3.5. **Quantities that are computable in** $\mathcal{A}_k(\mathrm{P})$

Assume now that we are given a polyhedron P and, for an integer $k \geq 1$, the VEM nodal space $\mathcal{A}_k(\mathrm{P})$ as defined in (3.29). Assume moreover that we are given the degrees of freedom (3.30)-(3.33) of an element $\boldsymbol{v} \in \mathcal{A}_k(\mathrm{P})$. The question is: what are the quantities, related to $\boldsymbol{v}$, that we can actually calculate on a computer, without solving a (system of) PDE's in P? As a general set-up of the problem, we assume that we can compute: the integral over edges, faces, and P of all polynomials of degree $\leq k$. But the elements of $\mathcal{A}_k(\mathrm{P})$ are **not** polynomials, in general, apart from very special cases (e.g., if P is a tetrahedron and $k \leq 2$). Or, to be more precise, all polynomials of $(\mathbb{P}_k)^3$ with constant divergence will belong to $\mathcal{A}_k(\mathrm{P})$, that however will contain other, non polynomial, functions.

To start with, using (3.30) and (3.31) we see that:

- The values of each component of $\boldsymbol{v}$ on every edge of P are computable. (3.38)

Then, on each face f we can use (3.24) to see that:

- For every face $f, \forall \boldsymbol{q} \in (\mathbb{P}_k(f))^3$ the moments $\int_f \boldsymbol{v} \cdot \boldsymbol{q} \, \mathrm{d}f$ are computable. (3.39)

In particular, on every face f we will be able to compute

$$\int_f \boldsymbol{v} \cdot \boldsymbol{n}_{\mathrm{P}} \, \mathrm{d}f, \tag{3.40}$$

where, on each f, $\boldsymbol{n}_{\mathrm{P}}$ is the (3-dimensional) unit vector normal to the face f. As the divergence of $\boldsymbol{v}$ is constant in P (see (3.29)), from (3.40) we immediately see that:

- The value of $\mathrm{div}\boldsymbol{v}$ in P is computable. (3.41)

We can also compute the moments of $\boldsymbol{v}$ against all (vector valued) polynomials of degree $\leq k-2$ in P. Indeed, given a $\boldsymbol{p}_{k-2} \in (\mathbb{P}_{k-2}(\mathrm{P}))^3$ we can use (2.4) and write it as

$$\boldsymbol{p}_{k-2} = \nabla q_{k-1} + \boldsymbol{x} \wedge \boldsymbol{q}_{k-3}$$

with $q_{k-1} \in \mathbb{P}_{k-1}(\mathrm{P})$ and $\boldsymbol{q}_{k-3} \in (\mathbb{P}_{k-3}(\mathrm{P}))^3$. Hence:

$$\begin{aligned}\int_{\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{p}_{k-2} \, \mathrm{dP} &= \int_{\mathrm{P}} \boldsymbol{v} \cdot (\nabla q_{k-1} + \boldsymbol{x} \wedge \boldsymbol{q}_{k-3}) \, \mathrm{dP} = \int_{\mathrm{P}} \boldsymbol{v} \cdot \nabla q_{k-1} \, \mathrm{dP} + \int_{\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{x} \wedge \boldsymbol{q}_{k-3} \, \mathrm{dP} \\ &= -\int_{\mathrm{P}} \mathrm{div}\boldsymbol{v} \, q_{k-1} \, \mathrm{dP} + \int_{\partial\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{n}_{\mathrm{P}} \, q_{k-1} \, \mathrm{d}S + \int_{\mathrm{P}} \boldsymbol{v} \cdot (\boldsymbol{x} \wedge \boldsymbol{q}_{k-3}) \, \mathrm{dP} \end{aligned} \tag{3.42}$$

and all the three terms of the last line are computable (the third using (3.33)). Hence:

- The values of $\int_{\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{q}_{k-2} \, \mathrm{dP} \;\; \forall \boldsymbol{q}_{k-2} \in (\mathbb{P}_{k-2}(\mathrm{P}))^3$ are computable. (3.43)

For $k = 1$, using $\boldsymbol{p}_0 = \nabla \boldsymbol{q}_1$ and proceeding as in (3.42) we obtain that

