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VIRTUAL ELEMENT METHOD FOR PLATE BENDING PROBLEMS

FRANCO BREZZI[1,2,3] AND L. DONATELLA MARINI[4,2]

ABSTRACT. We discuss the application of *Virtual Elements* to linear plate bending problems, in the Kirchhoff-Love formulation. As we shall see, in the Virtual Element environment the treatment of the C^1-continuity condition is much easier than for traditional Finite Elements. The main difference consists in the fact that traditional Finite Elements, for every element K and for every given set of degrees of freedom, require the use of a space of polynomials (or piecewise polynomials for composite elements) for which the given set of degrees of freedom is unisolvent. For Virtual Elements instead we only need unisolvence for a space of smooth functions that *contains* a subset made of polynomials (whose degree determines the accuracy). As we shall see the non-polynomial part of our local spaces does not need to be known in detail, and therefore the construction of the local stiffness matrix is simple, and can be done for much more general geometries.

Keywords: High-order MFD; Plate bending problems; Virtual Elements

1. INTRODUCTION

The Finite Element literature of the last fifty years contains a big variety of H^1-conforming (in practice, C^0) elements of various degrees, with different features. On the other hand, the list of available H^2-conforming (in practice, C^1) elements is much more slim. Among the most commonly used are the composite Hsieh-Clough-Tocher element (with cubic accuracy) and its reduced version (with quadratic accuracy), the Argyris element (with quintic accuracy) and its reduced Bell version (with quartic accuracy). But a much bigger effort has been devoted to alternative formulations that could avoid the use of C^1 finite elements, including mixed formulations (essentially, based on the Hellinger-Reissner principle) or the use of the Reissner-Mindlin model for thin plates and shells (instead of moderately thick). Dual hybrid elements of Pian and co-authors (see e.g. [32], [33]) could be seen as *intermediate* C^1 formulations, as the displacements are indeed C^1 but they are defined only at the interelement boundaries (see also [11] and in particular [17]). And several among the most common and useful applications of nonconforming elements (as for instance the Morley element) can indeed be seen as an effort to bypass C^1 continuity.

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We refer for instance to the books [1], [24], [6], [39] (or, for more mathematically oriented ones, to [18], [10], [9]) and to the references therein.

In this paper we want to show that the newly born Virtual Element Method [4] is able to tackle the construction of C^1-approximations in a much swifter way, and on very general polygonal elements, and we propose a few examples of elements that could surely be interesting in the approximation of the solution of plate bending problems in the Kirchhoff-Love formulation.

Virtual Element Methods could be seen as an evolution of Mimetic Finite Differences and related methods (see e.g. [23], [25], [28], [29], or the more recent [8], [19] [20], [37]). In particular they are strictly related to the more theoretical versions of MFD ([26], [14], [16] [15] [12]), and specially to their last versions concerning higher order methods ([31], [22] [3], [2]).

As we shall see, the *basic idea* consists in choosing first some degrees of freedom at the interelement boundaries that could identify in a unique way the traces of globally C^1 functions that are polynomials of degree $\leq r$ on each edge, with normal derivative of degree $\leq s$ on each edge. Then we add a suitable amount of internal degrees of freedom, and we define the discrete subspace inside the elements by means of a local plate bending problem. Needless to say, the actual solution of these local problems will not be required, not even in an approximate way.

Actually, we will show that, having constructed the discrete subspace V_h as above, if the boundary and internal degrees of freedom are chosen properly, then

a) in each element K, the restriction of the discrete space to K contains the space $\mathbb{P}_k$ of polynomials of degree $\leq k$, where k depends on r and s,

b) for every polynomial $p_k \in \mathbb{P}_k(K)$, and for every *non-polynomial* element v_h of V_h, the contribution of K to the energy bilinear form $a(p_k, v_h)$ can be computed *exactly*, using only the degrees of freedom of v_h (and not the fact that v_h solves a local plate problem),

c) we can then easily complete the computation of the energy bilinear form $a(v_h, w_h)$ (when neither v_h nor w_h is a polynomial in $\mathbb{P}_k$) in an almost arbitrary way, in order to ensure *stability*.

Note that, if we have a), b), and c) we can ensure a sort of $k-th$ order **patch test**, meaning that, on any patch of elements: if the true solution is a global polynomial of degree $\leq k$, then our discrete solution will coincide exactly with the true solution.

For other attempts to deal with elements of a general shape we refer to [5], [7], [21], [30], [34], [35], [36], [38] and the references therein. We point out, however, that here we *do not* use numerical integration.

An outline of the paper is as follows. In Section 2 we recall the continuous problem and fix some notation. In Section 3 we present in an abstract framework the Virtual Element approach, state the basic assumptions and prove a convergence theorem. In Section 4 we explicitly show how to construct a C^1 VEM-approximation of the plate

bending problem with the optimal convergence rate, namely, order k, with $k \geq 2$ whenever the discrete space V_h contains locally the polynomials of degree k.

Throughout the paper we shall use the common notation for the Sobolev spaces $H^m(\mathcal{D})$ for m a nonnegative integer and $\mathcal{D}$ an open bounded domain. In particular (see e.g. [27], [18]) the $L^2(\mathcal{D})$ scalar product and norm will be indicated by $(\cdot, \cdot)_{0,\mathcal{D}}$ or $(\cdot, \cdot)_{\mathcal{D}}$ and $\|\cdot\|_{0,\mathcal{D}}$ or $\|\cdot\|_{\mathcal{D}}$, respectively. Moreover, for m a nonnegative integer, the mth seminorm of the function φ will be defined by

$$(1.1) \qquad |\varphi|^2_{m,\mathcal{D}} := \sum_{|\boldsymbol{\alpha}|=m} \left\| \frac{\partial^{|\boldsymbol{\alpha}|}\varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right\|^2_{0,\mathcal{D}}$$

where for the nonegative multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ we denoted as usual $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$.

When $\mathcal{D} \equiv \Omega$ the subscript $\mathcal{D}$ will often be omitted.

