

# Equivalent Projectors for Virtual Element Methods

B. Ahmad

*KAU, Jeddah, Saudi Arabia*

A. Alsaedi

*KAU, Jeddah, Saudi Arabia*

F. Brezzi

*IUSS-Pavia and IMATI-CNR*

*27100 Pavia, Italy*

*KAU, Jeddah, Saudi Arabia*

*brezzi@imati.cnr.it*

L. D. Marini

*Dipartimento di Matematica, Università di Pavia, and IMATI-CNR,*

*Via Ferrata 1, 27100 Pavia, Italy*

*marini@imati.cnr.it*

A. Russo

*Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca*

*Via Cozzi 53, I-20153, Milano, Italy*

*and IMATI-CNR, Via Ferrata 1, 27100 Pavia, Italy*

*alessandro.russo@unimib.it*

---

## Abstract

In the original Virtual Element space with degree of accuracy  $k$ , projector operators in the  $H^1$ -seminorm onto polynomials of degree  $\leq k$  can be easily computed. On the other hand, projections in the  $L^2$  norm are available only on polynomials of degree  $\leq k - 2$  (directly from the degrees of freedom). Here we present a variant of VEM that allows the exact computations of the  $L^2$  projections on all polynomials of degree  $\leq k$ . The interest of this construction is illustrated with some simple examples, including the construction of three-dimensional Virtual Elements, the treatment of lower order terms, the treatment of the right-hand side, and the  $L^2$  error estimates.

*Keywords:* Virtual Elements, Mimetic Finite Differences

*2000 MSC:* 65N30, 65N12, 65G99, 76R99

---

## 1. Introduction

The Virtual Element Methods (VEM) have been introduced very recently (see [4]) and first applied to some simple two-dimensional elasticity problems (see [5]) and plate problems (see [17]).

The origin of Virtual Elements, historically, is clearly from the classical Mimetic Finite Difference Method (see for instance [30, 26, 18, 27, 28]), and from their subsequent mathematical frameworks and settings [14, 16, 9, 15].

In their more recent evolution Mimetic Finite Differences (MFD) were often presented either as a form of Cochain approximation or as a sort of Finite Element methods in which *only the degrees of freedom are used* (and, in principle, there are no test and trial functions attached to them) [33, 19, 1, 2, 20, 11, 22].

Further developments included the use of primal formulations (with nodal values, or 0-Cochains [12, 3]) as well as higher order methods [8, 25, 7, 6].

Still the lack of trial and test functions inside the elements (or even inside the faces) was making the presentation and the analysis rather complicated.

In the long run it became clear that life would be much simpler if the MFD unknowns were considered to be attached to trial functions suitably defined inside the elements, as is commonly done in Finite Element Methods, thus motivating the birth of Virtual Element Methods.

In order to preserve the great generality that MFD allow for the geometry of the elements, the Virtual Element Methods use local spaces of test and trial functions that, in addition to all the polynomials of the chosen degree (say,  $k$ ), contain some additional functions that are solution of suitable PDE problems inside each element.

In this respect the VEM are getting closer to other attempts to generalize Finite Elements on polygons, like the use of rational functions (see e.g. [34]), the *Polygonal Finite Element Methods* (see for instance [31, 32]) or the *Extended Finite Element Methods* (see [24] and the references therein).

One of the basic ideas of VEM is that even on elements  $K$  with a rather general geometry we can compute integrals of polynomials, essentially through formulae of the type

$$\int_K x^r dK = \int_{\partial K} \frac{x^{r+1}}{r+1} n_x dS \quad (1)$$

while the computation of the non-polynomial functions (and of their integrals) requires some additional trick (and could also be practically unfeasible). In particular in order to compute the contribution of each of these non-polynomial functions to the local stiffness matrix of an element  $K$ , one has first to compute a *local projector* (here denoted  $\Pi_k^\nabla$ ) on the space of polynomials of degree  $\leq k$ . These are, in general, projectors in the  $H_0^1(K)$  scalar product (with a suitable adjustment of the constant part). See [4, 5, 17].

In many applications, the explicit knowledge of the projector  $\Pi_k^\nabla$  is enough to complete the discretization process and to perform the analysis. However, there are obvious cases in which it would be very useful to have an explicit knowledge, together with  $\Pi_k^\nabla$ , also of the *local  $L^2$ -orthogonal projector*  $\Pi_k^0$  on the space of polynomials of degree  $\leq k$ .