- $\int_{\mathrm{P}} \boldsymbol{v} \cdot \boldsymbol{q}_0 \, \mathrm{dP}$ is computable. (3.44)

The moments of $\mathbf{grad}\boldsymbol{v}$ against all tensor valued polynomials of degree $\leq k-1$ are also computable. To see this, let $\boldsymbol{\tau}_{k-1} \in (\mathbb{P}_{k-1}(\mathrm{P}))^{3\times 3}$ and consider

$$\int_{\mathrm{P}} (\mathbf{grad}\boldsymbol{v}) : \boldsymbol{\tau}_{k-1} \, \mathrm{dP} = -\int_{\mathrm{P}} \boldsymbol{v} \cdot (\mathbf{div}(\boldsymbol{\tau}_{k-1})) \, \mathrm{dP} + \int_{\partial\mathrm{P}} \boldsymbol{v} \cdot (\boldsymbol{\tau}_{k-1} \cdot \boldsymbol{n}_{\mathrm{P}}) \, \mathrm{d}S. \tag{3.45}$$

In (3.45) $\mathbf{div}(\boldsymbol{\tau}_{k-1})$ is a vector in $(\mathbb{P}_{k-2}(\mathrm{P}))^3$, so that the first term is computable from (3.42). Similarly, $\boldsymbol{\tau}_{k-1}\cdot\boldsymbol{n}_{\mathrm{P}}$ is in $(\mathbb{P}_{k-1}(f))^3$ on each face, so that recalling (3.39) the second term is computable as well. Hence:

- The value of $\displaystyle\int_{\mathrm{P}}\mathbf{grad}\boldsymbol{v}:\boldsymbol{\tau}_{k-1}\,\mathrm{dP}$ is computable $\forall\boldsymbol{\tau}_{k-1}\in(\mathbb{P}_{k-1}(\mathrm{P}))^{3\times 3}$. (3.46)

Note that (3.46) implies that for every $\boldsymbol{v}\in\mathcal{A}_k$, for every component v_i, $(i=1,2,3)$ and for every index j, $(j=1,2,3)$:

- the $L^2(\mathrm{P})$-projection of $\dfrac{\partial v_i}{\partial x_j}$ onto $\mathbb{P}_{k-1}(\mathrm{P})$ is computable. (3.47)

Hence we can also compute, for $\boldsymbol{v}\in\mathcal{A}_k(\mathrm{P})$ and $\boldsymbol{q}\in(\mathbb{P}_k(\mathrm{P}))^3$, the quantities

$$\int_{\partial\mathrm{P}}(\boldsymbol{v}\wedge\boldsymbol{n})\cdot(\boldsymbol{q}\wedge\boldsymbol{n})\,\mathrm{d}S, \tag{3.48}$$

$$\int_{\mathrm{P}}(\mathrm{div}\boldsymbol{v})\,(\mathrm{div}\boldsymbol{q})\,\mathrm{dP}, \tag{3.49}$$

$$\int_{\mathrm{P}}\mathbf{curl}\boldsymbol{v}\cdot\mathbf{curl}\boldsymbol{q}\,\mathrm{dP}. \tag{3.50}$$

Introducing the restriction of the bilinear form a to P, as natural

$$a^{\mathrm{P}}(\boldsymbol{u},\boldsymbol{v}):=\int_{\mathrm{P}}\mu^{-1}\mathbf{curl}\boldsymbol{u}\cdot\mathbf{curl}\boldsymbol{v}\,\mathrm{dP}+\int_{\mathrm{P}}\mathrm{div}\boldsymbol{u}\,\mathrm{div}\boldsymbol{v}\,\mathrm{dP}\qquad\boldsymbol{u},\boldsymbol{v}\in\mathcal{A}_k(\mathrm{P}), \tag{3.51}$$

we also have as an immediate consequence that:

- $\forall\boldsymbol{v}\in\mathcal{A}_k(\mathrm{P})$ and $\forall\boldsymbol{q}\in(\mathbb{P}_k(\mathrm{P}))^3:\qquad a^{\mathrm{P}}(\boldsymbol{v},\boldsymbol{q})$ is computable. (3.52)