2. The Continuous Problem

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain occupied by the plate, let Γ be its boundary, and let $f \in L^2(\Omega)$ be a transversal load acting on the plate. The Kirchoff-Love model for thin plates (see e.g. [18]) corresponds to look for the transversal displacement w solution of

$$(2.1) \qquad D\Delta^2 w = f \qquad \text{in } \Omega,$$

where $D = Et^3/12(1 - \nu^2)$ is the bending rigidity, t the thickness, E the Young modulus, and ν the Poisson's ratio.

Assuming for instance the plate to be clamped all over the boundary, equation (2.1) is supplemented with the boundary conditions

$$(2.2) \qquad w = \frac{\partial w}{\partial n} = 0 \qquad \text{on } \Gamma.$$

The variational formulation of (2.1)-(2.2) is:

$$(2.3) \qquad \begin{cases} \text{Find } w \in V := H_0^2(\Omega) \text{ solution of} \\ a(w, v) = (f, v) \quad \forall v \in H_0^2(\Omega), \end{cases}$$

where, as we said, $(\cdot, \cdot)$ is the usual scalar product in $L^2(\Omega)$, and the energy bilinear form $a(\cdot, \cdot)$ is given by

$$(2.4) \qquad a(w, v) = D\left[(1 - \nu) \int_\Omega w_{/ij} v_{/ij} \, dx + \nu \int_\Omega \Delta w \Delta v \, dx\right].$$

In (2.4) $v_{/i} = \partial v / \partial x_i$, $i = 1, 2$, and we used the summation convention of repeated indices. Thus, for instance:

$$(2.5) \qquad w_{/ij} v_{/ij} = (w_{/11} v_{/11} + 2w_{/12} v_{/12} + w_{/22} v_{/22}).$$

Setting $\|v\|_V := |v|_{2,\Omega}$, it is easy to see that, thanks to the boundary conditions in V and to the Poincaré inequality, this is indeed a norm on V. Moreover

$$(2.6) \qquad \exists M > 0 \quad \text{such that} \quad a(u, v) \leq M\|u\|_V \|v\|_V \quad u, v \in V,$$

(2.7) $\exists\, \alpha > 0 \quad \text{such that} \quad a(v,v) \geq \alpha \|v\|_V^2 \quad v \in V.$

Hence, (2.3) has a unique solution, and (see, e.g. [27])

(2.8) $\|w\|_V \leq C \|f\|_{L^2(\Omega)}.$

Before going to the discretization of (2.3) let us fix some notation that will be used later on. On a domain $\mathcal{D} \subset \mathbb{R}^2$, with boundary $\partial\mathcal{D}$, we denote by $\mathbf{n} = (n_1, n_2)$ the outward unit normal vector to $\partial\mathcal{D}$, and by $\mathbf{t} = (t_1, t_2)$ the unit tangent vector in the counterclockwise ordering of the boundary. For $v \in H^2(\Omega)$, let $\mathbf{M} = M_{ij}(v)\ i,j = 1,2$ be the moment tensor given by the stress-strain relation

(2.9)
$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} v_{/11} \\ v_{/22} \\ v_{/12} \end{bmatrix},$$

and let $\mathbf{M}_n = M_{ij}n_j$ be its normal vector. We then denote by $M_{nn}(v) := M_{ij}n_in_j$ the normal bending moment, by $M_{nt}(v) := M_{ij}n_it_j$ the twisting moment, and by $Q_n(v) := M_{ij/i}n_j$ the normal shear force.

Always for a generic domain $\mathcal{D}$ we also set, as in (2.4)

(2.10) $a^{\mathcal{D}}(w,v) = D\Big[(1-\nu)\int_{\mathcal{D}} w_{/ij}v_{/ij}\,\mathrm{d}x + \nu \int_{\mathcal{D}} \Delta w \Delta v\,\mathrm{d}x\Big],$

keeping the simpler notation $a(w,v)$ only when $\mathcal{D} = \Omega$.

After integrating by parts twice we have

(2.11) $a^{\mathcal{D}}(w,v) = \int_{\mathcal{D}} D\Delta^2 w\, v\,\mathrm{d}x + \int_{\partial\mathcal{D}} M_{nn}(w)\frac{\partial v}{\partial n}\,\mathrm{d}t - \int_{\partial\mathcal{D}} \Big(Q_n(w) + \frac{\partial M_{nt}(w)}{\partial t}\Big)v\,\mathrm{d}t.$

When $\mathcal{D} = \Omega$ and $v \in V$ the last two terms in (2.11) disappear, due to the boundary conditions (2.2), but (2.11) will be useful later on when dealing with the discrete problem (and, in any case, whenever boundary conditions of a different type are considered on some part of the boundary). In the sequel we shall often use $v_{/n}$ and $v_{/t}$ for $\partial v/\partial n$ and $\partial v/\partial t$, respectively.

3. The discrete problem. Abstract framework

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into elements K, and let $\mathcal{E}_h$ be the set of edges e of $\mathcal{T}_h$. We want to construct a finite dimensional space $V_h \subset H_0^2(\Omega)$, a bilinear form $a_h(\cdot,\cdot): V_h \times V_h \to \mathbb{R}$, and an element $f_h \in V_h'$ such that the discrete problem:

(3.1)
$$\begin{cases} \text{Find } w_h \in V_h \text{ such that} \\ a_h(w_h, v_h) = <f_h, v_h> \qquad \forall\, v_h \in V_h, \end{cases}$$

has a unique solution w_h, and good approximation properties hold. Namely, if $k \geq 2$ is the target degree of accuracy, and the solution w of (2.3) is smooth enough, we want to have

(3.2) $\|w - w_h\|_V \leq C\, h^{k-1} |w|_{k+1,\Omega},$

where C, here and in the following formulae, is, as usual, a positive constant independent of h. Note that (3.2), as written, implies our $k - th$ order Patch-Test, since when $w \in \mathbb{P}_k$ then $|w|_{k+1} = 0$.

Let us first recall the basic assumptions that we need (see e.g. [12]).