The main purpose of this paper is to show that in a certain number of cases, just by changing slightly the definition of the non-polynomial local functions (that, in any case, are never computed!), one can have a local space in which the operator  $\Pi_k^0$  can be easily computed using  $\Pi_k^\nabla$  and the local degrees of freedom, so that having computed  $\Pi_k^\nabla$  one can get  $\Pi_k^0$  (almost) for free.

As we shall see, the knowledge of the operator  $\Pi_k^0$  is very useful in several circumstances: in particular it allows an extension of VEM to the three-dimensional case that is much cheaper than the obvious version that would come from MFD. Here we will discuss a few other examples of applications, dealing with the treatment of possible reaction terms and with a simpler treatment of the forcing term. But the range of application is clearly much wider.

Throughout the paper, we will follow the usual notation for Sobolev spaces and norms (see e.g. [21]). In particular, for an open bounded domain  $\mathcal{D}$ , we will use  $|\cdot|_{s,\mathcal{D}}$  and  $\|\cdot\|_{s,\mathcal{D}}$  to denote seminorm and norm, respectively, in the Sobolev space  $H^s(\mathcal{D})$ , while  $(\cdot, \cdot)_{0,\mathcal{D}}$  will denote the  $L^2(\mathcal{D})$  inner product. Often the subscript  $\mathcal{D}$  will be omitted when no confusion arises. For  $k$  a non negative integer,  $\mathbb{P}_k(\mathcal{D})$  will denote the space of polynomials of degree  $\leq k$  on  $\mathcal{D}$ . Conventionally,  $\mathbb{P}_{-1}(\mathcal{D}) = \{0\}$ . Sometimes, for the sake of clarity, we will denote by  $\Pi_k^{0,\mathcal{D}}$  and  $\Pi_k^{\nabla,\mathcal{D}}$  the projector operators  $\Pi_k^0$  and  $\Pi_k^\nabla$  related to  $\mathcal{D}$ . Finally,  $C$  will be a generic constant independent of the decomposition that could change from one occurrence to the other.

Concerning geometric objects (and related items) we will use the following notation. For a geometric object  $\mathcal{O}$  of dimension  $d$  ( $d = 1, 2, 3$ ), as an edge, or a face, or an element, we will denote by  $\mathbf{x}_\mathcal{O}$  its barycenter, by  $|\mathcal{O}|$  its measure (resp. length, area, or volume) and by  $h_\mathcal{O}$  its diameter. Moreover, for  $r \in \mathbb{N}$  we denote by  $\mathcal{M}_r(\mathcal{O})$  the set of polynomials

$$\mathcal{M}_r(\mathcal{O}) := \left\{ m \mid m = \left( \frac{\mathbf{x} - \mathbf{x}_\mathcal{O}}{h_\mathcal{O}} \right)^{\mathbf{s}} \text{ for } \mathbf{s} \in \mathbb{N}^d \text{ with } |\mathbf{s}| \leq r \right\} \quad (2)$$

where, for a multi-index  $\mathbf{s} = (s_1, \dots, s_d)$  we denoted, as usual,  $|\mathbf{s}| := s_1 + \dots + s_d$  and  $\mathbf{x}^{\mathbf{s}} := x_1^{s_1} \cdot \dots \cdot x_d^{s_d}$ . It is elementary to check that the elements in  $\mathcal{M}_r(\mathcal{O})$  form a basis for the space of polynomials of degree  $\leq r$  on  $\mathcal{O}$ , and their number equals  $(r+1) \cdot \dots \cdot (r+d)/d!$ . We will also make use of the set  $\mathcal{M}_r^*(\mathcal{O})$  defined by

$$\mathcal{M}_r^*(\mathcal{O}) := \left\{ m \mid m = \left( \frac{\mathbf{x} - \mathbf{x}_\mathcal{O}}{h_\mathcal{O}} \right)^{\mathbf{s}} \text{ for } \mathbf{s} \in \mathbb{N}^d \text{ with } |\mathbf{s}| = r \right\}. \quad (3)$$

The number of elements in  $\mathcal{M}_r^*(\mathcal{O})$  is  $d(d+1) \cdot \dots \cdot (d+r-1)/r!$ .