All this will allow us to compute a projection operator Π^A_k from smooth-enough vector valued functions onto $(\mathbb{P}_k(\mathrm{P}))^3$. For this we first introduce the space

$$\mathbb{H}_k:=\{\boldsymbol{q}_k\in(\mathbb{P}_k(\mathrm{P}))^3\text{ such that }\exists\varphi\in\mathbb{P}_{k+1}(\mathrm{P})\text{ with }\Delta\varphi=0\text{ and }\boldsymbol{q}_k=\nabla\varphi\} \tag{3.53}$$

of the gradients of the harmonic polynomials in $\mathbb{P}_{k+1}(\mathrm{P})$. We note that, as it can be easily checked:

$$\mathbb{H}_k\equiv\{\boldsymbol{q}_k\in(\mathbb{P}_k(\mathrm{P}))^3\text{ such that }a^{\mathrm{P}}(\boldsymbol{q}_k,\boldsymbol{q}_k)=0\}. \tag{3.54}$$

We also note that:

$$\forall\boldsymbol{q}\in\mathbb{H}_k:\qquad\{\boldsymbol{q}\wedge\boldsymbol{n}_{\mathrm{P}}=0\text{ on }\partial\mathrm{P}\}\Leftrightarrow\{\boldsymbol{q}\equiv\mathbf{0}\}. \tag{3.55}$$

Then we can introduce the following definition.

Definition 3.9. Given $\boldsymbol{v}$, for instance, in $(H^1(\mathrm{P}))^3$ we define its projection $\Pi^A_k\boldsymbol{v}$ onto $(\mathbb{P}_k(\mathrm{P}))^3$ as follows:

$$a^{\mathrm{P}}(\Pi^A_k\boldsymbol{v}-\boldsymbol{v},\boldsymbol{q}_k)=0\qquad\forall\boldsymbol{q}_k\in(\mathbb{P}_k(\mathrm{P}))^3, \tag{3.56}$$

$$\int_{\partial\mathrm{P}}[(\Pi^A_k\boldsymbol{v}-\boldsymbol{v})\wedge\boldsymbol{n}]\cdot[\boldsymbol{q}_k\wedge\boldsymbol{n}]\,\mathrm{d}S=0\qquad\forall\boldsymbol{q}_k\in\mathbb{H}_k. \tag{3.57}$$

Note that, due to (3.54) and (3.55), the solution of (3.56) -(3.57) is unique in $(\mathbb{P}_k(\mathrm{P}))^3$. □

Remark 3.10. Clearly, the projection operator Π^A_k is not $(L^2(\mathrm{P}))^3$-orthogonal, but, in some sense, is a^{P}-orthogonal. This, however, will not be a problem in what follows. □

Remark 3.11. The space $\mathcal{A}_k(\mathrm{P})$, as presented in (3.29), **does not** contain all polynomials in $(\mathbb{P}_k)^3$, but only the subspace made of those with constant divergence. In order to keep all polynomials of $(\mathbb{P}_k)^3$ inside, we should (obviously) take instead

$$\widetilde{\mathcal{A}_k}(\mathrm{P}) := \{\boldsymbol{v} \in (C^0(\overline{\mathrm{P}}))^3 \text{ s. t. } \boldsymbol{v}_{|\partial\mathrm{P}} \in \mathbb{B}_k(\partial\mathrm{P}),\ \mathbf{curl}(\Delta\boldsymbol{v}) \in (\mathbb{P}_{k-3}(\mathrm{P}))^3,\ \mathrm{div}\boldsymbol{v} \in \mathbb{P}_{k-1}(\mathrm{P})\}, \tag{3.58}$$

and, as degrees of freedom, add to (3.30)-(3.33) the natural ones

- (for $k \geq 2$) value of $\int_{\mathrm{P}} \mathrm{div}\boldsymbol{v}\, q^0_{k-1} \quad \forall q^0_{k-1} \in \mathbb{P}^0_{k-1}(\mathrm{P})$ (3.59)

as in [20]. □

At this point we are able to re-enter the more classical path of VEMs. In particular, we can define *the contribution of the element P* to the *global approximated bilinear form*: for $\boldsymbol{u}$ and $\boldsymbol{v}$ smooth enough (for instance to have each component in $H^1(\Omega) \cap C^0(\overline{\Omega})$ would be sufficient) we set

