

L^2 –estimates for the DG IIPG-0 scheme

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Abstract

We discuss the optimality in L^2 of a variant of the Incomplete Discontinuous Galerkin Interior Penalty method (IIPG) for second order linear elliptic problems. We prove optimal estimate, in two and three dimensions, for the lowest order case under suitable regularity assumptions on the data and on the mesh. We also provide numerical evidence, in one dimension, of the necessity of the regularity assumptions.

1 Introduction

Interior Penalty Discontinuous Galerkin methods for diffusion equations were introduced in the late seventies and in the beginning of the eighties (see [24] and [3]). They were not used for some time, but were revived in the late nineties, mostly in order to deal with the diffusive part of convection dominated problems.

At the beginning of the last decade, a considerable interest aroused for the use of nonsymmetric methods, even for discretizing symmetric operators as the Laplace operator. The interest was mainly addressed to the Baumann-Oden approach, in which the anti-symmetrization of the DG-consistency terms allowed a much simpler proof of a-priori estimates and stability results. Later on, other nonsymmetric approaches were proposed (in particular by Sun and Wheeler) based on encouraging numerical tests. The discrete- H^1 error estimates for all these methods are quite easy to prove, essentially using the same arguments used for dealing with the original symmetric scheme (then renamed "IP", then renamed again "SIPG"). See for instance [4] and the references therein.

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However, many questions remained open for the last ten years concerning the optimality of nonsymmetric methods in L^2 , even when considering the approximation of the simple Poisson problem. Indeed, on the one hand the classical Aubin-Nitsche duality technique for proving L^2 estimates cannot be applied, and on the other hand the numerical experiments are not always conclusive, as the quality of the results seems to depend heavily on *parity of the degree* of the local polynomials and/or on the regularity of the mesh and of the right-hand side.

In [17] ^{larson-niklasson} the authors showed optimal error estimates for the NIPG approximation for the one-dimensional problem on uniform grids (for odd degrees). Still in one-dimension, in [13] ^{chen} a superconvergence result for the error in the derivative at Gauss-nodes is shown for the NIPG and SIPG, always for uniform grids and odd degrees. As a consequence, the author could easily deduce an improved $k + 1/2$ rate of convergence in the L^2 -norm for the NIPG method (for uniform meshes and k odd).

More recently, in [16] ^{guzman-riviere} the authors provided numerical evidence of the sub-optimal convergence in L^2 of the NIPG approximation on some particular meshes (in 1D and in 2D) having some periodic pattern, while in [15] ^{otto} L^2 -optimality for the one-dimensional IIPG approximation is proved on quasi-uniform meshes. Furthermore, the authors establish some necessary conditions on the choice of the penalty parameter (depending on the lengths of the neighboring intervals and on the polynomial degree), for guaranteeing the L^2 -optimality for odd degrees.

Few results are known concerning the L^2 -optimality of non symmetric DG methods in several dimensions. In particular, in [12] ^{purman-stam} the authors consider a weakly-penalized NIPG (with strongly imposed boundary conditions) for linear elements, and in [23] ^{nws-wheeler} the discretization of a parabolic problem with the lowest order NIPG method is considered.

We also note that all the works providing some L^2 optimality results for nonsymmetric DG schemes require stronger regularity assumptions (on the right-hand side and/or on the mesh) than those normally used for the L^2 -error analysis of symmetric schemes (based on the Aubin-Nitsche technique).

In the present paper we want to add some additional steps on these issues. For the sake of simplicity, we consider the following very simple model problem. Let Ω be a bounded, convex, polygonal domain in \mathbf{R}^d , $d \geq 2$, and let $f \in L^2(\Omega)$. We look for $u \in H^2(\Omega)$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega. \end{cases} \quad (1.1) \quad \boxed{\text{mod0}}$$

With the same techniques more general linear elliptic second order operators could be considered, as well as more general boundary conditions.

We will analyze the lowest order (i.e. piecewise linear discontinuous) approximation

of the so called "Incomplete Interior Penalty Galerkin" method (IIPG) or, actually, a minor variant of it, penalizing only the mean value of the jumps (IIPG-0). We underline the fact that here we consider *weakly imposed* boundary conditions, as is typical and in some sense more natural for DG methods. For the IIPG-0 method we show that this can be done without introducing major difficulties in the estimates. For other methods (as NIPG or NIPG-0) optimal L^2 estimates, so far, can only be proved in the case of *strongly imposed* boundary conditions, where the variational formulation is restricted to piecewise polynomials that already satisfy the boundary condition, at least for the average on each boundary edge (face). See for instance [12] or our Remark 5.1 here below.

Our approach shares with previous works the idea of using a decomposition of the linear DG space (introduced in several dimensions in [12] and independently in [5] for the design of preconditioners).

We will show L^2 -optimal error estimates for the linear approximation on *1-strongly regular meshes* (roughly speaking: decompositions where the measure of any two neighboring elements is one order smaller than the measure of the elements themselves).

Our analysis will also require a better regularity of the right-hand side, namely, f in $H^1(\Omega) \cap L^\infty(\Omega)$. However (and this is an additional novelty presented in this paper) we demonstrate numerically that this "extra" regularity is indeed **necessary** for achieving the optimal order.

The outline of the paper is as follows. In Section 2 we describe the basic notation, we introduce the IIPG-0 method and revise some basic results that we need for the analysis. In Section 3 we report the error analysis in the energy norm. We study some further properties of the approximate solution to (1.1) in Section 4. In Section 5 we present the L^2 -error analysis of the IIPG-0 method and briefly discuss the extension to NIPG-0 in Remark 5.1, where we show that the results of [12] can be obtained here with less regularity assumptions on the mesh. Finally, Section 6 contains some numerical examples validating the presented theory. In the last part of this section, we give numerical evidence showing that the regularity assumptions required by our analysis (and all previous ones) are indeed essential for achieving L^2 -optimality.

All over the paper, the inequality

$$A \lesssim B$$

will be used to indicate that there exists a constant C , depending only on the minimum angle of the decomposition, such that $A \leq C B$. We will also use standard notation of Sobolev spaces [1].

2 The IIPG-0 Method

sec:2

Let \mathcal{T}_h be a shape-regular family of decompositions of Ω into triangles T (or tetrahedrons if $d = 3$); let h_T denote the diameter of T , and

$$h = \max_{T \in \mathcal{T}_h} h_T.$$

Following [\[4\]](#), we recall the usual DG-tools. Let \mathcal{E}_h° be the set of interior edges (faces if $d = 3$), and let $e \in \mathcal{E}_h^\circ$ be shared by the elements T_1 and T_2 . Define the unit normal vectors \mathbf{n}_1^e and \mathbf{n}_2^e on e , external to T_1 and T_2 , respectively. For a function ζ , piecewise smooth on \mathcal{T}_h , using the notation $\zeta^i := \zeta|_{T_i}$ we define *averages* and *jumps* as

$$\{\zeta\} = \frac{1}{2}(\zeta^1 + \zeta^2), \quad \llbracket \zeta \rrbracket = \zeta^1 \mathbf{n}_1^e + \zeta^2 \mathbf{n}_2^e \quad \text{on } e \in \mathcal{E}_h^\circ.$$

For a vector valued function $\boldsymbol{\tau}$, piecewise smooth on \mathcal{T}_h , with analogous meaning for $\boldsymbol{\tau}^1$ and $\boldsymbol{\tau}^2$, we define

$$\{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^1 + \boldsymbol{\tau}^2), \quad \llbracket \boldsymbol{\tau} \rrbracket = \boldsymbol{\tau}^1 \cdot \mathbf{n}_1^e + \boldsymbol{\tau}^2 \cdot \mathbf{n}_2^e \quad \text{on } e \in \mathcal{E}_h^\circ.$$

For $e \in \mathcal{E}_h^\partial$, the set of boundary edges, and \mathbf{n} = outward unit normal, we set

$$\llbracket \zeta \rrbracket = \zeta \mathbf{n}, \quad \{\zeta\} = \zeta, \quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}.$$

We shall also use the notation

$$(\nabla v, \nabla w)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \int_T \nabla v \cdot \nabla w dx \quad \langle v, w \rangle_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \int_e v w dl \quad \forall v, w, \text{ piecewise smooth.}$$

Let V^{DG} denote the discontinuous finite element space defined by:

$$V^{DG} := \{v \in L^2(\Omega) : v|_T \in \mathbb{P}^1(T) \forall T \in \mathcal{T}_h\}, \quad (2.1) \quad \text{defDG}$$

where $\mathbb{P}^1(T)$ is the space of polynomials of degree ≤ 1 on T . We note that, in general, the functions in V^{DG} will have no limit for \mathbf{x} tending to any point of the interelement boundaries. Therefore, to start with, we shall consider that they are defined only in the interior of each element. For $m \geq 1$ we denote by $H^m(\mathcal{T}_h)$ the broken H^m space, that is, the space of functions belonging to $H^m(T)$ for all $T \in \mathcal{T}_h$. We set

$$V(h) := V^{DG} + H^2(\Omega) \cap H_0^1(\Omega) \subset H^2(\mathcal{T}_h)$$

and for $v \in V(h)$ we define the seminorms and norms

$$|v|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{0,T}^2 \quad | [v]]_*^2 := \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v]] \|_{0,e}^2, \quad (2.2) \quad \boxed{\text{norm:semi}}$$

$$\|v\|^2 := |v|_{1,h}^2 + | [v]]_*^2 + \sum_{T \in \mathcal{T}_h} h_T^2 |v|_{2,T}^2, \quad (2.3) \quad \boxed{\text{norm:DG}}$$

(h_e being the length of the edge e for $d = 2$ and the diameter of the face e for $d = 3$). Occasionally, it might also be useful to separate the contribution to the norm $|\cdot|_*$ of internal and boundary edges, writing

$$| [v]]_*^2 = \sum_{e \in \mathcal{E}_h^\circ} h_e^{-1} \| [v]] \|_{0,e}^2 + \sum_{e \in \mathcal{E}_h^\partial} h_e^{-1} \| [v]] \|_{0,e}^2 =: | [v]]_{*,\mathcal{E}_h^\circ}^2 + | [v]]_{*,\mathcal{E}_h^\partial}^2$$