The layout of the paper is the following. In Section 2 we recall some basic features of Virtual Elements, and in particular the construction of the projector  $\Pi_k^\nabla$ , in two dimensions. In Section 3, always in two dimensions, we introduce a variant of the local spaces that allows an easy construction of the  $L^2$ -orthogonal projector  $\Pi_k^0$ . In Section 4 we show that using  $\Pi_k^0$  on each face one can construct a quite useful version of VEM in three dimensions, and in Section 5 we show how to use it on a simple model problem (Poisson). In Section 6 we discuss some additional applications, including the proof of optimal  $L^2$  error estimates, the

treatment of a model reaction-diffusion problem, and some numerical results. Some conclusions are drawn in Section 7.

#### Acknowledgements

This paper was funded by King Abdulaziz University, under grant N<sup>o</sup>(38-3-1432/HiCi). The authors, therefore, acknowledge technical and financial support of KAU.

## 2. The original projector $\Pi_k^\nabla$

Let  $k \geq 1$  be an integer denoting the degree of accuracy that we want to obtain. We briefly recall from [4] *the core idea* of the classical VEM and in particular of the construction of the projector  $\Pi_k^\nabla$  (we refer to [4] for more details):

- The trial and test functions contain, on each element, all the polynomials of degree  $\leq k$ , plus (possibly) other functions that, in general, are not polynomials.
- The degrees of freedom are carefully chosen so that the local stiffness matrix (or, actually, its associated bilinear form), can be computed *exactly*, whenever the *trial* entry is a polynomial of degree  $\leq k$ , using only the degrees of freedom of the *test* entry.

Using the above properties one can show that for the remaining part of the local stiffness bilinear form (when a non-polynomial encounters another non-polynomial) we only need to produce a result with *the right order of magnitude and right stability properties*.

In a sense, instead of using, in a more traditional way, a *nearly exact* value for all entries in the local stiffness bilinear form (as with the use of numerical integration formulae) we have *exact* values when one of the two entries is a polynomial, and only much rougher approximations in the other cases.

We remind that the properties above bring us quite close to the **Patch Test** used by Engineers, as they imply that the method gives the exact solution whenever this is a global polynomial of degree  $\leq k$ .

Let us see now the construction of the projector  $\Pi_k^\nabla$  in the two-dimensional case. Given a positive integer  $k$  (*the order of accuracy*) we define first, on each polygon  $K$ , the space

$$B_k(\partial K) := \{v : v \in C^0(\partial K) \text{ and } v|_e \in \mathbb{P}_k(e) \text{ for each edge } e \text{ of } \partial K\}. \quad (4)$$

Then the Virtual Element spaces are constructed, on each polygon  $K$ , as

$$V_k(K) := \{v : v|_{\partial K} \in B_k(\partial K) \text{ and } \Delta v \in \mathbb{P}_{k-2}(K)\}. \quad (5)$$

The corresponding degrees of freedom are chosen, always at the element level, prescribing for each  $v_h$  in  $V_k(K)$ :

- the values of  $v_h$  at the *vertices*, (6)

and for  $k \geq 2$  the moments

$$\bullet |e|^{-1}(m, v_h)_{0,e} \quad \forall m \in \mathcal{M}_{k-2}(e), \quad \forall \text{edge } e, \quad (7)$$

$$\bullet |K|^{-1}(m, v_h)_{0,K} \quad \forall m \in \mathcal{M}_{k-2}(K). \quad (8)$$

It is easy to see that both the dimension of  $V_k(K)$  and the number of the degrees of freedom (6)-(8) are equal to

$$N_V^{K,k} := \dim V_k(K) = k\ell + k(k-1)/2 \quad (9)$$

where  $\ell$  is the number of vertices of the polygon  $K$ . Moreover it was proved (see always [4]) that the above degrees of freedom are *unisolvent* in  $V_k(K)$ .

**Remark 1.** *It is clear that for  $k \geq 2$  the degrees of freedom interior to each edge can be chosen differently from (7). For instance, for  $k = 2$ , to assign the value at the midpoint would be as good as assigning the average (7).  $\square$*

**Remark 2.** *The generic elements of  $V_k(K)$  will be denoted by  $v_h$ , but also by  $v$  when a formula is too long and/or no confusion can occur.  $\square$*