$$a_h^{\mathrm{P}}(\boldsymbol{u}, \boldsymbol{v}) := a^{\mathrm{P}}(\Pi_k^A \boldsymbol{u}, \Pi_k^A \boldsymbol{v}) + S_h^{\mathrm{P}}((\mathrm{I} - \Pi_k^A)\boldsymbol{u}, (\mathrm{I} - \Pi_k^A)\boldsymbol{v}), \tag{3.60}$$

where $S_h^{\mathrm{P}}(\cdot\,,\cdot)$, as usual, is a symmetric bilinear form such that there exist two constants α_* and α^*, independent of P, with

$$\alpha_* a^{\mathrm{P}}(\boldsymbol{v}, \boldsymbol{v}) \leq a_h^{\mathrm{P}}(\boldsymbol{v}, \boldsymbol{v}) \leq \alpha^* |\boldsymbol{v}|^2_{1,\mathrm{P}} \quad \forall \boldsymbol{v} \in \mathcal{A}_k(\mathrm{P}). \tag{3.61}$$

Needless to say, (3.56) and (3.60) easily imply that

$$a_h^{\mathrm{P}}(\boldsymbol{u}, \boldsymbol{v}) \equiv a^{\mathrm{P}}(\boldsymbol{u}, \boldsymbol{v}) \qquad \text{whenever either } \boldsymbol{u} \text{ or } \boldsymbol{v} \text{ is in } (\mathbb{P}_k(\mathrm{P}))^3. \tag{3.62}$$

Remark 3.12. As typical in the VEM framework (see e.g. [11, 15]) we can take

$$S_h^{\mathrm{P}}(\boldsymbol{u}, \boldsymbol{v}) := \sum_i \sigma_i \delta_i(\boldsymbol{u}) \delta_i(\boldsymbol{v}). \tag{3.63}$$

where the $\delta_i(\boldsymbol{v})$ are the degrees of freedom of $\boldsymbol{v}$ in P, and the weights σ_i are suitable scaling factors. To fix ideas, let ϕ_i be the element in $\mathcal{A}_k$ such that $\delta_i(\phi_i) = 1$ and $\delta_j(\phi_i) = 0$ for $j \neq i$. Then σ_i should be *of the order* (in terms of powers of h_{P}) of $a^{\mathrm{P}}(\phi_i, \phi_i)$. For instance, if $\delta_i(\varphi)$ is the value of φ at a given node N_i, then $a^{\mathrm{P}}(\phi_i, \phi_i)$ will be of the order of h_{P} (taking into account that the volume of P scales like $(h_{\mathrm{P}})^3$ and the gradient of ϕ_i scales like h_{P}^{-1}). Hence one could take $\sigma_i \simeq h_{\mathrm{P}}$. Note that, with the choice (3.63), the regularity required on $\boldsymbol{u}$ and $\boldsymbol{v}$ in order to give sense to (3.60) is just the regularity needed to compute Π_k^A and the degrees of freedom δ_i. □

4. The global spaces and the discretized problem

4.1. The global spaces

From the *local* Virtual Element spaces, defined in each $\mathrm{P} \in \mathcal{T}_h$, we can now construct easily the *global* spaces in Ω. We set

$$\mathcal{A}_h \equiv \mathcal{A}_h(\Omega) := \{\boldsymbol{v} \in \mathcal{A} \text{ such that } \boldsymbol{v} \in \mathcal{A}_k(\mathrm{P}) \text{ for all element } \mathrm{P} \in \mathcal{T}_h\}. \tag{4.1}$$

On $\mathcal{A}_h$ we can define the *global* bilinear form a_h simply setting

$$a_h(\boldsymbol{u}, \boldsymbol{v}) := \sum_{\mathrm{P} \in \mathcal{T}_h} a_h^{\mathrm{P}}(\boldsymbol{u}, \boldsymbol{v}). \tag{4.2}$$