The norm $\boxed{\text{norm:DG}}$ (2.3) is the natural one for obtaining boundedness of typical DG-bilinear forms in spaces like $V(h)$. On the other hand, the weaker norm

$$v \mapsto \|v\|_{DG}^2 := (|v|_{1,h}^2 + | [v]]_*^2)^{1/2} \quad (2.4) \quad \boxed{\text{norm2}}$$

is the natural one for analyzing the stability. Restricted to $v \in V^{DG}$, the norms $\boxed{\text{norm:DG}}$ (2.3) and $\boxed{\text{norm2}}$ (2.4) are equivalent, as is evident from a local inverse inequality. We also remark that both $\boxed{\text{norm:DG}}$ (2.3) and $\boxed{\text{norm2}}$ (2.4) define norms, not just seminorms, on $V(h)$. Indeed, the discrete Poincaré inequality given in $\boxed{\text{arnoldBrenner03}}$ [3], or [7], implies the existence of a constant C for which

$$\|v\|_0 \leq C(|v|_{1,h}^2 + | [v]]_*^2)^{1/2} \quad \forall v \in V(h).$$

We recall the following trace inequality $\boxed{\text{Agm70a}}$ [2]

$$\|\varphi\|_{0,e}^2 \leq C_t (h_e^{-1} \|\varphi\|_{0,T}^2 + h_e \|\varphi\|_{1,T}^2) \quad \forall \varphi \in H^1(T), \quad (2.5) \quad \boxed{\text{trace0}}$$

where C_t is a constant depending only on the minimum angle of T . We observe that, denoting by K_e the union of elements having $e \in \mathcal{E}_h$ in common, the inequality $\boxed{\text{trace0}}$ (2.5) implies in particular

$$\| [[\varphi]] \|_{0,e}^2 + \| \{ \varphi \} \|_{0,e}^2 \leq 4 C_t (h_e^{-1} \|\varphi\|_{0,K_e}^2 + h_e \|\varphi\|_{1,K_e}^2) \quad \forall \varphi \in H^1(\mathcal{T}_h) \quad \forall e \in \mathcal{E}_h. \quad (2.6) \quad \boxed{\text{trace-jumps}}$$

We finally recall the useful formula

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} v \boldsymbol{\tau} \cdot \mathbf{n}_T = \sum_{e \in \mathcal{E}_h} \int_e [[v]] \cdot \{ \boldsymbol{\tau} \} + \sum_{e \in \mathcal{E}_h^\circ} \int_e \{ \boldsymbol{\tau} \} [[v]] . \quad (2.7) \quad \boxed{\text{magic}}$$

We further introduce, for every $e \in \mathcal{E}_h$, a penalty weight S_e of the form

$$S_e = \alpha_e h_e^{-1} \quad \text{with } \alpha^{**} \geq \alpha_e \geq \alpha^* > 0, \quad \forall e \in \mathcal{E}_h, \quad (2.8) \quad \boxed{\text{se}}$$

where α^{**} and α^* are values fixed once and for all. We also consider the operator \mathcal{S} from $L^2(\mathcal{E}_h)$ into itself, defined on each $e \in \mathcal{E}_h$ as $(\mathcal{S}v)|_e = S_e v|_e$. We can then consider the IIPG bilinear form $\tilde{\mathcal{A}}(\cdot, \cdot)$, defined by (see $\boxed{\text{WS}}$ [22]):

$$\tilde{\mathcal{A}}(v, w) = (\nabla v, \nabla w)_{\mathcal{T}_h} - \langle \{\nabla_h v\}, \llbracket w \rrbracket \rangle_{\mathcal{E}_h} + \langle \mathcal{S} \llbracket v \rrbracket, \llbracket w \rrbracket \rangle_{\mathcal{E}_h}. \quad (2.9) \quad \boxed{\text{ipA}}$$

The IIPG approximation to the solution of $\boxed{\text{mod0}}$ (I.1) reads:

$$\text{find } \tilde{u}_h \in V^{DG} \quad \text{such that } \tilde{\mathcal{A}}(\tilde{u}_h, w) = (f, w)_{\mathcal{T}_h}, \quad \forall w \in V^{DG}. \quad (2.10) \quad \boxed{\text{methodA}}$$

It is well known (see e.g. $\boxed{\text{WS}}$ [22] or $\boxed{\text{abcm}}$ [4]) that if α^* in (2.8) is *large enough* (depending of the minimum angle in the decomposition), then the bilinear form $\tilde{\mathcal{A}}(\cdot, \cdot)$ is coercive.

For each $e \in \mathcal{E}_h$ let $P_e^0 : L^2(e) \rightarrow \mathbb{P}^0(e)$ denote the L^2 -orthogonal projection onto constants. We denote by m_e the midpoint of the edge e or, in 3 dimensions, the barycenter of the face e . With an abuse of language, we will still call m_e "midpoint", and e "edge", even in 3 dimensions. We note that from elementary integration rules we have

$$P_e^0(v) := \frac{1}{|e|} \int_e v dl = v(m_e), \quad \forall v \in \mathbb{P}^1(e), \quad (2.11) \quad \boxed{\text{P0}}$$

where $|e|$ denotes the measure of e . We also consider the operator \mathcal{P} from $L^2(\mathcal{E}_h)$ into itself, that on each $e \in \mathcal{E}_h$ acts as P_e^0 . Using this projection we define the following bilinear form:

$$\mathcal{A}(v, w) = (\nabla v, \nabla w)_{\mathcal{T}_h} - \langle \{\nabla_h v\}, \llbracket w \rrbracket \rangle_{\mathcal{E}_h} + \langle \mathcal{S} \llbracket v \rrbracket, \mathcal{P}(\llbracket w \rrbracket) \rangle_{\mathcal{E}_h}, \quad (2.12) \quad \boxed{\text{ipA0}}$$

with \mathcal{S} defined as before. Note that the above bilinear form $\boxed{\text{ipA0}}$ (2.12) is nothing but what results upon performing numerical integration (with the midpoint rule) in the bilinear form given in $\boxed{\text{ipA}}$ (2.9).

We also recall the following well known inequality, whose proof can be found, for instance, in $\boxed{\text{abcm}}$ [4]:

$$\int_e \left| v \{\nabla w\} \cdot \nu^e \right| dl \lesssim \left(\frac{\alpha_e}{h_e} \int_e v^2 dl \right)^{1/2} \left(\|w\|_{1, K_e}^2 + h_T |w|_{2, K_e}^2 \right)^{1/2} \quad \forall e \in \mathcal{E}_h, \quad (2.13) \quad \boxed{\text{contint-bor}}$$

where ν^e is a unit normal to e , and K_e is again the union of triangles having e in common.

Using ^{lipA0}(2.12) we introduce the following variant of IIPG, that we call IIPG-0:

$$\text{find } u_h \in V^{DG} \text{ such that } \mathcal{A}(u_h, w) = (f, w)_{\mathcal{T}_h}, \quad \forall w \in V^{DG} \quad (2.14) \quad \text{methodA0}$$

that will be the object of most of the analysis of the present paper. This type of IP discretization (also called *weakly penalized*) has been considered before by other authors (see for instance ^{wonipgsip}[8], ^{az}[9], and ^{pcms}[5]). We note that, following ^{pcms}[10], $\mathcal{A}(\cdot, \cdot)$ can also be rewritten in the weighted residual framework as follows:

$$\mathcal{A}(v, w) = (-\Delta v, w)_{\mathcal{T}_h} + \langle \llbracket \nabla v \rrbracket, \{w\} \rangle_{\mathcal{E}_h^0} + \langle \mathcal{S}[\llbracket v \rrbracket], \mathcal{P}(\llbracket w \rrbracket) \rangle_{\mathcal{E}_h}, \quad \forall v, w \in V(h). \quad (2.15) \quad \text{resid}$$

Remark 2.1. Other variants of the original IIPG formulation ^{methodA}(2.10) could be considered. For instance one could use the so-called **strong boundary conditions**, that amounts to use, instead of V^{DG} , the smaller space

$$V_0^{DG} := \{v \in V^{DG} \text{ such that } P_e^0(v) = 0 \quad \forall e \in \mathcal{E}_h^0\} \quad (2.16) \quad \text{strongbc}$$

as done, for instance, in ^{burman-stam}[12] for the NIPG method. Another possible variant would be to use a sort of **superpenalty** in the definition of the penalty weight S_e , taking in ^{se}(2.8)

$$S_e = \alpha h_e^{-p}$$

where p (instead of being 1 as in ^{se}(2.8)) is bigger than 1, as done for instance in ^{wonipgsip}[8], [9]. Both variants are interesting, but somehow, lack the traditional flavor of DG methods (moving them toward the more classical conforming or nonconforming Finite Element methods). In other words, unless one has a specific need for these types of variants, "it's not Cricket".

3 Error estimates in the DG norm

sec:3

We now recall a result that will be used later on.

Lemma 3.1. For $w \in H^1(\mathcal{T}_h)$ it holds

$$|\mathcal{P}(\llbracket w \rrbracket)|_*^2 \leq |\llbracket w \rrbracket|_*^2 \lesssim (|w|_{1,h}^2 + |\mathcal{P}(\llbracket w \rrbracket)|_*^2) \equiv \|w\|_{DG}^2. \quad (3.1) \quad \text{equiv-norms}$$

Proof. The result is well known (see for instance ^{bhmm05}[11], or ^{az}[5]). We sketch the proof for convenience of the reader. Equation ^{equiv-norms}(3.1) can be expanded to

$$\sum_{e \in \mathcal{E}_h} h_e^{-1} \|P_e^0(\llbracket w \rrbracket)\|_{0,e}^2 \leq \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\llbracket w \rrbracket\|_{0,e}^2 \lesssim (|w|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|P_e^0(\llbracket w \rrbracket)\|_{0,e}^2)$$

The first inequality follows from the L^2 -boundedness of the projection \mathcal{P}_e^0 . For the second one, we observe that on each $e \subset \partial T$ and for each $\varphi \in H^1(T)$, adding and subtracting $P_e^0(\varphi)$, extending $P_e^0(\varphi)$ inside T , applying (2.5) and classical interpolation estimates we have

$$\begin{aligned} h_e^{-1} \|\varphi\|_{0,e}^2 &\leq h_e^{-1} \|\varphi - P_e^0(\varphi)\|_{0,e}^2 + h_e^{-1} \|P_e^0(\varphi)\|_{0,e}^2 \\ &\leq h_e^{-1} C_t (h_e^{-1} \|\varphi - P_e^0(\varphi)\|_{0,T}^2 + h_e |\varphi|_{1,T}^2) + h_e^{-1} \|P_e^0(\varphi)\|_{0,e}^2 \\ &\lesssim (|\varphi|_{1,T}^2 + h_e^{-1} \|P_e^0(\varphi)\|_{0,e}^2). \end{aligned}$$

Applying the above procedure to the jumps of w , as done for instance in (2.6), and summing over e we conclude the proof. \square

For the original IIPG approximation (2.10) optimal error estimates in the norm $\|\cdot\|_{DG}$ have been proved (see for instance [22]). For the solution of (2.14), optimal convergence in the DG norm can also be easily shown.