We remark that for  $k \geq 2$  the  $L^2(K)$ -projection  $\Pi_{k-2}^{0,K} v_h$  of  $v_h \in V_k(K)$  onto  $\mathbb{P}_{k-2}(K)$  is immediately given by the degrees of freedom (8). However, the moments  $(v_h, q)_{0,K}$  are, practically, incomputable when  $q$  is a polynomial of degree  $\geq k-1$ . Hence the  $L^2(K)$  orthogonal projection operator  $\Pi_k^0$  from  $V_k(K)$  onto the space  $\mathbb{P}_k(K)$  cannot be computed from the degrees of freedom (6)-(8) of  $V_k(K)$ . On the other hand, considering the Green formula

$$\int_K \nabla v_h \cdot \nabla q \, dx = - \int_K v_h \Delta q \, dx + \int_{\partial K} v_h \frac{\partial q}{\partial n} \, ds, \quad (10)$$

we observe first that knowing the degrees of freedom (6)-(7) of a function  $v_h$  in  $V_k(K)$  we can easily compute the value of  $v_h$  on the whole boundary  $\partial K$ . Next we observe that, for  $k \geq 2$ , if  $q$  in (10) is a polynomial in  $\mathbb{P}_k$  then  $\Delta q$  is in  $\mathbb{P}_{k-2}$ , and hence the first term in the right-hand side of (10) can be computed using the degrees of freedom (8) of  $v_h$ . We conclude that using (10) we can compute the integral

$$\int_K \nabla v_h \cdot \nabla q \, dx \quad (11)$$

for every  $v_h \in V_k(K)$  and every  $q \in \mathbb{P}_k(K)$  using just the degrees of freedom (6)-(8). It follows immediately that for every  $v_h \in V_k(K)$  we can define  $\Pi_k^\nabla v_h$  as the unique element in  $\mathbb{P}_k(K)$  that satisfies

$$(\nabla(v_h - \Pi_k^\nabla v_h), \nabla q)_{0,K} = 0 \quad \forall v_h \in V_k(K), \quad \forall q \in \mathbb{P}_k(K) \quad (12)$$

plus (to take care of the constant part of  $\Pi_k^\nabla v_h$ ):

$$\int_{\partial K} (v_h - \Pi_k^\nabla v_h) \, ds = 0 \quad \text{for } k = 1 \quad (13)$$

or

$$\int_K (v_h - \Pi_k^\nabla v_h) \, d\mathbf{x} = 0 \quad \text{for } k \geq 2. \quad (14)$$

We notice that the operator  $\Pi_k^\nabla$  is easily computable using the degrees of freedom (6)-(8). It is also easy to check that whenever  $v_h \in \mathbb{P}_k(K)$  then  $\Pi_k^\nabla v_h \equiv v_h$ , showing that  $\Pi_k^\nabla$  is indeed a projection onto  $\mathbb{P}_k(K)$ .

**Remark 3.** We note that (12) and (13) (or (14)) define the operator  $\Pi_k^\nabla$  on the whole space  $H^1(K)$ , and not only on  $V_k(K)$ , but of course it is not computable in general.  $\square$

**Remark 4.** Note that, once  $\Pi_k^\nabla v_h$  is known, all its moments are known as well, but of course these moments do not coincide in general with those of  $v_h$  (apart from the mean value, i.e. the moment of order zero, for  $k \geq 2$ ). In particular we point out that for  $k > 2$  the moments of  $v_h$  up to the order  $k - 2$  are used in order to compute  $\Pi_k^\nabla v_h$  (see (10)) but, in general, we will have

$$\int_K v_h m \, d\mathbf{x} \neq \int_K \Pi_k^\nabla v_h m \, d\mathbf{x} \quad \text{for } m \in \mathcal{M}_r^*(K), \text{ with } 0 < r \leq k - 2. \quad (15)$$

$\square$

In a certain number of cases (in particular in two dimensions and when dealing with elliptic operators having only the principal part, as in [4, 5, 17]), the knowledge of the projector  $\Pi_k^\nabla$  is sufficient to construct the whole discretized problem and produce an interesting method. On the other hand, in many other cases (as for instance for three-dimensional problems) the explicit knowledge of the  $L^2(K)$ -orthogonal projection  $\Pi_k^0$  will be useful and allow a cheaper discretization. In other cases (as for instance in the presence of nonlinearities) this knowledge might substantially improve the quality of the method. Here below we will show that with a minor change of point of view and a minor additional work one can explicitly construct the operator  $\Pi_k^0$  using the knowledge of  $\Pi_k^\nabla$  and the degrees of freedom (8).