Finally, we also define, in each element P

$$\Big(\boldsymbol{j}_h\Big)_{|\mathrm{P}} := \begin{cases} (L^2(\mathrm{P}))^3\text{-orthogonal projection of } \boldsymbol{j} \text{ onto } (\mathbb{P}_0(\mathrm{P}))^3 \text{ for } k = 1 \\ (L^2(\mathrm{P}))^3\text{-orthogonal projection of } \boldsymbol{j} \text{ onto } (\mathbb{P}_{k-2}(\mathrm{P}))^3 \text{ for } k \geq 2, \end{cases} \tag{4.3}$$

and we note that the integral

$$\int_\Omega \boldsymbol{j}_h \cdot \boldsymbol{v}\,\mathrm{d}\Omega$$

is computable for every $\boldsymbol{v} \in \mathcal{A}_h$ (due to (3.43) and (3.44)).

4.2. **The discretized problem**

The discretized version of (3.6) will now read

$$\begin{cases} \text{find } \boldsymbol{A}_h \in \mathcal{A}_h \text{ such that:} \\ a_h(\boldsymbol{A}_h, \boldsymbol{v}) = \displaystyle\int_\Omega \boldsymbol{j}_h \cdot \boldsymbol{v}\,\mathrm{d}\Omega. \quad \forall \boldsymbol{v} \in \mathcal{A}_h. \end{cases} \tag{4.4}$$

It is very easy to see that a_h is symmetric, and satisfies the two fundamental properties of VEM approximations of linear elliptic problems, namely:

$$a(\boldsymbol{v}, \boldsymbol{q}) = a_h(\boldsymbol{v}, \boldsymbol{q}) \quad \text{ for all } \boldsymbol{v} \in \mathcal{A}_h, \text{ and for all } \boldsymbol{q} \text{ piecewise in } (\mathbb{P}_k)^3 \tag{4.5}$$

and

$$\exists\, \alpha_* \text{ and } \alpha^* \text{ in } \mathbb{R} \text{ such that:} \qquad \alpha_* a(\boldsymbol{v}, \boldsymbol{v}) \le a_h(\boldsymbol{v}, \boldsymbol{v}) \le \alpha^* a(\boldsymbol{v}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{A}_h. \tag{4.6}$$

Note that the constants α_* and α^* will also depend on μ. We point out that in deriving (4.6) from the local (3.61) we are now able to use (3.5) (that needs the boundary conditions on $\partial\Omega$ but not on $\partial\mathrm{P}$) and use $a(\boldsymbol{v}, \boldsymbol{v})$ on the right-hand side instead of $|\boldsymbol{v}|_1^2$ as we had in (3.61). Indeed using (3.61) and (3.5) we have

$$a_h(\boldsymbol{v}, \boldsymbol{v}) = \sum_{\mathrm{P}} a_h^{\mathrm{P}}(\boldsymbol{v}, \boldsymbol{v}) \le \alpha^* \sum_{\mathrm{P}} |\boldsymbol{v}|_{1,\mathrm{P}}^2 = \alpha^* \|\boldsymbol{v}\|_{1,\Omega}^2 \le \alpha^*\, c_2^2\, \|\boldsymbol{v}\|_{\mathcal{A}}^2. \tag{4.7}$$

We also note that the symmetry of a_h and (4.6) easily imply the continuity of a_h with

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) &\le \Big(a_h(\boldsymbol{u}, \boldsymbol{u})\Big)^{1/2} \Big(a_h(\boldsymbol{v}, \boldsymbol{v})\Big)^{1/2} \\ &\le \alpha^* \Big(a(\boldsymbol{u}, \boldsymbol{u})\Big)^{1/2} \Big(a(\boldsymbol{v}, \boldsymbol{v})\Big)^{1/2} \le \alpha^* \|\boldsymbol{u}\|_{\mathcal{A}} \|\boldsymbol{v}\|_{\mathcal{A}} \end{aligned} \tag{4.8}$$

for all $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathcal{A}_h$.