Theorem 3.2. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of (1.1), and let u_h be the solution of (2.14). There exists a constant $\bar{\alpha} > 0$, depending only on the minimum angle of the decompositions, such that for every choice of \mathcal{S} with $\alpha^* \geq \bar{\alpha}$ we have*

$$\|u - u_h\|_{DG} \lesssim h \|u\|_{2,\Omega}. \quad (3.2) \quad \text{est:DG}$$

Proof. Thanks to (3.1) and (2.13) one can easily check that there exist a constant C_b and (for α^* large enough) a constant C_s such that

$$\begin{aligned} \mathcal{A}(v, w) &\leq C_b \|v\| \|w\| & \forall v, w \in H^2(\mathcal{T}_h) \\ \mathcal{A}(v, v) &\geq C_s \|v\|_{DG}^2 & \forall v \in V^{DG}. \end{aligned}$$

Therefore, continuity and stability being satisfied, using standard arguments (see [4] for details), one can easily get the a-priori error estimate (3.2). \square

4 Additional properties of the discrete solution

sec:4

We shall discuss here some additional properties of the solution u_h of (2.14) that will be useful in proving L^2 error estimates.

For some of the results of the paper, we shall need to assume further regularity on the family of partitions \mathcal{T}_h . The next condition has been frequently used in the superconvergence analysis of conforming finite element methods (see for instance [18], [6]).

Definition: We say that a shape regular finite element partition \mathcal{T}_h is s -strongly regular if, for any pair of adjacent elements $T_1, T_2 \in \mathcal{T}_h$, the following condition is satisfied:

$$|T_1| - |T_2| \lesssim h^{d+s} \quad s > 0, \quad \forall T_1, T_2 \in \mathcal{T}_h \quad T_1 \cap T_2 \neq \emptyset. \quad (4.1) \quad \boxed{\text{strong:reg}}$$

We shall consider partitions that satisfy (4.1) with $s = 1$. Observe also that any shape regular partition satisfies (4.1) with $s = 0$.

cfrreg

Remark 4.1. We explicitly point out that our parameter s does not coincide with the parameter ζ in the definition of asymptotically ζ -uniform decomposition in [12] that in our notation would become, instead of (4.1),

$$|T_1| - |T_2| \lesssim h^{d+\zeta(d-1)}.$$

Hence, in a sense, $s = \zeta(d - 1)$

Following [5], we briefly review a decomposition of the space V^{DG} defined in (2.1), which will play a key role in our subsequent analysis.

In [12, 5] it was shown that

$$V^{DG} = V^{CR} \oplus \mathcal{Z},$$

where V^{CR} is the space of nonconforming piecewise linear elements (Crouzeix-Raviart), and \mathcal{Z} is a space of piecewise linear discontinuous elements having average with zero-meanvalue. More precisely:

$$V^{CR} = \{v \in V^{DG} \text{ such that } \mathcal{P}(\llbracket v \rrbracket) = 0\}, \quad (4.2) \quad \boxed{\text{defCR}}$$

$$\mathcal{Z} = \{z \in V^{DG} \text{ such that } \mathcal{P}_e^0(\{v\}) = 0 \forall e \in \mathcal{E}_h\}.$$

Note that every function $\varphi \in V^{CR}$ has a finite limit at every midpoint m_e , so that we can assign the value

$$\varphi(m_e) := \lim_{\mathbf{x} \rightarrow \mathbf{m}_e} \varphi(\mathbf{x})$$

making the functions in V^{CR} continuous at the midpoints of internal edges (by virtue of (2.11)), and vanishing at the midpoints m_e of boundary edges. On the other hand, every function $\psi \in \mathcal{Z}$ is such that $|\psi|$ has a finite limit at every midpoint m_e , so that we can give a meaning to the quantity

$$|\psi|(m_e) := \lim_{\mathbf{x} \rightarrow \mathbf{m}_e} |\psi(\mathbf{x})|,$$

and we note that

$$\llbracket \psi \rrbracket(m_e) = 2|\psi|(m_e) \quad \forall e \in \mathcal{E}_h^\circ, \quad \llbracket \psi \rrbracket(m_e) = |\psi|(m_e) \quad \forall e \in \mathcal{E}_h^\partial. \quad (4.3) \quad \text{jumpinZ}$$

In a sense, the functions in \mathcal{Z} could be considered, somehow, as “high frequency”.

It is quite natural to choose for both spaces V^{CR} and \mathcal{Z} a basis associated to the midpoints of the edges. Let T be an element with edges e_i , and corresponding midpoints m_{e_i} , $i = 1, \dots, d+1$. To T we associate $d+1$ basis functions satisfying (see Fig. [4.1](#))

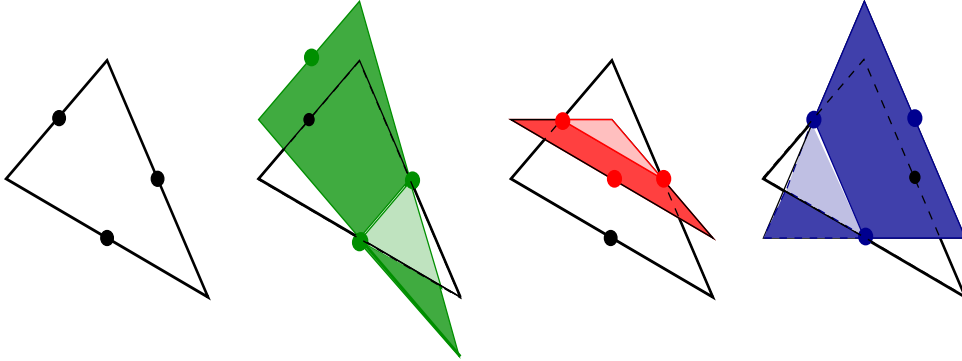


Figure 4.1: Local basis functions

fig1

$$\chi_T^{e_i}(x) \in \mathbb{P}^1(T) : \quad \chi_T^{e_i}(m_{e_j}) = \delta_{ij} \quad i, j = 1, \dots, d+1,$$

and being identically zero outside T . For any $e \in \mathcal{E}_h^\circ$, $e = \partial T_1 \cap \partial T_2$, we define

$$\varphi^e(x) = \chi_{T_1}^e(x) + \chi_{T_2}^e(x).$$

Note that the limit of φ^e , at every point of e , will be equal to 1 (see Fig. [4.2](#), right), so that

$$\{\varphi^e\}|_e = 1, \quad \llbracket \varphi^e \rrbracket|_e = 0.$$

For any edge $e \in \mathcal{E}_h^\circ$, $e = \partial T_1 \cap \partial T_2$, we denote by ν^e one of the two normal directions, chosen once and for all. We then define

$$\psi^e(x) = \chi_{T_1}^e(x) \mathbf{n}_1^e \cdot \nu^e + \chi_{T_2}^e(x) \mathbf{n}_2^e \cdot \nu^e, \quad (4.4) \quad \text{basis-zeta-}$$

and for $e \in \mathcal{E}_h^\partial$, $e \subset \partial T$, we take

$$\psi^e(x) = \chi_T^e(x). \quad (4.5) \quad \text{basis-zeta-}$$

We note that for every $e \in \mathcal{E}_h^\circ$ we have (see Fig. [4.2](#), left),

$$\{\psi^e\}_{|e} = 0, \quad |\psi^e|_{|e} = 1, \quad \llbracket \psi^e \rrbracket_{|e} = 2\nu^e, \quad (4.6) \quad \text{jumpofzint}$$

while for $e \in \mathcal{E}_h^\partial$

$$\psi^e|_e = 1, \quad \llbracket \psi^e \rrbracket_{|e} = \mathbf{n}. \quad (4.7) \quad \text{jumpofzest}$$

We observe that in $L^2(\Omega)$ the functions χ_T^e , φ^e , and ψ^e are orthogonal bases for V^{DG} ,

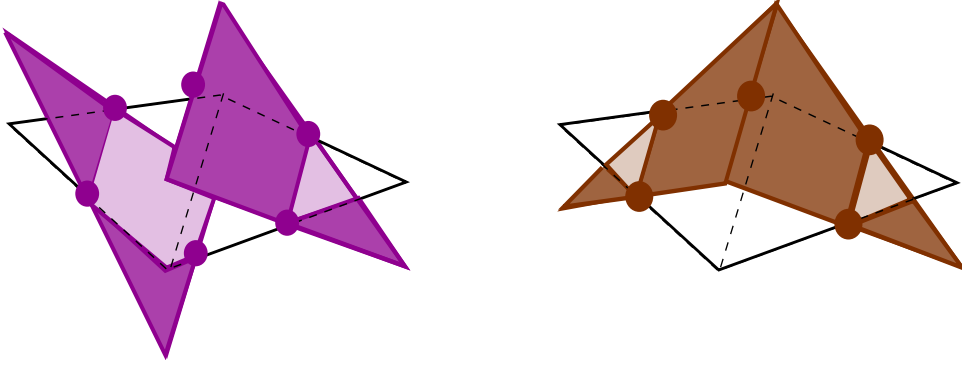


Figure 4.2: Global basis functions of \mathcal{Z} (left) and V^{CR} (right)

[fig2](#)

V^{CR} , and \mathcal{Z} , respectively.

We finally point out that for the functions $z \in \mathcal{Z}$, together with [\(3.1\)](#), we have some additional properties, shown in the following lemma.

Lemma 4.2. *For every $z \in \mathcal{Z}$ we have*

$$|z|_{1,h}^2 \lesssim \sum_{e \in \mathcal{E}_h} h_e^{-2} \|z\|_{0,K_e}^2 \simeq |\mathcal{P}(\llbracket z \rrbracket)|_*^2,$$

where K_e is the union of the elements of \mathcal{T}_h having e as an edge (or "face" in 3 dimensions).