H0 - There exists an integer N and a positive real number γ such that for every h and for every $K \in \mathcal{T}_h$:

- the number of edges of K is $\leq N$,
- the ratio between the shortest edge and the diameter h_K of K is bigger than γ, and
- K is star-shaped with respect to every point of a ball of radius γh_K. $\square$

H1 - We assume that we are given, for each h:

- a space $V_h \subset V$; for each element K we will denote

(3.3) $$V_h^K = \text{ restriction of } V_h \text{ to } K$$

- a symmetric bilinear form a_h from $V_h \times V_h$ to $\mathbb{R}$ that can be split as

(3.4) $$a_h(u_h, v_h) = \sum_K a_h^K(u_h, v_h) \qquad \forall u_h,\, v_h \in V_h,$$

  where a_h^K is a bilinear symmetric form on $V_h^K \times V_h^K$;
- an element $f_h \in V_h'$. $\square$

Similarly, we split the bilinear form $a(\cdot, \cdot)$ and the norm $\| \cdot \|_V$ as

(3.5) $$a(u, v) = \sum_K a^K(u, v) \qquad \forall\, u,\, v \in V, \quad \|v\|_V = (\sum_K |v|_{V,K}^2)^{1/2} \quad \forall v \in V.$$

Since in what follows we shall also deal with functions in $H^2(\mathcal{T}_h) := \prod_K H^2(K)$, we also need to define a broken H^2 semi-norm:

(3.6) $$|v|_{h,V} := \left(\sum_K |v|_{V,K}^2 \right)^{1/2} \quad \forall v \in \prod_K H^2(K).$$

3.1. **An abstract convergence theorem.** Together with **H0** and **H1** we further assume the following properties.

H2 - There exists an integer $k \geq 2$ (that will determine our order of accuracy) such that for all h, and for all K in $\mathcal{T}_h$,

- *k-Consistency*: $\forall p \in \mathbb{P}_k, \forall v_h \in V_h$

(3.7) $$a_h^K(p, v_h) = a^K(p, v_h).$$

- *Stability*: $\exists$ two positive constants α_* and α^*, independent of h and of K, such that

(3.8) $$\forall v_h \in V_h \qquad \alpha_* \, a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* \, a^K(v_h, v_h).$$

$\square$

We note that the symmetry of a_h, (3.8) and the continuity of a^K easily imply the continuity of a_h with

$$(3.9) \qquad a_h^K(u,v) \le \left(a_h^K(u,u)\right)^{1/2}\left(a_h^K(v,v)\right)^{1/2} \le \alpha^*\left(a^K(u,u)\right)^{1/2}\left(a^K(v,v)\right)^{1/2}$$
$$\le \alpha^* M \|u\|_{V,K} \|v\|_{V,K} \quad \text{for all } u \text{ and } v \text{ in } V_h.$$

In turn, (3.8) and (3.9) easily imply

$$(3.10) \qquad\qquad \forall v \in V_h \qquad \alpha_* a(v,v) \le a_h(v,v) \le \alpha^* a(v,v).$$

and

$$(3.11) \qquad\qquad a_h(u,v) \le \alpha^* M \|u\|_V \|v\|_V \quad \text{for all } u \text{ and } v \text{ in } V_h.$$

We have the following convergence theorem.

Theorem 3.1. *Under the Assumptions* **H1-H2** *the discrete problem: Find $w_h \in V_h$ such that*

$$(3.12) \qquad\qquad a_h(w_h, v_h) = < f_h, v_h > \qquad \forall v_h \in V_h,$$

has a unique solution w_h. Moreover, for every approximation w_I of w in V_h and for every approximation w_π of w that is piecewise in $\mathbb{P}_k$, we have

$$\|w - w_h\|_V \le C\left(\|w - w_I\|_V + \|w - w_\pi\|_{h,V} + \|f - f_h\|_{V_h'}\right)$$

where C is a constant depending only on α, α_*, α^*, M and, with the usual notation, the norm in V_h' is defined as

$$(3.13) \qquad\qquad \|f - f_h\|_{V_h'} := \sup_{v_h \in V_h} \frac{< f - f_h, v_h >}{\|v_h\|_V}.$$

Proof. Existence and uniqueness of the solution of (3.12) is a consequence of (3.8) and (2.7). Next, setting

$$(3.14) \qquad\qquad \delta_h := w_h - w_I$$

we have

$$
\begin{aligned}
\alpha_*\alpha\|\delta_h\|_V^2 &\le (\ \text{use (2.7) and (3.10)} \\
&\le \alpha_* a(\delta_h, \delta_h) \le \alpha_* a(\delta_h, \delta_h) \ \text{use (3.14)} \\
&= a_h(w_h, \delta_h) - a_h(w_I, \delta_h)(\text{use (3.12) and (3.4)}) \\
&= < f_h, \delta_h > -\sum_K a_h^K(w_I, \delta_h) \ (\text{use } \pm w_\pi) \\
&= < f_h, \delta_h > -\sum_K \left(a_h^K(w_I - w_\pi, \delta_h) + a_h^K(w_\pi, \delta_h)\right) \ (\text{use (3.7)}) \\
&= < f_h, \delta_h > -\sum_K \left(a_h^K(w_I - w_\pi, \delta_h) + a^K(w_\pi, \delta_h)\right) \ (\text{use } \pm w \text{ and (3.5)}) \\
&= < f_h, \delta_h > -\sum_K \left(a_h^K(w_I - w_\pi, \delta_h) + a^K(w_\pi - w, \delta_h)\right) - a(w, \delta_h)
\end{aligned}
$$

$$(3.15)$$

Now we can use (2.3) in (3.15) to get

$$
\begin{aligned}
\alpha_* \alpha \|\delta_h\|_V^2 &\leq \\
&\leq <f_h, \delta_h> -\sum_K \left(a_h^K(w_I - w_\pi, \delta_h) + a^K(w_\pi - w, \delta_h) \right) - (f, \delta_h) \\
&= <f_h, \delta_h> -(f, \delta_h) - \sum_K \left(a_h^K(w_I - w_\pi, \delta_h) + a^K(w_\pi - w, \delta_h) \right).
\end{aligned}
$$
(3.16)

Finally we use (3.13), (3.9), and the continuity of each a^K in (3.16) to obtain

$$
\|\delta_h\|_V^2 \leq C \left(\|f - f_h\|_{V_h'} + \|w_I - w_\pi\|_{h,V} + \|w - w_\pi\|_{h,V} \right) \|\delta_h\|_V
$$
(3.17)

for some constant C depending only on α, α_*, α^*, and M. Then the result follows easily by the triangle inequality. $\square$