5. **Error Estimates**

In the two previous sections we allowed μ to be piecewise constant in Ω. This was more than enough in order to let us construct the VEM spaces and to design the discretised problem. In this Section, however, we also have to deal with the exact solution of the vector potential problem (3.6), and in particular with its regularity. We recall that one typical difficulty of magnetostatic problems is the possible lack of regularity of its solution. Note that this does not depend on the use of the Vector Potential formulation, and even less on the fact that we use a VEM discretisation. The (unavoidable) problem is related to the fact that (1.1) is a so-called *div-curl* system, and that for it there are several occurrences where the solution (let it be $\boldsymbol{B}$, or $\boldsymbol{H}$, or the Vector Potential $\boldsymbol{A}$) is not too regular. Many of these occur as well for simpler problems as $\mathrm{div}(c(\boldsymbol{x})\nabla u) = f$ for a discontinuous coefficient c, but for *div-curl* systems worse cases can occur, since the solution of the magneto-static problem might fail to be in $(H^1)^3$. Much worse: for non-convex polyhedra $(H^1)^3$ could be a *closed* subspace of $\mathcal{A}$, and hence H^1-conforming approximations cannot be used.

We point out that, in practice, the method proposed here, and described so far, will still make sense and be applicable in a certain number of more general cases. The troubles arrive when one wants to prove error estimates, expressed (as usual) in terms of powers of $|h|$ and norms of $\boldsymbol{A}$ in spaces with a certain regularity.

To make a long story short, for the sake of simplicity when proving error estimates we just *assume* here that the solution $\boldsymbol{A}$ is at least in $(H^1)^3$ and point out that, after all, for the same accuracy, we do not require more regularity than other methods.

It will surely be very interesting to make further investigations in order to be able to circumvent this or that difficulty, as it has been done for Finite Element Methods in the last 25 years, starting from [47], and then (among others) with [7, 32, 29, 59, 37, 10, 53, 28], down to the present times (as in [54]). It would also be interesting to investigate the case of unbounded domains: either mimicking the *Infinite Elements* (as e.g.[63, 58]), or creating an artificial boundary at a certain distance from the region of interest, as done in Finite Elements with the Perfectly Matching Layer technique (see e.g. [24, 31]).

Here however, since this (as far as we know) is the first VEM approach to Vector Potential formulations, we decided to Keep It Simple, and stick with the most elementary cases.

5.1. **The convergence theorem**

We start our discussion with an abstract convergence result, that bounds the error $|\boldsymbol{A}-\boldsymbol{A}_h|$ in terms of suitable interpolation errors for $\boldsymbol{A}$ and in terms of the error $|\boldsymbol{j}-\boldsymbol{j}_h|$ in the right-hand side.

Theorem 5.1. *The discrete problem* (4.4) *has a unique solution* $\boldsymbol{A}_h$. *Moreover, for every approximation* $\boldsymbol{A}_I$ *of* $\boldsymbol{A}$ *in* $\mathcal{A}_h$ *and for every approximation* $\boldsymbol{A}_\pi$ *of* $\boldsymbol{A}$ *that is piecewise in* $(\mathbb{P}_k)^3$, *we have*

$$\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\mathcal{A}} \le C\Big(\|\boldsymbol{A}-\boldsymbol{A}_I\|_{1,h}+\|\boldsymbol{A}-\boldsymbol{A}_\pi\|_{1,h}+\|\boldsymbol{j}-\boldsymbol{j}_h\|_{\mathcal{A}_h'}\Big), \tag{5.1}$$

where:

- C *is a constant depending only on* α_*, α^*, μ,
- $\|\boldsymbol{v}\|_{1,h} := \Big(\sum_{P\in\mathcal{T}_h} \|\boldsymbol{v}\|^2_{1,P}\Big)^{1/2}$
- $\|\boldsymbol{j}-\boldsymbol{j}_h\|_{\mathcal{A}_h'}$ *is defined as the smallest constant* $\mathfrak{C}$ *such that*

$$(\boldsymbol{j},\boldsymbol{v}) - (\boldsymbol{j}_h,\boldsymbol{v}) \le \mathfrak{C}\,|\boldsymbol{v}|_{\mathcal{A}} \quad \forall\, \boldsymbol{v}\in\mathcal{A}_h. \tag{5.2}$$