Proof. The first inequality follows from the usual inverse inequality. For the second we just recall the orthogonality in $L^2(\Omega)$ of the basis functions ψ^e and note that, denoting by $|z_e|$ the value of $|z|$ at the midpoint m_e of each $e \in \mathcal{E}_h$, we have:

$$\sum_{e \in \mathcal{E}_h} h_e^{-2} \|z\|_{0,K_e}^2 \simeq \sum_{e \in \mathcal{E}_h} h_e^{-2} |z_e|^2 \|\psi^e\|_{0,K_e}^2 \simeq \sum_{e \in \mathcal{E}_h} h_e^{-1} |z_e|^2 |e| \simeq |\mathcal{P}(\llbracket z \rrbracket)|_*^2$$

where in the last step we also used [\(4.3\)](#). □

[jumpinz](#)

The following result can be found in [5].

properties

Proposition 4.3. *For any $v \in V^{DG}$ there exist a unique $v^{cr} \in V^{CR}$ and a unique $v^z \in \mathcal{Z}$ such that $v = v^{cr} + v^z$. Moreover, the following properties hold for the bilinear form (2.12):*

$$\begin{aligned} \mathcal{A}(v^{cr}, v^z) &= 0 \quad \forall v^{cr} \in V^{CR}, \quad \forall v^z \in \mathcal{Z}, \\ \mathcal{A}(v^{cr}, w^{cr}) &= (\nabla v^{cr}, \nabla w^{cr})_{\mathcal{T}_h} \quad \forall v^{cr}, w^{cr} \in V^{CR}, \\ \mathcal{A}(v^z, v^{cr}) &= (\nabla v^z, \nabla v^{cr})_{\mathcal{T}_h} \quad \forall v^z \in \mathcal{Z}, \quad \forall v^{cr} \in V^{CR}, \\ \mathcal{A}(v^z, w^z) &= \langle \mathcal{S}[[v^z]], \mathcal{P}([[w^z]]) \rangle_{\mathcal{E}_h} \quad \forall v^z, w^z \in \mathcal{Z}. \end{aligned}$$

Proof. (Sketch) The uniqueness of the decomposition follows, by looking at the basis functions. The second and third equalities simply follow from (2.12) using the properties of functions in V^{CR} and \mathcal{Z} . The first and fourth follow from (2.15) using again the properties of V^{CR} and \mathcal{Z} . \square

As a consequence of Proposition 4.3, problem (2.14) can be written as:

$$\left\{ \begin{array}{l} \text{Find } u_h = u^{cr} + u^z \text{ such that:} \\ i) \quad \langle \mathcal{S}[[u^z]], \mathcal{P}([[v^z]]) \rangle_{\mathcal{E}_h} = (f, v^z)_{\mathcal{T}_h} \quad \forall v^z \in \mathcal{Z} \\ ii) \quad (\nabla u^{cr}, \nabla v^{cr})_{\mathcal{T}_h} = (f, v^{cr})_{\mathcal{T}_h} - (\nabla u^z, \nabla v^{cr})_{\mathcal{T}_h} \quad \forall v^{cr} \in V^{CR} \end{array} \right. \quad (4.8)$$

pb-decouple

Observe that this last result implies that the solution of (2.14) reduces to solve two smaller and decoupled subproblems, one after the other.

The next Lemma provides a useful estimate, based on the fact that the linear system associated with (4.8) *i)* is diagonal.

le:z1

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, let $f \in L^2(\Omega)$, and let $u_h \in V_h^{DG}$ be the solution of (2.14). Let $u^{cr} \in V^{CR}$ and $u^z \in \mathcal{Z}_h$ be such that $u_h = u^{cr} + u^z$. Then we have*

$$\begin{aligned} \frac{2\alpha_e |e|}{h_e} [[u^z]](m_e) \cdot \boldsymbol{\nu}^e &= \int_{\Omega} f \psi^e dx \quad \forall e \in \mathcal{E}_h^o; \\ \frac{\alpha_e |e|}{h_e} [[u^z]](m_e) \cdot \mathbf{n} &= \frac{\alpha_e |e|}{h_e} u^z(m_e) = \int_{\Omega} f \psi^e dx \quad \forall e \in \mathcal{E}_h^\partial, \end{aligned}$$

where ψ^e is the basis function associated to the edge e as defined in (4.4)-(4.5).

basis-basis-eta-est

Proof. The proof follows immediately from (4.8) ^{pb-decoupled} v , taking $v^z = \psi^e$ and using (2.8) ^{se} together with the properties of the basis functions of \mathcal{Z} ; (4.6), and (4.7). \square

It is clear, from the above result, that it will be convenient to estimate quantities like

$$\int_{\Omega} f \psi^e dx$$

where f is a smooth enough function and ψ^e is one of the basis functions of \mathcal{Z} , associated to an edge e .

(i) If f is constant and \mathcal{T}_h is uniform, then

$$\int_{\Omega} f \psi^e dx = 0 \quad \forall e \in \mathcal{E}_h^{\circ}$$

since ψ^e is antisymmetric with respect to the edge e .

(ii) For $f \in H^1(\Omega)$, if \mathcal{T}_h is uniform, for all $e \in \mathcal{E}_h^{\circ}$ setting $K_e := \text{supp}(\psi^e)$ and $\bar{f} :=$ average of f over K_e we get

$$\left| \int_{\Omega} f \psi^e dx \right| = \left| \int_{K_e} (f - \bar{f}) \psi^e dx \right| \lesssim h_e \|f\|_{1,K_e} \|\psi^e\|_{0,K_e} \lesssim h_e^{1+d/2} \|f\|_{1,K_e}.$$

(iii) For $f \in H^1(\Omega) \cap L^{\infty}(\Omega)$, if \mathcal{T}_h is s -strongly-regular, with $s > 0$ as defined in (4.1) ^{strong:reg}, then for all $e \in \mathcal{E}_h^{\circ}$, denoting by T_1 and T_2 the elements having e in common, we have

$$\begin{aligned} \left| \int_{\Omega} f \psi^e dx \right| &= \left| \int_{K_e} (f - \bar{f}) \psi^e dx + \int_{K_e} \bar{f} \psi^e dx \right| \\ &\lesssim h_e^{1+d/2} \|f\|_{1,K_e} + \|f\|_{0,\infty,K_e} \left| |T_1| - |T_2| \right| \\ &\lesssim h_e^{1+d/2} \|f\|_{1,K_e} + h_e^{d+s} \|f\|_{0,\infty,\Omega}. \end{aligned} \quad (4.9) \quad \boxed{\text{ze:1}}$$

(iv) For $f \in L^{\infty}(\Omega)$, for all $e \in \mathcal{E}_h^{\partial}$ and always with $K_e := \text{supp}(\psi^e)$, we have

$$\left| \int_{\Omega} f \psi^e dx \right| \lesssim |K_e| \|f\|_{0,\infty,K_e} \lesssim h_e^d \|f\|_{0,\infty,\Omega}. \quad (4.10) \quad \boxed{\text{ze:2}}$$

We collect in particular the results (4.9) ^{ze:1}, for $s = 1$, and (4.10) ^{ze:2} in the following theorem, that we are going to use for the L^2 error estimates.

to:use

Theorem 4.5. Let $\Omega \subset \mathbb{R}^d, d \geq 1$, let $f \in H^1(\Omega) \cap L^\infty(\Omega)$, and let \mathcal{T}_h be an s -strongly regular finite element partition of Ω , as defined in (4.1). Let moreover $u_h = u^{cr} + u^z$ be the solution of (4.8). Then we have

$$|\mathcal{P}(\llbracket u^z \rrbracket)|_{*,\mathcal{E}_h^\circ}^2 = \sum_{e \in \mathcal{E}_h^\circ} \frac{|e|}{h_e} |\llbracket u^z \rrbracket(m_e)|^2 \lesssim h^4 \|f\|_{1,\Omega}^2 + h^{2+2s} \|f\|_{0,\infty,\Omega}^2, \quad (4.11) \quad \text{ze:1a}$$

and for boundary edges:

$$\|\mathcal{P}(\llbracket u^z \rrbracket)\|_{0,\partial\Omega}^2 = \sum_{e \in \mathcal{E}_h^\partial} |e| |u^z(m_e)|^2 \lesssim h^4 \|f\|_{0,\infty,\Omega}^2. \quad (4.12) \quad \text{ze:1b}$$

Proof. The proof of (4.11) is immediate, using (4.9) from Lemma 4.4 and the fact that $\sum_{e \in \mathcal{E}_h^\circ} h_e^d \simeq |\Omega|$ and $|e| \simeq h_e^{d-1}$:

$$\begin{aligned} \sum_{e \in \mathcal{E}_h^\circ} \frac{|e|}{h_e} |\llbracket u^z \rrbracket(m_e)|^2 &= \sum_{e \in \mathcal{E}_h^\circ} \left(\frac{|e|}{h_e} |\llbracket u^z \rrbracket(m_e)| \right)^2 \frac{h_e}{|e|} \lesssim \sum_{e \in \mathcal{E}_h^\circ} \left(\frac{|e|}{h_e} |\llbracket u^z \rrbracket(m_e)| \right)^2 h_e^{2-d} \\ &\lesssim \sum_{e \in \mathcal{E}_h^\circ} h_e^{2-d} h_e^{2+d} \|f\|_{1,K_e}^2 + \sum_{e \in \mathcal{E}_h^\circ} h_e^{2-d} h_e^{2d+2s} \|f\|_{0,\infty,K_e}^2 \\ &\lesssim h^4 \sum_{e \in \mathcal{E}_h^\circ} \|f\|_{1,K_e}^2 + h_e^{2+2s} \|f\|_{0,\infty,\Omega}^2 \sum_{e \in \mathcal{E}_h^\circ} h_e^d \\ &\lesssim h^4 \|f\|_{1,\Omega}^2 + h^{2+2s} \|f\|_{0,\infty,\Omega}^2, \end{aligned}$$

while the proof of (4.12) uses (4.10) again from Lemma 4.4 and the fact that $\sum_{e \in \mathcal{E}_h^\partial} h_e^{d-1} \simeq |\partial\Omega|$:

$$\begin{aligned} \sum_{e \in \mathcal{E}_h^\partial} |e| |\llbracket u^z \rrbracket(m_e)|^2 &= \sum_{e \in \mathcal{E}_h^\partial} \left(\frac{|e|}{h_e} |\llbracket u^z \rrbracket(m_e)| \right)^2 \frac{h_e^2}{|e|} \lesssim \sum_{e \in \mathcal{E}_h^\partial} \left(\frac{|e|}{h_e} |\llbracket u^z \rrbracket(m_e)| \right)^2 h_e^{3-d} \\ &\lesssim \sum_{e \in \mathcal{E}_h^\partial} h_e^{3-d} h_e^{2d} \|f\|_{0,\infty,\Omega}^2 \lesssim h^4 \|f\|_{0,\infty,\Omega}^2 \sum_{e \in \mathcal{E}_h^\partial} h_e^{d-1} \lesssim h^4 \|f\|_{0,\infty,\Omega}^2. \end{aligned}$$