### 3. The modified VEM and the projector $\Pi_k^0$

The basic idea of the modified VEM is now to introduce a new space  $W_k(K)$  to be used in place of  $V_k(K)$  in such a way that

- the degrees of freedom (6)–(8) can still be used for  $W_k(K)$ ,
- all the polynomials of degree  $\leq k$  are still included in  $W_k(K)$ ,
- the projection  $\Pi_k^\nabla$  from  $W_k(K)$  to  $\mathbb{P}_k$  can still be computed exactly,

but, for the elements  $v_h \in W_k(K)$  the moments of order  $k - 1$  and  $k$  of  $v_h$  and of  $\Pi_k^\nabla v_h$  coincide. Note that, as we have seen in Remark 4, this last property does not hold in  $V_k(K)$ .

To construct  $W_k(K)$  we proceed as follows:

a) We **enlarge** first  $V_k(K)$  to, say,

$$\tilde{V}_k(K) := \{v_h : v_h|_{\partial K} \in B_k(\partial K) \text{ and } \Delta v_h \in \mathbb{P}_k(K)\}. \quad (16)$$

b) We **restrict**  $\tilde{V}_k(K)$  to a subspace  $W_k(K)$  having the same dimension (and *the same degrees of freedom!*) as the original  $V_k(K)$ , but where the moments of degree  $k-1$  and  $k$  of  $v$  and  $\Pi_k^\nabla v$  coincide. More precisely, we set

$$W_k(K) := \{w_h \in \tilde{V}_k(K) : (w_h - \Pi_k^\nabla w_h, q^*)_{0,K} = 0 \\ \forall q^* \in \mathcal{M}_{k-1}^*(K) \cup \mathcal{M}_k^*(K)\}. \quad (17)$$

**Proposition 1.** *The dimension of  $\tilde{V}_k(K)$  is*

$$\tilde{N}_V^{K,k} = \ell k + (k+1)(k+2)/2,$$

and as degrees of freedom in  $\tilde{V}_k(K)$  we can take the traces on  $\partial K$  (piecewise in  $\mathbb{P}_k$ ) and the moments, in  $K$ , up to the order  $k$ .

*Proof.* The proof is virtually identical to that given in [4] for  $V_k(K)$  and is based on the observation that for functions in  $H_0^1(K)$  with Laplacian in  $\mathbb{P}_r(K)$  ( $r \in \mathbb{N}$ ) there is a one-to-one mapping between the moments of the functions up to the order  $r$  and their Laplacian.  $\square$

**Proposition 2.** *The dimension  $N_W^{K,k}$  of  $W_k(K)$  is equal to that of  $V_k(K)$ , that is, as in (9),*

$$N_W^{K,k} := \dim W_k(K) = \ell k + k(k-1)/2. \quad (18)$$

As degrees of freedom in  $W_k(K)$  we can take the same as in  $V_k(K)$  (that is, (6)-(8)).

*Proof.* It is immediate to check that the dimension of  $\mathcal{M}_{k-1}^*(K) \cup \mathcal{M}_k^*(K)$  is equal to  $2k+1$ . Hence, without checking the independence of the additional  $2k+1$  conditions in (17), we can at least be assured that the dimension  $N_W^{K,k}$  of  $W_k(K)$  verifies

$$N_W^{K,k} \geq \tilde{N}_V^{K,k} - (2k+1) = \ell k + (k+1)(k+2)/2 - (2k+1) = \ell k + k(k-1)/2. \quad (19)$$

We now observe that a function  $w_h \in W_k(K)$  that vanishes on  $\partial K$  and has zero moments up to the order  $k-2$  is identically zero. Indeed, it is immediate to see from (12)-(14) that in this case  $\Pi_k^\nabla w_h$  would be zero, implying that all its moments are zero, implying (since  $w_h \in W_k(K)$ ) that all the moments of order  $k-1$  and  $k$  of  $w_h$  are also zero. Using Proposition 1 this implies that  $w_h$  is zero.