Proof. The proof follows exactly the same lines as the original one in [11]. Existence and uniqueness of the solution of (4.4) are a consequence of (4.6) and (3.7). Next, setting $\boldsymbol{\delta}_h := \boldsymbol{A}_h-\boldsymbol{A}_I$ and starting from (4.6) we have:

$$\begin{aligned}
&\alpha_*\|\boldsymbol{\delta}_h\|^2_{\mathcal{A}} = \alpha_*\, a(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h) \le a_h(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h) = a_h(\boldsymbol{A}_h,\boldsymbol{\delta}_h) - a_h(\boldsymbol{A}_I,\boldsymbol{\delta}_h)\\
&\text{[use (4.4) and (4.2))]} \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} a_h^{\mathrm{P}}(\boldsymbol{A}_I,\boldsymbol{\delta}_h)\\
&\text{[use } \pm\boldsymbol{A}_\pi] \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I-\boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a_h^{\mathrm{P}}(\boldsymbol{A}_\pi,\boldsymbol{\delta}_h)\Big)\\
&\text{[use (3.62)]} \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I-\boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a^{\mathrm{P}}(\boldsymbol{A}_\pi,\boldsymbol{\delta}_h)\Big)\\
&\text{[use } \pm a(\boldsymbol{A},\boldsymbol{\delta}_h)] \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I-\boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a^{\mathrm{P}}(\boldsymbol{A}_\pi-\boldsymbol{A},\boldsymbol{\delta}_h)\Big) - a(\boldsymbol{A},\boldsymbol{\delta}_h)\\
&\text{[use (3.6)]} \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I-\boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a^{\mathrm{P}}(\boldsymbol{A}_\pi-\mathcal{A},\boldsymbol{\delta}_h)\Big) - (\boldsymbol{j},\boldsymbol{\delta}_h)\\
&\text{[re-order]} \quad = (\boldsymbol{j}_h,\boldsymbol{\delta}_h) - (\boldsymbol{j},\boldsymbol{\delta}_h) - \textstyle\sum_{\mathrm{P}} \Big(a_h^{\mathrm{P}}(\boldsymbol{A}_I-\boldsymbol{A}_\pi,\boldsymbol{\delta}_h) + a^{\mathrm{P}}(\boldsymbol{A}_\pi-\boldsymbol{A},\boldsymbol{\delta}_h)\Big).
\end{aligned}$$

Now use (5.2), (4.8), and the continuity of each a^{P} to obtain

$$\|\boldsymbol{\delta}_h\|^2_{\mathcal{A}} \le C\Big(\|\boldsymbol{j}-\boldsymbol{j}_h\|_{\mathcal{A}_h'} + \|\boldsymbol{A}_I-\boldsymbol{A}_\pi\|_{1,h} + \|\boldsymbol{A}-\boldsymbol{A}_\pi\|_{1,h}\Big)\,\|\boldsymbol{\delta}_h\|_{\mathcal{A}} \tag{5.3}$$

for some constant C depending only on α_*, α^* and μ. Then the result follows easily by the triangle inequality. ■

From the estimate (5.1), given the sequence of decompositions $\{\mathcal{T}_h\}_h$ one can then deduce an error estimate in terms of powers of $|h|$ (as defined in (3.9)), of some regularity constant for the polyhedrons, and of the regularity of the solution $\boldsymbol{A}$. For this we need suitable interpolation estimates.

5.2. Interpolation estimates

Theorem 5.2. *Assume that the sequence of decompositions $\{\mathcal{T}_h\}_h$ satisfies the following assumptions (that are quite standard in the VEM literature). There exists a positive constant γ, independent of h, such that for every h all polyhedrons P of $\mathcal{T}_h$ satisfy:*

- **D1**) *P is star-shaped with respect to a sphere of radius bigger than γh_P;*
- **D2**) *every face $f \in \partial P$ is star-shaped with respect to a disk of radius bigger than γh_P, and every edge of P has length bigger than γh_P.*

Then if the spaces $\mathcal{A}_h$ are defined as in (4.1) *for some integer $k \geq 1$ we have*

$$\|\boldsymbol{A} - \boldsymbol{A}_I\|_{1,h} + \|\boldsymbol{A} - \boldsymbol{A}_\pi\|_{1,h} \leq C_1\, |h|^s \|\boldsymbol{A}\|_{s+1,\Omega}, \quad 0 \leq s \leq k \tag{5.4}$$

and

$$\|\boldsymbol{j} - \boldsymbol{j}_h\|_{\mathcal{A}_h'} \leq C_2\, |h|^s \|\boldsymbol{j}\|_{s-1,\Omega}, \quad 1 \leq s \leq k \tag{5.5}$$

where C_1 and C_2 are constants that depend only on $\gamma, \alpha_*, \alpha^*, \mu$ and on the regularity of $\boldsymbol{A}$ and $\boldsymbol{j}$, respectively.