□

Remark 4.6. When using strong boundary conditions (see (2.16)), the estimate (4.11) would easily imply

$$\|u^z\|_{DG} \lesssim h^2 \|f\|_{1,\Omega} + h^{1+s} \|f\|_{0,\infty,\Omega}. \quad (4.13) \quad \text{stimauzsbc}$$

On the other hand, for our case, combining (4.11) and (4.12) one does not get anything better than

$$\|u^z\|_{DG} \lesssim (h^{\frac{3}{2}} + h^{1+s})(\|f\|_{1,\Omega}^2 + \|f\|_{0,\infty,\Omega}^2)^{1/2}. \quad (4.14) \quad \text{stimauzwb}$$

5 L^2 -Error Analysis

sec:5
teo1

Theorem 5.1. Let $\Omega \subset \mathbb{R}^d, d \geq 1$ be a convex domain. Let $f \in H^1(\Omega)$ and let u be the solution of the Poisson problem (1.1). Let \mathcal{T}_h be an s -strongly regular finite element partition of Ω , as defined in (4.1), and let $u_h \in V_h^{DG}$ be the solution of (2.14) (or, equivalently, of (4.8)). Then, the following error estimate holds

$$\|u - u_h\|_{0,\Omega} \lesssim (h^2 + h^{1+s})(\|f\|_{1,\Omega}^2 + \|f\|_{0,\infty,\Omega}^2)^{1/2}. \quad (5.1) \quad \text{aprovar}$$

Proof. We proceed by standard duality arguments. Let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the dual problem

$$-\Delta\psi = u - u_h \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega.$$

The convexity of the domain Ω guarantees that the solution ψ satisfies the a-priori estimate

$$\|\psi\|_{2,\Omega} \lesssim \|u - u_h\|_{0,\Omega}.$$

Let ψ^I be the continuous piecewise linear interpolant of ψ . Standard approximation properties guarantee that (see [14]):

$$\|\psi - \psi^I\|_{0,\Omega} + h|\psi - \psi^I|_{1,h} \lesssim h^2\|\psi\|_{2,\Omega} \lesssim \|u - u_h\|_{0,\Omega}, \quad (5.2) \quad \text{aprox1}$$

as well as

$$\left\| \frac{\partial\psi^I}{\partial\mathbf{n}} \right\|_{0,\partial\Omega} \lesssim \|\psi\|_{2,\Omega} \lesssim \|u - u_h\|_{0,\Omega}. \quad (5.3) \quad \text{aprox11}$$

We also observe that Galerkin orthogonality, the definition (2.12) and $[\![\psi^I]\!] = 0$, imply

$$\mathcal{A}(u - u_h, \psi^I) \equiv (\nabla(u - u_h), \nabla\psi^I)_{\mathcal{T}_h} = 0. \quad (5.4) \quad \text{Galpsi}$$

Using the definition of the L^2 -norm, integrating by parts, using (2.7) and the regularity of ψ , then adding and subtracting ψ^I and using (5.4), and finally separating internal and

boundary edges, we have

$$\begin{aligned}
\|u - u_h\|_{0,\Omega}^2 &= (u - u_h, u - u_h)_{\mathcal{T}_h} = (u - u_h, -\Delta\psi)_{\mathcal{T}_h} \\
&= (\nabla(u - u_h), \nabla\psi)_{\mathcal{T}_h} - \langle \llbracket u - u_h \rrbracket, \{\nabla\psi\} \rangle_{\mathcal{E}_h} - \langle \{u - u_h\}, \llbracket \nabla\psi \rrbracket \rangle_{\mathcal{E}_h^\circ} \\
&= (\nabla(u - u_h), \nabla\psi)_{\mathcal{T}_h} - \langle \llbracket u - u_h \rrbracket, \{\nabla\psi\} \rangle_{\mathcal{E}_h} \\
&= (\nabla(u - u_h), \nabla(\psi - \psi^I))_{\mathcal{T}_h} - \langle \llbracket u - u_h \rrbracket, \{\nabla(\psi - \psi^I)\} \rangle_{\mathcal{E}_h} - \langle \llbracket u - u_h \rrbracket, \{\nabla\psi^I\} \rangle_{\mathcal{E}_h} \\
&= (\nabla(u - u_h), \nabla(\psi - \psi^I))_{\mathcal{T}_h} - \langle \llbracket u - u_h \rrbracket, \{\nabla(\psi - \psi^I)\} \rangle_{\mathcal{E}_h} \\
&\quad - \langle \llbracket u - u_h \rrbracket, \{\nabla\psi^I\} \rangle_{\mathcal{E}_h^\circ} - \langle \llbracket u - u_h \rrbracket, \{\nabla\psi^I\} \rangle_{\mathcal{E}_h^\partial} =: I + II + III + IV.
\end{aligned}$$

We then get, using Cauchy-Schwarz, [\(3.2\)](#) ^{est:DG}, and [\(5.2\)](#) ^{approx1}:

$$\begin{aligned}
|I| &:= |(\nabla(u - u_h), \nabla(\psi - \psi^I))_{\mathcal{T}_h}| \leq |u - u_h|_{1,h} |\psi - \psi^I|_{1,h} \\
&\lesssim h^2 \|u - u_h\|_{0,\Omega}.
\end{aligned} \tag{5.5} \quad \boxed{\text{pezzoI}}$$

On the other hand, using [\(2.13\)](#) ^{contintleborDG}, [\(3.2\)](#), and [\(5.2\)](#) ^{approx1}:

$$\begin{aligned}
|II| &:= |\langle \llbracket u - u_h \rrbracket, \{\nabla(\psi - \psi^I)\} \rangle_{\mathcal{E}_h}| \lesssim \|u - u_h\|_* (\|\psi - \psi^I\|_{1,h}^2 + h^2 |\psi - \psi^I|_{2,h}^2)^{1/2} \\
&\lesssim h^2 \|u - u_h\|_{0,\Omega}.
\end{aligned} \tag{5.6} \quad \boxed{\text{pezzoII}}$$

To deal with *III* and *IV* we note first that $\llbracket u \rrbracket = 0$. Next, since $\{\nabla\psi^I\}$ is constant, $\llbracket u_h \rrbracket$ can be replaced by $\mathcal{P}(\llbracket u_h \rrbracket)$. Moreover, $\mathcal{P}(\llbracket u_h \rrbracket) = \mathcal{P}(\llbracket u^{cr} + u^z \rrbracket) = \mathcal{P}(\llbracket u^z \rrbracket)$ since, by definition [\(4.2\)](#) ^{defCR} of V^{CR} , $\mathcal{P}(\llbracket u^{cr} \rrbracket) \equiv 0$. Hence:

$$III + IV = -\langle \mathcal{P}(\llbracket u^z \rrbracket), \{\nabla\psi^I\} \rangle_{\mathcal{E}_h^\circ} - \langle \mathcal{P}(\llbracket u^z \rrbracket), \{\nabla\psi^I\} \rangle_{\mathcal{E}_h^\partial}.$$

From Lemma [4.4](#) ^{le:z1} we have:

$$\begin{aligned}
|III| &= |\langle \mathcal{P}(\llbracket u^z \rrbracket), \{\nabla\psi^I\} \rangle_{\mathcal{E}_h^\circ}| = \left| \sum_{e \in \mathcal{E}_h^\circ} \int_e \llbracket u^z \rrbracket(m_e) \cdot \{\nabla\psi^I\} dl \right| \\
&= \left| \sum_{e \in \mathcal{E}_h^\circ} \left(\llbracket u^z \rrbracket(m_e) \cdot \nu^e \right) \left(|e| \{\nabla\psi^I\} \cdot \nu^e \right) \right| \\
&= \left| \sum_{e \in \mathcal{E}_h^\circ} \left(\frac{h_e}{2\alpha_e} \int_\Omega f \psi^e dx \right) \{\nabla\psi^I\} \cdot \nu^e \right| =: \left| \int_\Omega f g dx \right|,
\end{aligned} \tag{5.7} \quad \boxed{\text{pezzoIII-a}}$$

having set

$$g(x) := \sum_{e \in \mathcal{E}_h^\circ} \frac{h_e}{2\alpha_e} (\{\nabla\psi^I\} \cdot \nu^e) \psi^e(x).$$

Let \bar{f} be the piecewise constant approximation of f on \mathcal{T}_h . Then, by adding and subtracting \bar{f} , using the Cauchy-Schwarz inequality and classical approximation estimates we have:

$$\int_{\Omega} f g dx = \int_{\Omega} (f - \bar{f}) g dx + \int_{\Omega} \bar{f} g dx \quad (5.8)$$

$$\lesssim \left(\sum_{T \in \mathcal{T}_h} h_T^{-2} \|f - \bar{f}\|_{0,T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|g\|_{0,T}^2 \right)^{1/2} \quad (5.9)$$

$$+ \left(\sum_{T \in \mathcal{T}_h} |T| (\bar{f}|_T)^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} |T|^{-1} \left(\int_T g dx \right)^2 \right)^{1/2} \quad (5.10)$$

$$\lesssim \|f\|_{1,\Omega} \left(\sum_{T \in \mathcal{T}_h} \left(h_T^2 \|g\|_{0,T}^2 + |T|^{-1} \left(\int_T g dx \right)^2 \right) \right)^{1/2}. \quad (5.11)$$

$$\|g\|_{0,T}^2 = \int_T \sum_{e \subset \partial T} \left(\frac{h_e}{2\alpha_e} \{ \nabla \psi^I \} \cdot \boldsymbol{\nu}^e \psi^e(x) \right)^2 = \sum_{e \subset \partial T} \left(\frac{h_e}{2\alpha_e} \{ \nabla \psi^I \} \cdot \boldsymbol{\nu} \right)^2 \frac{|T|}{d+1} \quad (5.12)$$

$$\lesssim h_T \sum_{e \subset \partial T} h_e^2 \| \{ \nabla \psi^I \} \|_{0,e}^2 \lesssim h_T \sum_{e \subset \partial T} h_e \| \psi \|_{2,K_e}. \quad (5.13)$$