This (together with (19)) implies that the dimension of  $W_k(K)$  is actually equal to  $\ell k + k(k-1)/2$ , and that in  $W_k(K)$  the degrees of freedom (6)-(8) are unisolvent.  $\square$

**Remark 5.** More generally, we could set, for  $r \geq k - 1$ ,

$$\tilde{V}_{k,r}(K) := \{v_h : v_h|_{\partial K} \in B_k(\partial K) \text{ and } \Delta v_h \in \mathbb{P}_r(K)\}$$

and then define (always for  $r \geq k - 1$ ):

$$W_{k,r}(K) := \{w_h \in \tilde{V}_{k,r}(K) : (w_h - \Pi_k^\nabla w_h, q^*)_{0,K} = 0 \\ \forall q^* \in \mathcal{M}_s^*(K) \text{ for all } s \text{ with } k - 1 \leq s \leq r\}. \quad (20)$$

For a general  $r \geq k - 1$  we could again take, as degrees of freedom in  $W_{k,r}(K)$ , the values on  $\partial K$  and the moments up to the degree  $k - 2$  in  $K$ . Hence, for a function  $w_h \in W_{k,r}(K)$  the moments up to  $k - 2$  are among the degrees of freedom, and the moments of order  $s$  with  $k - 2 < s \leq r$  are computable (since we can compute  $\Pi_k^\nabla w_h$  explicitly). However, the maximum available accuracy is still  $k$  (we did not touch the values on  $\partial K!$ ), which seems to render the choice  $r > k$  totally useless. □

To summarize, what we have obtained is the following:

- Every set of  $\ell k + k(k - 1)/2$  real numbers, interpreted as degrees of freedom (6)–(8), uniquely defines a function  $v_h \in V_k(K)$  or a function  $w_h \in W_k(K)$ . These functions are clearly different, but they share the same degrees of freedom. Note however that if, by chance, the  $\ell k + k(k - 1)/2$  values of (6)–(8) are taken from a polynomial  $p \in \mathbb{P}_k$ , then  $p \equiv v_h \equiv w_h$ .
- Since the computation of  $\Pi_k^\nabla$  only requires the use of the d.o.f. (6)–(8), we obviously have  $\Pi_k^\nabla v_h \equiv \Pi_k^\nabla w_h$ .
- If  $w_h \in W_k(K)$ , the moments of order  $k - 1$  and  $k$  of  $w_h$  and of  $\Pi_k^\nabla w_h$  are equal. This information, together with (8), allows us to compute all the moments of  $w_h$  up to the order  $k$ . This can provide, if needed, the  $L^2$ -projection of  $w_h$  onto  $\mathbb{P}_k(K)$ , as we see here below.

We now show in detail how the operator  $\Pi_k^0$  can be explicitly computed. To start with, we point out that  $\Pi_k^0 v_h$  can be presented as the unique element in  $\mathbb{P}_k(K)$  such that

$$\int_K \Pi_k^0 v_h m \, d\mathbf{x} = \int_K v_h m \, d\mathbf{x} \quad \text{for all } m \in \mathcal{M}_k(K). \quad (21)$$

Clearly (21) is a linear system, whose unknowns are the coefficients of  $\Pi_k^0 v_h$  in the monomial basis  $\mathcal{M}_k(K)$ . The matrix associated to (21) can be computed through (1), so that all the difficulties in the computation of  $\Pi_k^0$  are in the computation of the right-hand side.

For  $k \geq 2$  part of the components of the right-hand side of (21) are immediately available from the degrees of freedom (8), and the others, for  $v_h$  in  $W_k(K)$ , can be obtained from

$$\int_K v_h m \, d\mathbf{x} = \int_K \Pi_k^\nabla v_h m \, d\mathbf{x} \quad \text{for } m \in \mathcal{M}_{k-1}^*(K) \cup \mathcal{M}_k^*(K), \quad (22)$$



once the projector  $\Pi_k^\nabla$  has been computed. To clarify this point, we distinguish the following two cases.

**The case  $k \leq 2$ .**

In this case we always have  $\Pi_k^\nabla = \Pi_k^0$ . Indeed, for  $k = 1$  we have from (22)

$$\int_K v_h m \, d\mathbf{x} = \int_K \Pi_1^\nabla v_h m \, d\mathbf{x} \quad \text{for } m \in \mathcal{M}_0^*(K) \cup \mathcal{M}_1^*(K), \quad (23)$$

so that from (21)

$$\int_K \Pi_1^0 v_h m \, d\mathbf{x} = \int_K \Pi_1^\nabla v_h m \, d\mathbf{x} \quad \text{for } m \in \mathcal{M}_1(K). \quad (24)$$

Hence,  $\Pi_1^\nabla = \Pi_1^0$ . On the other hand, for  $k = 2$  we still have  $\Pi_2^\nabla = \Pi_2^0$ . Indeed from (14) and (22) we immediately have that the moments of  $v_h$  and  $\Pi_2^\nabla v_h$  coincide for  $k = 0, 1, 2$ .