Proof. From known results on polynomial approximation (see e.g. [34]), one can first get easily

$$\|\boldsymbol{A} - \boldsymbol{A}_\pi\|_{1,h} \leq c_{ext}\,, |h|^s \|\boldsymbol{A}\|_{s+1,\Omega}, \quad 0 \leq s \leq k \tag{5.6}$$

and,

$$(\boldsymbol{j} - \boldsymbol{j}_h, \boldsymbol{v}) \leq c_{ext}\, |h|^s \|\boldsymbol{j}\|_{s-1,\Omega} \|\boldsymbol{v}\|_{1,\Omega}, \quad 1 \leq s \leq k \tag{5.7}$$

for some constant c_{ext} depending on k and on the maximum (over the polygons P) of the constants that bound the extension of a function φ from P to a sphere of diameter $2\,h_{\mathrm{P}}$ containing P. Note that these constants, themselves, can also be uniformly bounded in terms of the γ appearing in **D1** and **D2**. Then we define $\boldsymbol{A}_I$ as the interpolant of $\boldsymbol{A}$, locally, in $\widetilde{\mathcal{A}_k}(\mathrm{P})$ as defined in (3.58). At first sight, such an $\boldsymbol{A}_I$ might fail to belong to $\mathcal{A}_k(\mathrm{P})$: indeed, $\mathcal{A}_k(\mathrm{P})$, being made of vectors with constant divergence, is *smaller* than $\widetilde{\mathcal{A}_k}(\mathrm{P})$ which is made of vectors having divergence in $\mathbb{P}_{k-1}$. But we recall that $\boldsymbol{A}$ has *zero* divergence, and it is easy to see that the degrees of freedom of $\widetilde{\mathcal{A}_k}(\mathrm{P})$ are such that the interpolant of a solenoidal vector is itself solenoidal. Now we make profit of the fact that $\widetilde{\mathcal{A}_k}(\mathrm{P})$ contains *all* vector polynomials of degree $\leq k$, and with the (nowadays) classical instruments of Virtual Element approximation theory (see e.g. [19, 21, 33, 35, 41, 65]) it is not difficult to see that we also have

$$\|\boldsymbol{A} - \boldsymbol{A}_I\|_{1,h} \leq c\, |h|^s \|\boldsymbol{A}\|_{s+1,\Omega}, \quad 0 \leq s \leq k \tag{5.8}$$

for some constant c that depends on k and on the constant γ in **D1** and **D2**. ■

Then we have the final convergence Theorem.

Theorem 5.3. *Under the assumptions of Theorem 5.2 we have:*

$$\|\boldsymbol{H} - \boldsymbol{H}_h\|_{0,\Omega} + \|\boldsymbol{B} - \boldsymbol{B}_h\|_{0,\Omega} \leq C\, |h|^s (\|\boldsymbol{A}\|_{s+1,\Omega} + \|\boldsymbol{j}\|_{\max\{0,s-1\},\Omega}), \quad 0 \leq s \leq k \tag{5.9}$$

for a constant C that depends only on $\gamma, \alpha_*, \alpha^*, \mu$ and k.

Proof. Setting $\boldsymbol{B}_h := \mathbf{curl}\,\boldsymbol{A}_h$ and $\boldsymbol{H}_h := \mu^{-1}\boldsymbol{B}_h$, the result follows by inserting estimates (5.4) and (5.5) into (5.1). ■

Acknowledgments

The first author was partially supported by the European Research Council through the H2020 Consolidator Grant (grant no. 681162) CAVE – Challenges and Advancements in Virtual Elements. This support is gratefully acknowledged.

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