Consequently,

$$\sum_{T \in \mathcal{T}_h} h_T^2 \|g\|_{0,T}^2 \lesssim h^4 \| \psi \|_{2,\Omega}. \quad (5.14) \quad \boxed{\text{normalL2-g}}$$

$$\int_T g dx = \sum_{e \subset \partial T} \frac{h_e}{2\alpha_e} \{ \nabla \psi^I \} \cdot \boldsymbol{\nu} (\mathbf{n}_e \cdot \boldsymbol{\nu}^e) \frac{|T|}{d+1} \lesssim h_T \sum_{e \subset \partial T} h_e \int_e \{ \nabla \psi^I \} \cdot \mathbf{n}_e \quad (5.15)$$

$$= h_T \sum_{e \subset \partial T} h_e \left(\int_e [\{ \nabla \psi^I \} - \nabla \psi] \cdot \mathbf{n}_e dl + \int_e \nabla \psi \cdot \mathbf{n}_e dl \right) \quad (5.16)$$

$$\lesssim h_T \sum_{e \subset \partial T} h_e |e|^{1/2} \| \{ \nabla \psi^I \} - \nabla \psi \|_{0,e} + h_T^2 \int_{\partial T} \frac{\partial \psi}{\partial \mathbf{n}} dl \quad (5.17)$$

$$\lesssim h_T \sum_{e \subset \partial T} h_e^{1/2} |e|^{1/2} h^{1/2} \| \psi \|_{2,K_e} + h_T h_e |T|^{1/2} \| \Delta \psi \|_{0,T} \quad (5.18)$$

$$\lesssim h^{2+d/2} \sum_{e \subset \partial T} \| \psi \|_{2,K_e} + h^{2+d/2} \| \psi \|_{2,T} \quad (5.19)$$

Hence,

$$\sum_{T \in \mathcal{T}_h} |T|^{-1} \left(\int_T g \, dx \right)^2 \simeq \sum_{T \in \mathcal{T}_h} h_T^{-d} \left(\int_T g \, dx \right)^2 \lesssim h^4 \left(\sum_{e \subset \partial T} \|\psi\|_{2, K_e}^2 + \|\psi\|_{2, T}^2 \right) \quad (5.20)$$

$$\lesssim h^4 \|\psi\|_{2, \Omega} \quad (5.21)$$

Collecting everything we deduce

$$|III| \lesssim h^2 \|f\|_{1, \Omega} \|\psi\|_{2, \Omega} \lesssim h^2 \|u - u_h\|_{0, \Omega} \quad (5.22) \quad \text{pezzoIII}$$

Finally, we estimate IV using [\(4.12\)](#) and [\(5.3\)](#)

$$\begin{aligned} |IV| &= |\langle \mathcal{P}(\llbracket u^z \rrbracket), \{\nabla \psi^I\} \rangle_{\mathcal{E}_h^\partial}| \lesssim \|\mathcal{P}(\llbracket u^z \rrbracket)\|_{0, \partial \Omega} \left\| \frac{\partial \psi^I}{\partial \mathbf{n}} \right\|_{0, \partial \Omega} \\ &\lesssim h^2 \|f\|_{0, \infty, \Omega} \left\| \frac{\partial \psi^I}{\partial \mathbf{n}} \right\|_{0, \partial \Omega} \lesssim h^2 \|u - u_h\|_{0, \Omega}, \end{aligned} \quad (5.23) \quad \text{pezzoIV}$$

Collecting [\(5.5\)](#)–[\(5.6\)](#) and [\(5.22\)](#)–[\(5.23\)](#) we conclude the estimate. \square

Remark 5.1. *The above approach could also be applied to deal with the NIPG-0 scheme (see e.g. [\[12\]](#)). We recall that the NIPG-0 scheme could be written as: find $u_h \in V^{DG}$ such that*

$$\mathcal{A}^N(u_h, w) = (f, w)_{\mathcal{T}_h}, \quad \forall w \in V^{DG} \quad (5.24) \quad \text{NIPG0}$$

where \mathcal{A}^N is defined by

$$\mathcal{A}^N(v, w) := \mathcal{A}(v, w) + \langle \llbracket v \rrbracket, \{\nabla_h w\} \rangle_{\mathcal{E}_h} \quad \forall v, w \in V^{DG} \quad (5.25) \quad \text{defAN}$$

and \mathcal{A} is still the bilinear form defined in [\(2.12\)](#). We note that for the NIPG-0 case we would still have a lower block triangular system similar to [\(4.8\)](#), but we could not localize the estimates on u^z as it was done in Lemma [4.4](#). However, from [\(5.25\)](#), [\(2.12\)](#), and then [\(5.24\)](#) we could easily have

$$|u^z|_{1, h}^2 + \langle \mathcal{S}\mathcal{P}(\llbracket u^z \rrbracket), \mathcal{P}(\llbracket u^z \rrbracket) \rangle_{\mathcal{E}_h} = \mathcal{A}^N(u^z, u^z) = (f, u^z)_{\mathcal{T}_h}$$

that, together with the estimates [\(4.9\)](#) and [\(4.10\)](#), would still allow us to get [\(4.14\)](#) (or even [\(4.13\)](#) if strong boundary conditions were used). This, together with the known (optimal) error estimates for NIPG-0 in the DG-norm (see e.g. [\[12\]](#)), would still allow to follow the lines of the proof of Theorem [5.1](#) and get, for an s -strongly regular decomposition

$$\|u - u_h\|_{0, \Omega} \lesssim (h^{\frac{3}{2}} + h^{1+s}) (\|f\|_{1, \Omega}^2 + \|f\|_{0, \infty, \Omega}^2)^{1/2}, \quad (5.26) \quad \text{aprovarN}$$

and

$$\|u - u_h\|_{0,\Omega} \lesssim (h^2 + h^{1+s}) (\|f\|_{1,\Omega}^2 + \|f\|_{0,\infty,\Omega}^2)^{1/2} \quad (5.27)$$

if strong boundary conditions were also used. Note that, in view of Remark 4.1, the term h^{1+s} appearing in (5.26) and (5.27) requires less regular decompositions than the corresponding h^s appearing in [12, Theorem 8.13].

aprovarNs

6 Numerical Experiments

sec:6

In this section we present some numerical experiments that validate the analysis for the IIPG-0 discretization. Moreover, some tests are devoted to compare, at least in a simple test case, the performance of the IIPG-0 and IIPG methods. For completeness, we also provide comparison with the Symmetric Interior Penalty method (SIPG [3]) and its weakly penalized version, SIPG-0. The non-symmetric Interior Penalty method (NIPG [19, 20]), and its weakly penalized version, were also considered. However, as the results are very similar to those obtained with the IIPG and IIPG-0 methods, to keep the clarity of the graphics we have chosen not to report them here.

The experiments are performed with a simple test case on the unit square $\Omega = (0, 1)^2$, using piecewise linear approximations on triangular grids, structured and unstructured. The forcing term f is chosen so that the analytical solution of (1.1) is given by $u(x, y) = \sin(2\pi x) \sin(2\pi y)$. For all the methods, the parameter α entering in the definition of the penalty weight S defined in (2.8) is set to $\alpha = 5$.

In all the graphics the original IP methods are represented with continuous line, and the corresponding IP-0 methods with dashed lines. More precisely, the IIPG method is represented by $-\square-$; the IIPG-0 method by $-\cdot-\circ-\cdot-\cdot-$; the SIPG method by $-\triangle-$, and the SIPG-0 method by $-\cdot-\nabla-\cdot-\cdot-$.

We first study the convergence of the IIPG and IIPG-0 methods. In Fig. 6.1 are depicted the convergence diagrams for the IIPG and IIPG-0 methods in several norms, on structured meshes. From the graphics it can be seen that both methods attain second order convergence in the L^2 -norm (left diagram), and first order in the $\|\cdot\|_{DG}$ -norm (right diagram). As it should be expected, in the “jump”-seminorm $|\cdot|_*$ the original IIPG method outperforms the IIPG-0 method. However, since the error in the H^1 -broken seminorm $|\cdot|_{1,h}$ is the dominant term in the error $\|u - u_h\|_{DG}$ (which is somehow natural since the exact solution is very smooth), both methods produce approximation with the same accuracy in this norm. In Fig. 6.3, we represent the convergence diagrams on unstructured meshes for four methods: IIPG, IIPG-0, SIPG, SIPG-0. Three of the unstructured meshes used in our computations are given in Fig. 6.2: the coarsest mesh (left figure) and two successive refinements of it (center and right figure). Notice that

fig1a

fig2a

fig0

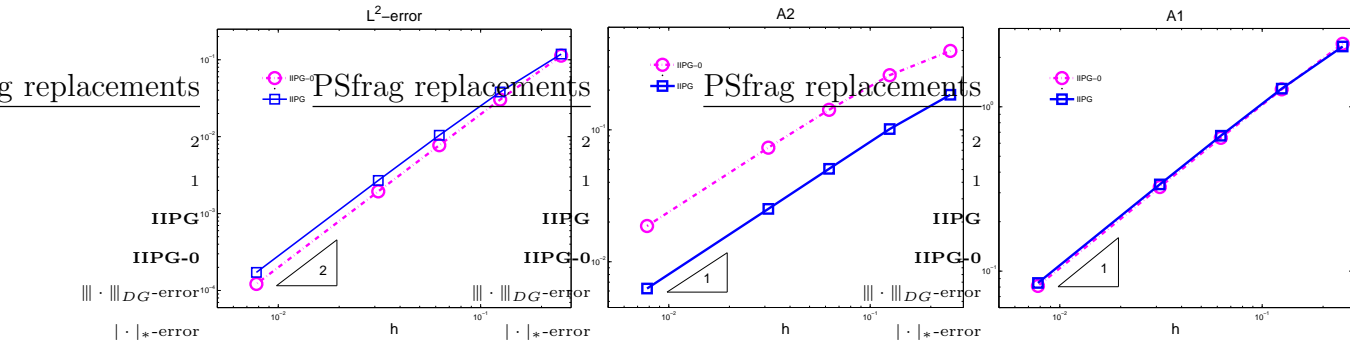


Figure 6.1: Convergence diagrams on structured meshes for the IIPG and IIPG-0 methods in several norms: L^2 -norm (left); $|\cdot|_*$ seminorm (center), and $|||\cdot|||_{DG}$ -norm (right).