4. CONSTRUCTION OF V_h, a_h, AND f_h

We shall construct now a space V_h, a bilinear form a_h, and a right-hand side f_h satisfying assumptions $\mathbf{H1} - \mathbf{H2}$. Our family of elements will depend on three integer indices (r, s, m), related to the degree of accuracy $k \geq 2$ by:

$$
r = \max\{3, k\}, \quad s = k - 1, \quad m = k - 4.
$$
(4.1)

Remark 4.1. *As we shall see in a while, the indices r and s will be related, respectively, to the polynomial degree of the functions in V_h, and to the polynomial degree of their normal derivative, on each edge of the decomposition. On the other hand, the index m will be related (in a way to be made precise) to degrees of freedom internal to each element.* $\square$

Remark 4.2. *As it will be clear when constructing the spaces, condition (4.1) will not allow, for instance, to take the simple choice*

$$
r = 5, \, s = 3, \, m = 0 \text{ and } k = 4
$$
(4.2)

(that might produce a Bell-like type of element). What we really need is actually

$$
r \geq \max\{3, k\}, \quad s \geq k - 1, \quad m \geq k - 4.
$$
(4.3)

On the other hand, the present choice allows to consider a family of discretizations that depends only on one parameter (here, k). Choosing (4.3), instead, we should actually deal with a family of spaces V_h depending on four parameters r, s, m, and k, with a considerable complication in the notation and little conceptual gain. However, in developing our theory we should keep an eye on more general cases, like (4.3) or other possible approaches for which our theory would apply almost unchanged (but whose comprehensive treatment would imply a much more complicated notation). $\square$

Remark 4.3. *We can easily attach to each vertex $\boldsymbol{\xi}$ a characteristic length $h_{\boldsymbol{\xi}}$, taken, for instance, as the average of the diameters of the elements having $\boldsymbol{\xi}$ as a vertex. To an edge e we can attach, as characteristic length h_e the length $|e|$ of the edge itself, while to an element K we can attach as characteristic length h_K its diameter.* $\square$

4.1. **Local construction of V_h.** Let K be a generic polygon in $\mathcal{T}_h$, let $k \geq 2$, and let r, s, m be given by (4.1). We define V_h^K (that will be the *local* version of our discretized space V_h) as

(4.4) $\quad V_h^K := \{v \in H^2(K) \text{ s.t. } \Delta^2 v \in \mathbb{P}_m(K), \ v_{|e} \in \mathbb{P}_r(e), \ (v_{/n})_{|e} \in \mathcal{P}_s(e), \ \forall e \in \partial K\}.$

We shall make use of the following notation: for t a nonnegative integer, and e an edge with midpoint x_e, we denote by $\mathcal{M}_t^e$ the set of $t+1$ normalized monomials

(4.5) $$\mathcal{M}_t^e := \left\{1, \frac{x - x_e}{h_e}, \left(\frac{x - x_e}{h_e}\right)^2, ..., \left(\frac{x - x_e}{h_e}\right)^t\right\}.$$

Similarly, for a two-dimensional domain K with diameter h_K and barycenter $\mathbf{x}_K$ we denote by $\mathcal{M}_t^K$ the set of $(t+1)(t+2)/2$ normalized monomials

(4.6) $$\mathcal{M}_t^K := \left\{\left(\frac{x - \mathbf{x}_K}{h_K}\right)^{\boldsymbol{\beta}}, \quad |\boldsymbol{\beta}| \leq t\right\},$$

where, as usual, for a nonnegative multiindex $\boldsymbol{\beta} = (\beta_1, \beta_2)$ we set $|\boldsymbol{\beta}| = \beta_1 + \beta_2$ and $\mathbf{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2}$. In K we define the following degrees of freedom:

(4.7) $\qquad \bullet$ The value of $v(\boldsymbol{\xi}) \quad \forall$ *vertex* $\boldsymbol{\xi}$

(4.8) $\qquad \bullet$ The values of $h_{\boldsymbol{\xi}} v_{/1}(\boldsymbol{\xi})$ and $h_{\boldsymbol{\xi}} v_{/2}(\boldsymbol{\xi}) \quad \forall$ *vertex* $\boldsymbol{\xi}$

(4.9) $\qquad \bullet$ For $r > 3$, the moments $\dfrac{1}{h_e} \displaystyle\int_e q(\xi) v(\xi) \mathrm{d}\xi \quad \forall q \in \mathcal{M}_{r-4}^e, \quad \forall$ *edge* e

(4.10) $\qquad \bullet$ For $s > 1$, the moments $\displaystyle\int_e q(\xi) v_{/n}(\xi) \mathrm{d}\xi \quad \forall q \in \mathcal{M}_{s-2}^e \quad \forall$ *edge* e

(4.11) $\qquad \bullet$ For $m \geq 0$, the moments $\dfrac{1}{h_K^2} \displaystyle\int_K q(x) v(x) \, \mathrm{d}x \quad \forall q \in \mathcal{M}_m^K.$

Note that all the quantities (4.7)-(4.11) have the same dimension, and hence in a homothetic blow up of K would *scale* in the same manner.

Figure 1 shows the first two elements of the family determined by the d.o.f. (4.7)-(4.11): for both elements we have that $v \in \mathbb{P}_3(e)$ on each edge, but $\dfrac{\partial v}{\partial n}_{|e} \in \mathbb{P}_1(e)$ for $k = 2$, and $\dfrac{\partial v}{\partial n}_{|e} \in \mathbb{P}_2(e)$ for $k = 3$. These elements are the extension to polygonal domains of the Hsieh-Clough-Tocher triangle ($k = 3$) and of its reduced version ($k = 2$).