fig1a

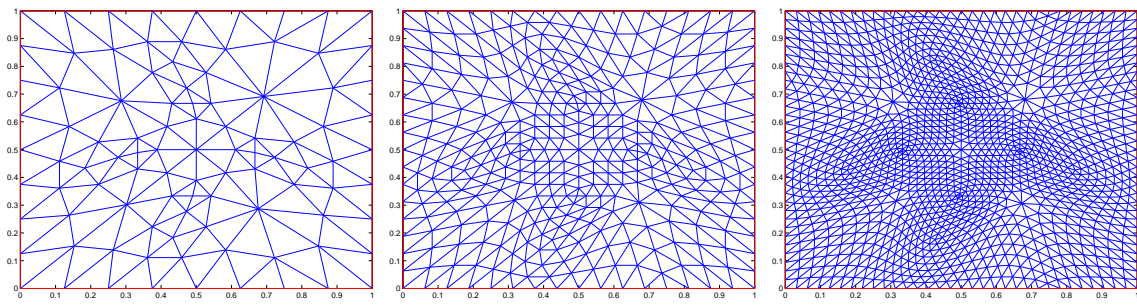


Figure 6.2: Unstructured meshes used in the computations: the coarsest mesh (left figure); first refinement (center) and second refinement (right figure).

fig0

these meshes are not 1-strongly regular (see definition (4.1)), and therefore our Theorem 5.1 does not apply directly. However we point out that when using a sequence of uniform refinements of a given coarse mesh we are approaching, asymptotically, the situation of 1-strongly regular meshes, as the number of interelement edges where the condition is *not* satisfied grows like $O(h)$ whereas the total number of internal edges grows like $O(h^2)$. Indeed, as it can be appreciated, also in this case the IIPG-0 (and actually all the other methods) converges in L^2 with second order.

From the results in the diagrams it can be also observed that all the weakly penalized methods give slightly smaller errors than the corresponding original ones in the L^2 -norm. As it happened before, in the $|\cdot|_*$ -seminorm the approximations with the original IP methods are clearly more accurate. Analogous results, although not reported here, were obtained on structured meshes for all the methods.

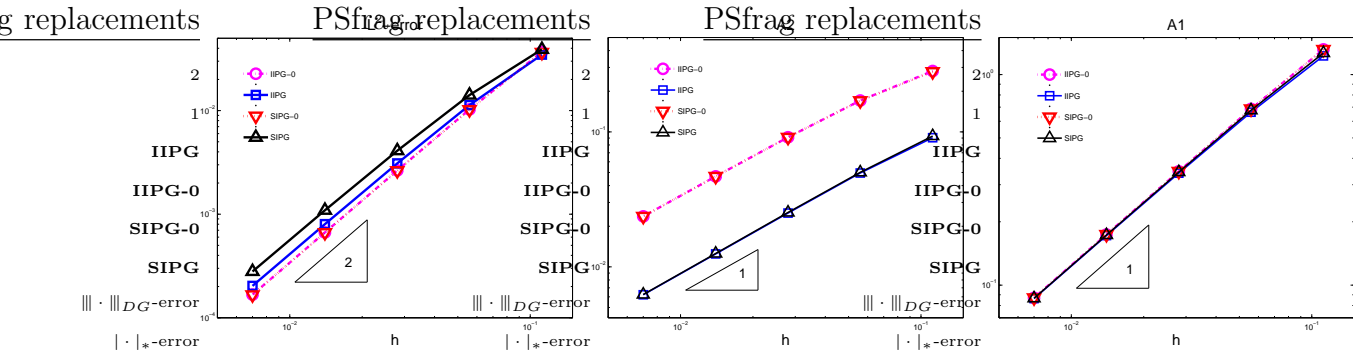


Figure 6.3: Convergence diagrams on unstructured meshes for IIPG, IIPG-0, SIPG and SIPG-0 in several norms: L^2 -norm (left); $|\cdot|_*$ semi-norm (center); $||| \cdot |||_{DG}$ -norm (right).

fig2a

6.1 Sharpness of the optimal L^2 -estimate: a counterexample

We finally present a simple numerical experiment demonstrating that the regularity of the right-hand side f assumed in our analysis (and more precisely in (5.1)) is somehow necessary to obtain optimal L^2 -order of convergence. We recall that in general, see e.g. [4], one expects an L^2 estimate of the form

$$\exists C > 0 \quad \text{such that} \quad \forall h > 0, \forall f \in L^2(\Omega) \quad \|u(f) - u_h(f)\|_{0,\Omega} \leq Ch^2 \|f\|_{0,\Omega}, \quad (6.1)$$

conv:teo

where $u(f)$ and $u_h(f)$ are the exact and (respectively) the approximate solution of our problem (1.1) having f as right-hand side. The aim of this section is to give numerical

evidence that denies ^(conv:teo)(6.1) for all the non symmetric methods IIPG, IIPG-0, NIPG, and NIPG-0. We consider a simple one-dimensional example on the unit interval $[0, 1]$:

$$-u_{xx} = f \quad \text{in } [0, 1], \quad u = 0 \quad \text{at } \{0\} \text{ and } \{1\}. \quad (6.2) \quad \boxed{\text{mod1D}}$$

We start by noting that in one-dimension the two methods IIPG and IIPG-0 coincide, and hence produce the same approximate solutions. In a similar way NIPG and NIPG-0 also coincide. Hence in what follows we will simply refer to IIPG and NIPG. We also included for comparison the corresponding results obtained with the SIPG discretizations. We aim at showing that for the IIPG and NIPG approximations it holds:

$$\forall C_0 > 0 \quad \exists f \in L^2([0, 1]) \quad \text{and} \quad \exists h > 0, \quad \text{s.t.} \quad \|u(f) - u_h(f)\|_0 > C_0 h^2 \|f\|_0.$$

Actually, we will show something a bit stronger. Namely, we show that

$$\forall C_0 > 0 \quad \forall h > 0 \quad \exists f^* = f^*(h) \in L^2([0, 1]) \quad \text{s.t.} \quad \mathcal{Q}_2 := \frac{\|u(f^*) - u_h(f^*)\|_0}{h^2 \|f^*\|_0} > C_0. \quad (6.3) \quad \boxed{\text{nego0}}$$

In particular, we will show that the quotient \mathcal{Q}_2 grows linearly as h decreases, and cannot be uniformly bounded, contrary to the behavior of the SIPG. In other words:

$$\begin{aligned} \mathcal{Q}_2^{IIPG}(h, f^*(h)) &\longrightarrow \infty \quad \text{as } h \rightarrow 0, \\ \mathcal{Q}_2^{NIPG}(h, f^*(h)) &\longrightarrow \infty \quad \text{as } h \rightarrow 0, \\ \mathcal{Q}_2^{SIPG}(h, f^*(h)) &\simeq 1. \end{aligned} \quad (6.4) \quad \boxed{\text{aiC0}}$$

Moreover, we will show that *for the above f^* 's and for the corresponding solutions $u(f^*)$ and approximate solutions $u_h(f^*)$* one has instead the following experimental behavior, clearly suggesting first order convergence in L^2 :

$$\exists C_1 > 0, \quad \text{s.t.} \quad \lim_{h \rightarrow 0} \mathcal{Q}_1(h, f^*(h)) \equiv \lim_{h \rightarrow 0} \frac{\|u(f^*) - u_h(f^*)\|_0}{h \|f^*\|_0} = C_1. \quad (6.5) \quad \boxed{\text{nego01}}$$

We now describe the numerical test. We take for \mathcal{T}_h a family of uniform partitions of $[0, 1]$ with mesh size $h = 3^{-1}2^{-k}$, and $k = 2, 3, \dots, 12$. Associated to each mesh we construct a family of functions $\{f^*(h)\}$. Each f^* is a piecewise linear polynomial on each mesh:

$$f^*(h) = \begin{cases} -\frac{(x-x_i)}{h} & 0 \leq x_i < x < x_{i+1} \leq 1/4, \\ \frac{(x-x_i)}{h} & 1/4 \leq x_i < x < x_{i+1} \leq 3/4, \\ -\frac{(x-x_i)}{h} & 3/4 \leq x_i < x < x_{i+1} \leq 1, \end{cases}$$