We note that he degrees of freedom (4.7) and (4.8) are always needed to ensure C^1−continuity at the vertices. Moreover, on each edge, they provide v and $v_{/t}$ at the endpoints of the edge: enough information to identify uniquely a polynomial of degree ≤ 3. This, together with (4.9), explains the requirement $r \geq 3$. Indeed, to identify a polynomial of degree r, together with (4.7)-(4.8) we need $r - 3$ additional information (as for instance (4.9)). At the same time, conditions (4.8) provide, for every edge, $v_{/n}$ at the endpoints of the edge, thus allowing to identify uniquely a polynomial of degree ≤ 1. For $s > 1$, to identify a polynomial of degree $\leq s$, we would need the
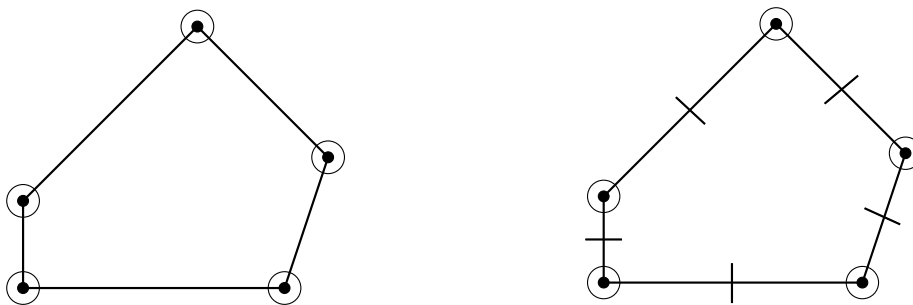
FIGURE 1. Local d.o.f. for the lowest-order element: $k = 2$ (left), and next to the lowest $k = 3$ (right)

additional $s-1$ conditions (4.10). On the other hand, the d.o.f. (4.11) are equivalent to prescribe $P_m^K v$ in K, where, for m a nonnegative integer,

(4.12) $\qquad P_m^K v$ is the $\mathrm{L}^2(\mathrm{K}) -$ projector operator onto the space $\mathbb{P}_\mathrm{m}(\mathrm{K})$.

We summarize the above discussion in the following proposition

Proposition 4.1. *In each element K the d.o.f. (4.7), (4.8), and (4.9) uniquely determine a polynomial of degree $\leq r$ on each edge of K, the degrees of freedom (4.8) and (4.10) uniquely determine a polynomial of degree $\leq s$ on each edge of K, and the d.o.f. (4.11) are equivalent to prescribe $P_m^K v$ in K.*

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Let ℓ be the number of edges (and hence of vertices) of K, and let $\mathcal{G}^K$ be the set of degrees of freedom corresponding to K, whose total number amounts to

(4.13)
$$T := T_\partial + T_0; \quad \text{with } T_\partial = \{3\ell + \ell(r-3) + \ell(s-1)\} \equiv \ell(r+s-1),$$
$$\text{and } T_0 = \left\{ \frac{(m+1)(m+2)}{2} \right\} \equiv \frac{m^2 + 3m + 2}{2}.$$

Now we have to show how to associate to the above degrees of freedom a function in K. It is important to point out from the very beginning that we do not need to be able to *compute* such a function, but just to show that it exists and it has certain properties.

Proposition 4.2. *The degrees of freedom (4.7)–(4.11) are unisolvent in V_h^K (as defined in (4.4)).*

Proof. According to Proposition 4.1, to prove the proposition it is enough to show that, for each $K \in \mathcal{T}_h$, a function u in V_h^K such that

(4.14)
$$u = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial K$$

and

(4.15)
$$P_m^K u = 0 \quad \text{in } K$$

is actually identically zero in K. To this end, we show that $\Delta^2 u = 0$ in K, which joined with (4.14) gives $u \equiv 0$. We first solve, for every $q \in \mathbb{P}_m(K)$, the following auxiliary problem: *Find* $\psi = \psi(q) \in H_0^2(K)$ *such that:*

$$(4.16) \qquad a^K(\psi, v) = (q, v)_{0,K} \qquad \forall v \in H_0^2(K),$$

that could also be written as

$$(4.17) \quad D\Delta^2\psi = q \;\; \text{in } K, \quad \psi_{|\partial K} = \frac{\partial\psi}{\partial n}_{|\partial K} = 0, \qquad \text{or else } \psi = (D\Delta_{0,K}^2)^{-1}(q).$$

Next, we consider the map R, from $\mathbb{P}_m(K)$ into itself, defined by

$$(4.18) \qquad R(q) := P_m^K((D\Delta_{0,K}^2)^{-1}(q)) \equiv P_m^K\psi(q) \qquad \forall q \in \mathbb{P}_m(K).$$

We claim that the operator R defined by (4.18) is an isomorphism from $\mathbb{P}_m(K)$ into itself. Indeed, from (4.18) and the definition of P_m^K we have, for every $q \in \mathbb{P}_m(K)$:

$$(R(q), q)_{0,K} = (P_m^K(D\Delta_{0,K}^2)^{-1}(q), q)_{0,K} = (P_m^K\psi(q)), q)_{0,K}$$
$$= (\psi(q), q)_{0,K} = a^K(\psi(q), \psi(q)).$$

Since $\psi \in H_0^2(K)$ we have then that

$$(4.19) \qquad \{R(q) = 0\} \Rightarrow \{a^K(\psi(q), \psi(q)) = 0\} \Rightarrow \{\psi(q) = 0\} \Rightarrow \{q = 0\}.$$

We note that, if $u \in V_h^K$ is such that $u = \partial u/\partial n = 0$ on ∂K, then

$$(4.20) \qquad P_m^K u = P_m^K((D\Delta_{0,K}^2)^{-1}(D\Delta^2 u)) = R(D\Delta^2 u).$$

Hence, if $P_m^K u = 0$ as in (4.15), we can say that

$$P_m^K u = 0 \Longrightarrow R(D\Delta^2 v) = 0 \Longrightarrow D\Delta^2 v = 0,$$

and the proof is concluded. $\square$

Remark 4.4. *We point out once more that we will not try to compute the solutions of the local bi-harmonic problems involved in the definition of (4.4). On the other hand we will always assume that we can compute polynomials on ∂K, and hence also integral of polynomials over K. Indeed, for instance,*

$$(4.21) \qquad \int_K x_1^2 = \frac{1}{3}\int_{\partial K} x_1^3 n_1$$

and so on. $\square$

4.2. Global construction of V_h.

For every decomposition $\mathcal{T}_h$, and for every $k \geq 2$ we are now ready to define the space V_h as

$$(4.22) \;\; V_h = \{v \in V : v_{|e} \in \mathbb{P}_r(e), \frac{\partial v}{\partial n}_{|e} \in \mathbb{P}_s(e) \;\forall e \in \mathcal{E}_h, \; \Delta^2 v_{|K} \in \mathbb{P}_m(K) \;\forall K \in \mathcal{T}_h\},$$

where r, s, and m are always given by (4.1) (although, actually, they could be any triple that satisfies (4.3)). It follows from the above discussion that the global degrees of freedom in V_h could then be taken as

(4.23) The value of $v(\boldsymbol{\xi})$ $\forall$ internal vertex $\boldsymbol{\xi}$

(4.24) The values of $h_{\boldsymbol{\xi}} v_{/1}(\boldsymbol{\xi})$ and $h_{\boldsymbol{\xi}} v_{/2}(\boldsymbol{\xi})$ $\forall$ internal vertex $\boldsymbol{\xi}$

(4.25) For $r > 3$, the moments $\dfrac{1}{h_e} \displaystyle\int_e q(\xi) v(\xi) \mathrm{d}\xi \quad \forall q \in \mathcal{M}_{r-4}^e, \quad \forall$ internal edge e

(4.26) For $s > 1$, the moments $\displaystyle\int_e q(\xi) v_{/n}(\xi) \mathrm{d}\xi \quad \forall q \in \mathcal{M}_{s-2}^e \quad \forall$ internal edge e

(4.27) For $m \geq 0$, the moments $\dfrac{1}{h_K^2} \displaystyle\int_K q(x) v(x) \, \mathrm{d}x \quad \forall q \in \mathcal{M}_m^K \quad \forall$ element K.

The dimension of V_h, that is, the total number of degrees of freedom in V_h is then given by

(4.28) $$\mathcal{G} = 3N_V + N_E(r - 3 + s - 1) + N_K T_0,$$

where N_V is the number of internal vertices of $\mathcal{T}_h$, N_E the number of internal edges, N_K the number of elements, and T_0 is still given by (4.13). Needless to say, in the construction of $v_h \in V_h$ one should take into account the values of v_h and $v_{h/n}$ at the boundary as well, setting them equal to zero.

Remark 4.5. *At this point we can order (in a rather arbitrary way) the degrees of freedom of V_h*

(4.29) $$g_1, \; g_2, ..., g_{\mathcal{G}}.$$

Accordingly we will have a characteristic length h_i attached to each degree of freedom g_i, depending on its nature, according to Remark 4.3. It is obvious that, in the computer code, we will have chosen, once and for all, an orientation for the normal unit vector, say $\mathbf{n}_e$, to each internal edge e. □

Remark 4.6. *It follows easily from the above construction that for every smooth enough w there exists a unique element $w_I \in V_h$ such that*

(4.30) $$g_i(w - w_I) = 0 \qquad \forall i = 1, 2, ...\mathcal{G}.$$

Moreover, by the usual Bramble-Hilbert/Deny-Lions technique (see e.g. [18]) and using a scaling argument to get around the variability in the geometry (see e.g. [13]) it is not difficult to see that, for α and β nonnegative indices, we can prove that

(4.31) $$\|w - w_I\|_{\alpha,\Omega} \leq C \, h^{\beta - \alpha} \, |w|_{\beta,\Omega} \qquad \alpha = 0, 1, 2, \quad \alpha \leq \beta \leq k + 1$$

(with a constant C independent of h) as in the usual Finite Element framework. □

We point out that (as it happens also for the classical finite elements), different choices of degrees of freedom might be used for the same space. However a different choice of the local degrees of freedom can induce a different choice of the global degrees of freedom, that in turn could alter the continuity property of the resulting functions, giving rise to a different global space. Let us see a couple of examples that, in our opinion, are particularly meaningful. Let us consider the value $k = 5$ (and, according to (4.1), $(r, s, m) = (5, 4, 1)$), and the corresponding degrees of freedom in our construction (4.7)-(4.11). They are: the values of w, w_x, w_y at the vertices, plus, on every edge, the moments up to the order two of w and of w_n, plus the moments up to the order one of w inside the element. Taking instead $k = 4$ with $r = 5$, $s = 3$, and $m = 0$ (using this time (4.3)) we would have the same degrees of freedom *minus* the moments of order one of w_n on each edge. The two cases are presented in Figure 2, with obvious meaning of the symbols.
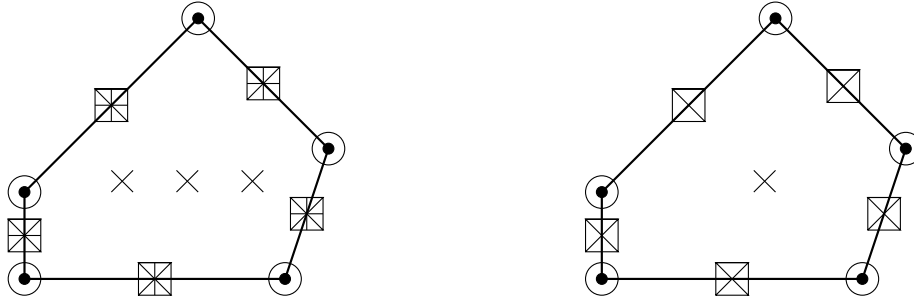
FIGURE 2. Local d.o.f. for the (5,4,1) element with $k = 5$ (left), and for the (5,3,0) element with $k = 4$ (right)

Note that, however, we could as well have used as local degrees of freedom the ones in Figure 3, where (with classical notation) the circles at the vertices represent the *six* degrees of freedom "values of $(w, w_x, w_y, w_{xx}, w_{xy}, w_{yy})$ at the vertices.
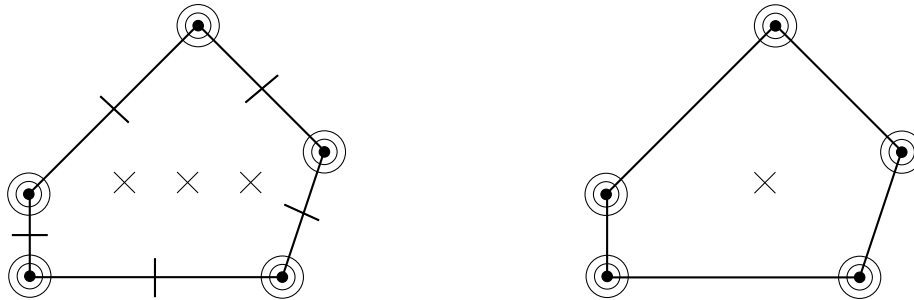

FIGURE 3. Alternative d.o.f. for the elements of Figure 2. We have an "Argyris-like" element on the left and a "Bell-like" element on the right

.