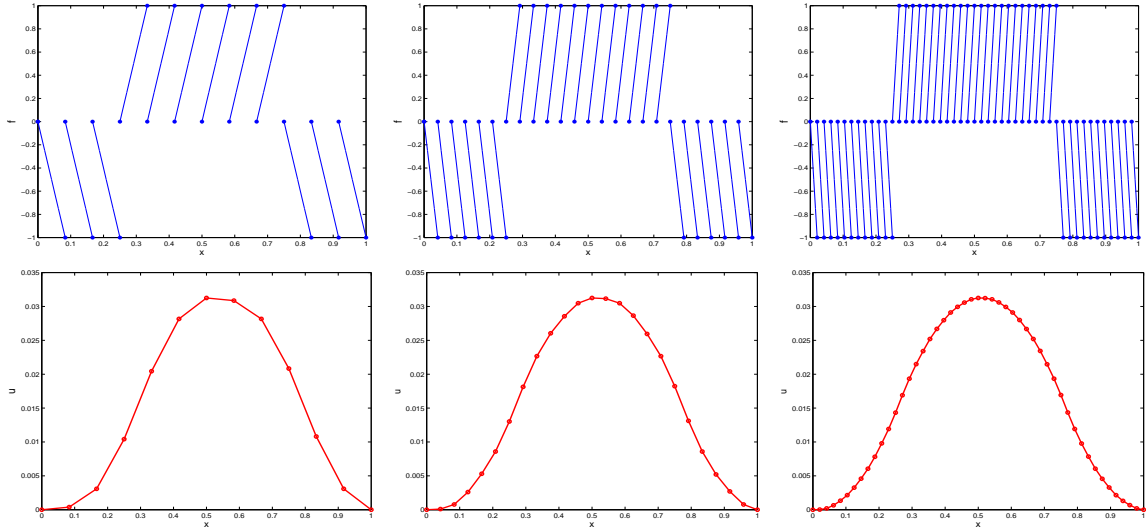


Figure 6.4: Graphic representations of $f^*(h)$ (top figures) and the corresponding exact solution $u(f^*)$ (bottom figures), computed on uniform meshes with mesh size $h = 1/12$ (left figures), $h = 1/24$ (center) and $h = 1/48$ (rightmost figures). fig:uf

where x_i are the nodes and $h = x_{i+1} - x_i$. The corresponding exact solution $u(f^*)$ of the problem (6.2) (with right hand side f^*) is computed analytically. Fig. 6.4 shows f^* (top figures) and the corresponding exact solutions $u(f^*)$ (bottom figures) for the first three uniform meshes used in the computations. Clearly f^* is in $L^2([0, 1])$ (and actually also in $L^\infty([0, 1])$) but $f^* \notin H^1([0, 1])$. We wish to stress that both the L^2 -norm and the L^∞ -norm of $f^*(h)$ are actually independent of h :

$$\|f^*(h)\|_{L^2([0,1])} = \frac{1}{\sqrt{3}} \quad \|f^*(h)\|_{L^\infty([0,1])} = 1.$$

The convergence diagrams in the L^2 -norm for all the methods are given in Fig. 6.5. For the IIPG and NIPG methods, only first order is attained, while SIPG converges with second order, as expected from the classical theory [3, 4] together with the regularity of the test problem, $f \in L^2([0, 1])$.

To verify (and support) numerically the statements (6.3)-(6.4) and (6.5), we have represented in a log-log scale the values of the (error constants) quotients \mathcal{Q}_2 and \mathcal{Q}_1 defined in (6.3) and in (6.5), respectively, together with the quotient

$$\mathcal{Q}_{1/2} := \frac{\|u(f^*) - u_h(f^*)\|_0}{h^{1/2}\|f^*\|_0}.$$

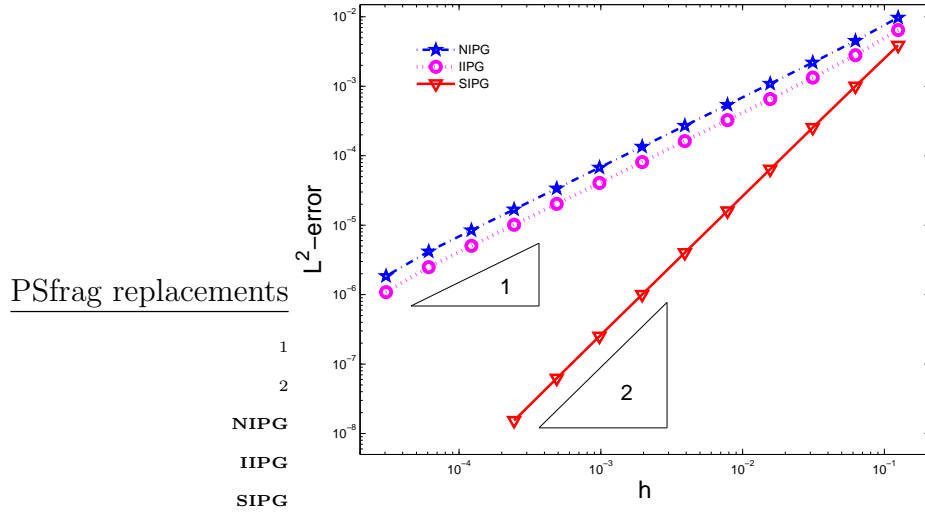


Figure 6.5: Convergence diagrams in the L^2 -norm. The IIPG method is represented by $\dots \circ \dots$; the NIPG by $-\cdot-\star\cdot-\cdot-$, and the SIPG by $-\nabla-$.

fig10

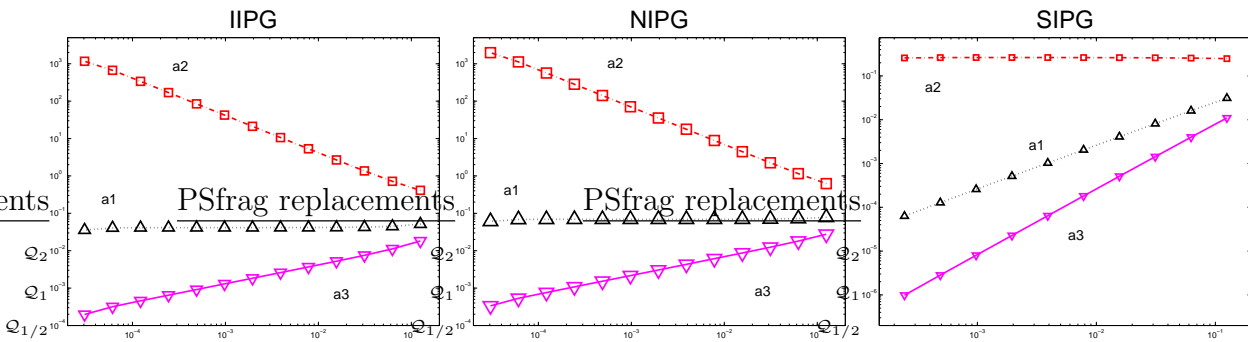


Figure 6.6: Lower bounds for the error best constants: \mathcal{Q}_2 ($-\cdot-\square-\cdot-$), \mathcal{Q}_1 ($\cdot\cdot\Delta\cdot\cdot$), and $\mathcal{Q}_{1/2}$ ($-\nabla-$) as functions of the mesh size, for IIPG (left), NIPG (center), and SIPG (right).

fig3

Since all the methods are at least first order convergent in L^2 (see [4]), we clearly expect $\mathcal{Q}_{1/2} \rightarrow 0$ as $h \rightarrow 0$. This can indeed be seen in Fig. 6.6, where the diagrams for the three methods are depicted.

Observe that the behavior predicted in (6.4) for \mathcal{Q}_2 can be easily checked in the graphics. While for the SIPG method it remains constant, for the IIPG and NIPG methods it increases linearly as h decreases. In contrast, \mathcal{Q}_1 remains constant for IIPG and NIPG methods, which confirms (6.5) and supports our conclusion that the methods are at most first order convergent if the data f is only in L^2 . In Table 6.1 we also report the computed values of the quotients \mathcal{Q}_2 (left table) and \mathcal{Q}_1 (right table).

All the experiments have been carried out with MATLAB on a Mac-Book Pro with 8Gb of Ram memory.

Remark 6.1. *One might argue about our construction of the numerical test, since the L^2 -suboptimality is demonstrated for a sequence of mesh dependent functions $f^*(h)$. Indeed we showed (numerically) (6.3) rather than producing the most common type of (numerical) counterexample:*

$$\exists f^* \in L^2([0,1]) \text{ such that not } \left\{ \exists C_0 > 0 \quad \forall h > 0 \quad \frac{\|u(f^*) - u_h(f^*)\|_0}{\|f^*\|_0} < C_0 h^2 \right\}. \quad (6.6)$$

nego0com

However we point out that, in the first place, (6.3) easily implies the falseness of (6.1) and hence it must be considered as a legitimate counterexample. Moreover, using the uniform boundedness principle (also known as Banach-Steinhaus theorem¹, see for instance [21, Theorem 2.5 & 2.6]) it is not difficult to see that (6.3) actually implies (6.6). More precisely, we can define a family of linear and continuous operators $E_h : L^2([0,1]) \rightarrow L^2([0,1])$ by

$$E_h(f) := h^{-2}(u(f) - u_h(f)) \quad \forall f \in L^2([0,1]), \quad (6.7)$$

def:Eh

and we denote with $\|E_h\|_{\mathcal{L}(L^2[0,1],L^2[0,1])}$ its norm. Then (6.3) implies that

$$\sup_h \|E_h\|_{\mathcal{L}(L^2[0,1],L^2[0,1])} = +\infty,$$

¹Uniform boundedness principle: let X, Y be two Banach spaces, let $\{E_h\}$ be a collection of continuous linear mappings $E_h : X \rightarrow Y$ and let $\|\cdot\|_{\mathcal{L}(X,Y)}$ denote the operator norm.

$$\text{if } \sup_h \|E_h x\|_Y < \infty, \quad \forall x \in X \quad \implies \quad \sup_h \|E_h\|_{\mathcal{L}(X,Y)} < \infty.$$

(a) Computed $\mathcal{Q}_2 = \frac{\|u(f^*) - u_h(f^*)\|_0}{h^2 \|f^*\|_0}$ table:cp0-a $= \frac{\|u(f^*) - u_h(f^*)\|_0}{h \|f^*\|_0}$ table:cp0-b

N	SIPG	IIPG	NIPG
12	0.4039	0.66	0.49
24	0.4311	0.71	1.08
48	0.4440	1.24	2.00
96	0.4503	2.36	3.90
192	0.4534	4.65	7.71
384	0.4549	9.23	15.36
768	0.4557	18.42	30.68
1536	0.4560	36.80	61.30
3072	0.4566	73.54	122.56
6144	0.4540	147.05	245.07
12288	0.4465	294.14	490.17

N	IIPG	NIPG
12	0.16404	0.12277
24	0.08921	0.13501
48	0.07756	0.12508
96	0.07381	0.12173
192	0.07257	0.12052
384	0.07212	0.12004
768	0.07194	0.11983
1536	0.07186	0.11973
3072	0.07182	0.11969
6144	0.07180	0.11966
12288	0.07181	0.11967

Table 6.1: Numerical Computed values of the quotients \mathcal{Q}_2 (left table) and \mathcal{Q}_1 (right table). table:cp0

that is just the negation of the **thesis** of the uniform boundedness principle. Therefore we conclude that E_h does not satisfy the hypothesis of the Theorem and therefore

$$\exists f \in L^2([0, 1]) \quad \forall C > 0 \quad \exists h > 0 \quad \text{such that } \|E_h(f)\|_{L^2[0,1]} > C,$$

that is exactly (6.6) nego0com

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