The theoretical treatment of the case of Figure 3 could be done in a way practically identical to the one followed here. As already pointed out in Remark 4.2 we chose to skip the comprehensive treatment of all possible cases allowed by our theory, in order to avoid a too cumbersome notation.

4.3. **Construction of** a_h**.** We are left to show how to construct a (computable!) symmetric bilinear form a_h satisfying (3.7) and (3.8). We shall work on a generic element $K \in \mathcal{T}_h$, and we recall that we denoted by V_h^K the restriction of V_h to K. First of all, we observe that the local degrees of freedom allow us to compute exactly $a^K(p, v)$ for any $p \in \mathbb{P}_k(K)$ and for any $v \in V_h^K$. Indeed, recalling (2.11), we have

$$(4.32) \qquad a^K(p, v) = D \int_K \Delta^2 p \, v \, \mathrm{d}x + \int_{\partial K} [M_{nn}(p) \frac{\partial v}{\partial n} + (Q_n(p) + \frac{\partial M_{nt}(p)}{\partial t})v] \, \mathrm{d}t.$$

We note then that $\Delta^2 p$ belongs to $\mathbb{P}_{k-4}(K)$; for $m \geq k - 4$, the integral over K that appears in (4.32) is then computable using only the values of the internal degrees of freedom of v, without actually knowing v. On the other hand, on each edge of K both $M_{nn}(p)$ (which belongs to $\mathbb{P}_{k-2}(e)$) and $Q_n(p) + \partial M_{nt}(p)/\partial t$ (which belongs to $\mathbb{P}_{k-3}(e)$) are polynomials, as well as v and $v_{/n}$. From the degrees of freedom on the

boundary we can easily compute both v and $v_{/n}$, and in conclusion all the terms in the right-hand side of (4.32) can be easily computed without knowing v inside.

Now for every $\varphi \in C^0(\overline{K})$ we define its *quasi-average* $\widehat{\varphi}$ as the constant function (over K) whose value

$$(4.33) \qquad \widehat{\varphi} := \frac{1}{\ell} \sum_{i=1}^{\ell} \varphi(\mathbf{x}^i)$$

is given by the average of the values that φ assumes at the ℓ vertices $\boldsymbol{\xi}^i$, $(i = 1, 2, ..., \ell)$ of K. Next, we introduce the operator $\Pi_k^K : V_h^K \longrightarrow \mathbb{P}_k(K) \subset V_h^K$ defined as the solution of

$$(4.34) \qquad \begin{cases} a^K(\Pi_k^K \psi, q) = a^K(\psi, q) & \forall \psi \in V_h^K, \ \forall q \in \mathbb{P}_k(K) \\ \widehat{\Pi_k^K \psi} = \widehat{\psi}, \quad \widehat{\nabla \Pi_k^K \psi} = \widehat{\nabla \psi}. \end{cases}$$

We note that for $v \in \mathbb{P}_k(K)$ the first equation in (4.34) implies $(\Pi_k^K v)_{/ij} = v_{/ij}$ for $i, j = 1, 2$, that joined with the second equation gives easily

$$(4.35) \qquad \Pi_k^K v = v \quad \forall v \in \mathbb{P}_k(K).$$

Choosing $a_h^K(u, v) = a^K(\Pi_k^K u, \Pi_k^K v)$ would now ensure property (3.7), but not, in general, property (3.8). Hence we need to add a suitable term capable to ensure (3.8). Let then $S^K(u, v)$ be a symmetric positive definite bilinear form, to be chosen to verify

$$(4.36) \qquad c_0 \, a^K(v, v) \leq S^K(v, v) \leq c_1 \, a^K(v, v), \qquad \forall v \in V_h^K \text{ with } \Pi_k^K v = 0,$$

for some positive constants c_0, c_1 independent of K and h_K. Then set

$$(4.37) \qquad a_h^K(u, v) := a^K(\Pi_k^K u, \Pi_k^K v) + S^K(u - \Pi_k^K u, v - \Pi_k^K v).$$

Proposition 4.3. *The bilinear form* (4.37) *constructed with the above procedure satisfies both the consistency property* (3.7) *and the stability property* (3.8).

Proof. Property (3.7) follows immediately from (4.35) and (4.34): for $p \in \mathbb{P}_k(K)$, (4.35) implies $S^K(p - \Pi_k^K p, v - \Pi_k^K v) = 0 \ \forall v$, and hence

$$(4.38) \qquad a_h^K(p, v) = a^K(\Pi_k^K p, \Pi_k^K v) = a^K(\Pi_k^K p, v) = a^K(p, v).$$

Moreover, from (4.34) we have $a^K(\psi - \Pi_k^K \psi, q) = 0$ for all $q \in \mathbb{P}_k(K)$ and for all $\psi \in V_h^K$, so that by Pythagoras theorem we have

$$(4.39) \qquad a^K(v - \Pi_k^K v, v - \Pi_k^K v) + a^K(\Pi_k^K v, \Pi_k^K v) = a^K(v, v) \quad \forall v \in V_h^K.$$

Property (3.8) follows then easily from (4.36) and (4.39). Indeed setting $\alpha^* := \max\{1, c_1\}$ we have

$$(4.40) \qquad \begin{aligned} a_h^K(v, v) \ &\leq a^K(\Pi_k^K v, \Pi_k^K v) + c_1 a^K(v - \Pi_k^K v, v - \Pi_k^K v) \\ &\leq \max\{1, c_1\} \Big(a^K(\Pi_k^K v, \Pi_k^K v) + a^K(v - \Pi_k^K v, v - \Pi_k^K v) \Big) \\ &= \alpha^* a^K(v, v), \end{aligned}$$

and similarly, setting $\alpha_* := \min\{1, c_0\}$:

(4.41)
$$a_h^K(v,v) \geq \min\{1, c_0\}\Big(a^K(\Pi_k^K v, \Pi_k^K v) + a^K(v - \Pi_k^K v, v - \Pi_k^K v)\Big)$$
$$= \alpha_* a^K(v,v).$$

□

4.4. **Choice of S^K.** In general, the choice of the bilinear form S^K might depend on the problem and on the degrees of freedom. From (4.36) it is clear that S^K must scale like $a^K(\cdot, \cdot)$ on the kernel of Π_k^K. Since in (4.23)-(4.27)we took care to have degrees of freedom of the same dimension, then all the elements of the canonical basis $\varphi_1, ..., \varphi_T$ defined by

(4.42)
$$g_i(\varphi_j) = \delta_{ij}, \quad i, j = 1, T,$$

will scale in the same way (and in particular as the ones associated to "point value" degrees of freedom). Hence the choice

$$S^K(v, w) = D \sum_{i=1}^{T} g_i(v) g_i(w) (h_i)^{-2},$$

(with h_i as in Remark 4.5) would clearly do the job.

Remark 4.7. *Clearly, is we decide to use degrees of freedom involving second derivatives, as the ones in Figure 3, we should scale them by h_ξ^2 in order to apply the above argument* □

4.5. **Construction of the right-hand side.** In order to build the loading term $< f_h, v_h >$ for $v_h \in V_h$ in a simple and easy way it is convenient to have internal degrees of freedom in V_h, and this means, according to (4.1) and (4.11), that we need $k \geq 4$. In this case we define f_h on each element K as the $L^2(K)$−projection of the load f onto the space of piecewise polynomials of degree $m = k - 4$, that is,

$$f_h = P_{k-4}^K f \qquad \text{on each } K \in \mathcal{T}_h.$$

Then, always for $k \geq 4$, the associated loading term

$$< f_h, v_h > = \sum_{K \in \mathcal{T}_h} \int_K f_h\, v_h \,\mathrm{d}x \equiv \sum_{K \in \mathcal{T}_h} \int_K (P_{k-4}^K f)\, v_h \,\mathrm{d}x = \sum_{K \in \mathcal{T}_h} \int_K f\, (P_{k-4}^K v_h) \,\mathrm{d}x$$

can be exactly computed using the degrees of freedom for V_h that represent the internal moments, see (4.11).

For $k > 4$, that is, $m = k - 4 \geq 1$, standard L^2 orthogonality and approximation estimates on star-shaped domains yield

(4.43) $< f_h, v_h > -(f, v_h) = \sum_{K \in \mathcal{T}_h} \int_K (P_{k-4}^K f - f)(v_h - P_1^K v_h) \,\mathrm{d}x$

$$\leq C \sum_{K \in \mathcal{T}_h} h_K^{k-3} |f|_{k-3,K}\, h_K^2 \|v_h\|_{2,K} \leq C\, h^{k-1} \Big(\sum_{K \in \mathcal{T}_h} |f|_{k-3,K}^2\Big)^{1/2} \|v_h\|_V,$$

and thus, recalling (3.13),

$$(4.44) \qquad \|f - f_h\|_{V_h'} \le Ch^{k-1} \Big(\sum_{K \in \mathcal{T}_h} |f|^2_{k-3,K} \Big)^{1/2}.$$

Hence, for $k > 4$ Theorem 3.1 ensures the optimal $O(h^{k-1})$ error bound. For $k \le 4$ we need some additional tricks. In view of Remark 4.4 we assume that we are able to compute exactly the integral

$$(4.45) \qquad \int_K p_1 \, q_1 \, \mathrm{d}x$$

for every pair p_1 and q_1 of polynomials of degree ≤ 1 on K. In each case $k = 2, 3, 4$ the term $< f_h, v_h >$ will then be defined by

$$(4.46) \qquad < f_h, v_h >_K = \int_K \widetilde{P}_1^k f \; \overline{P}_1^k v_h$$

where $\widetilde{P}_1^k$ and $\overline{P}_1^k$ are projectors on the space of polynomials of degree ≤ 1 that will change from one value of k to another.

For $k = 2$ and for $k = 3$ we take

$$(4.47) \quad f_{h|K} \equiv \widetilde{P}_1^k f := P_{k-2}^K f, \quad \overline{P}_1^k v_h \equiv \widetilde{v}_h := \widehat{v}_h + (k-2)(\mathbf{x} - \mathbf{x}_K) \cdot \widehat{\nabla v_h} \quad \text{on each } K.$$

Note that for both values $k = 2$ and $k = 3$ we have that $\widetilde{v}_h \in \mathbb{P}_{k-2}(K)$. Hence on each K we have

$$(4.48) \qquad \begin{aligned} < f_h, v_h >_K -(f, v_h)_K &= (P_{k-2}^K f, \widetilde{v}_h)_K - (f, v_h)_K \\ &= (P_{k-2}^K f - f, \widetilde{v}_h)_K + (f, \widetilde{v}_h - v_h)_K \\ &= (f, \widetilde{v}_h - v_h)_K \le Ch_K^{k-1} \|f\|_{0,K} \|v_h\|_{k-1,K}. \end{aligned}$$

For $k = 4$ we take instead

$$(4.49) \quad f_{h|K} \equiv \widetilde{P}_1^k f := P_1^K f, \quad \overline{P}_1^k v_h \equiv \widetilde{v}_h := P_0^K v_h + (\mathbf{x} - \mathbf{x}_K) \cdot \widehat{\nabla v_h} \quad \text{on each } K.$$

We have on each K

$$(4.50) \qquad \begin{aligned} < f_h, v_h >_K -(f, v_h)_K &= (P_1^K f, \widetilde{v}_h)_K - (f, v_h)_K \\ &= (P_1^K f - f, \widetilde{v}_h)_K + (f, \widetilde{v}_h - v_h)_K \\ &= (f - P_0^K f, \widetilde{v}_h - v_h)_K \le Ch_K^3 |f|_{1,K} \|v_h\|_{2,K} \end{aligned}$$

where, going from second to third line we took advantage of the fact that $\widetilde{v}_h - v_h$ has zero mean value over K. Note that, for $k = 4$, we have $m = 0$ in (4.1), and from (4.11) we have the right to use $P_0^K v_h$ instead of $\widehat{v}_h$, since $P_0^K v_h$ is now part of the degrees of freedom.

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