**The case  $k > 2$ .**

In this case we cannot ensure that  $\Pi_k^0 = \Pi_k^\nabla$  but we can always compute all the moments in the right-hand side of (21). Indeed, the first  $k - 2$  moments are given to us by (8), while the moments of order  $k - 1$  and  $k$  are provided by (22).

**Warning:** In the lowest order case ( $k = 1$ ) the degrees of freedom identify uniquely a function  $g_h$  on the boundary. To  $g_h$  we can uniquely associate a function  $v_h$  (harmonic in  $K$ ) in  $V_1(K)$  and a function  $w_h$  (with constant Laplacian) in  $W_1(K)$ . These two functions are different, but they share the same boundary values and hence (for  $k = 1$ ) the same projection, meaning that  $\Pi_1^\nabla v_h = \Pi_1^\nabla w_h$ . Note that we use (13) in order to take care of the *constant part* of  $\Pi_1^\nabla$ . The mean value of  $w_h$  on  $K$  is easily computable. As  $w_h \in W_1(K)$  we obviously have

$$\int_K w_h \, d\mathbf{x} = \int_K \Pi_1^\nabla w_h \, d\mathbf{x}.$$

Instead the mean value of  $v_h$  on  $K$  cannot be computed, unless we *think that (from the same degrees of freedom!) we are using  $w_h$* . In particular one *should not* be lazy and use

$$\frac{1}{|\partial K|} \int_{\partial K} v_h \, ds \quad (25)$$

which is already known by (13). Indeed, (25) is only a first order approximation of the average of  $v_h$  over  $K$  that in many cases could not be good enough.  $\square$

## 4. The three-dimensional spaces and projectors

### 4.1. Three-dimensional Virtual spaces

We consider now a polyhedron  $K$  with  $n_V$  vertices,  $n_f$  faces, and  $n_e$  edges. For every integer  $k \geq 1$  and for every face  $f$  of  $K$  we recall the definition (4) of the space  $B_k(\partial f)$ , and we observe that  $B_k(\partial f)$  is a linear space of dimension

$$\nu^f + \nu^f(k - 1) = \nu^f k,$$

where  $\nu^f$  is the number of edges (and the number of vertices) of  $f$ . Then we denote by  $W_k(f)$  the space defined in (17) with respect to the polygon  $f$ . We recall from (18) that the dimension of  $W_k(f)$  is

$$N_W^{f,k} = \nu^f k + k(k-1)/2.$$

At this point we can define for each polyhedron  $K$

$$W_k(\partial K) := \{v \in C^0(\partial K) : v|_f \in W_k(f) \forall \text{ face } f \subset \partial K\}. \quad (26)$$

We finally consider the finite dimensional space  $\mathbb{U}_k(K)$  defined as

$$\mathbb{U}_k(K) := \{v \in H^1(K) : v|_{\partial K} \in W_k(\partial K), \Delta v|_K \in \mathbb{P}_{k-2}(K)\}. \quad (27)$$

It is not difficult to check that the dimension of  $\mathbb{U}_k(K)$  is given by

$$n_V + n_e(k-1) + n_f k(k-1)/2 + k(k^2-1)/6 \quad (28)$$

where the last term corresponds to the dimension of polynomials of degree  $\leq k-2$  in three dimensions.

**Remark 6.** *Note that in particular, for  $k=1$  the local dimension will be equal to the number of vertices, and for  $k=2$  to the number of vertices, plus the number of edges, plus the number of faces plus 1. As an example, for a cube-shaped hexahedron the local space will have dimension 8 for  $k=1$  and 27 for  $k=2$  (in both cases, the same as you would get for  $\mathbb{Q}_k(K)$ ) while for  $k=3$  the local space will have dimension 54 (instead of 64 as for  $\mathbb{Q}_3(K)$ ).  $\square$*

In  $\mathbb{U}_k(K)$  we can choose the following *degrees of freedom*:

$$\text{- the values of } v_h \text{ at the vertices of } K, \quad (29)$$

and for  $k \geq 2$  the moments:

$$\text{- } |e|^{-1}(m, v_h)_{0,e} \forall m \in \mathcal{M}_{k-2}(e) \text{ on each edge } e \text{ of } K, \quad (30)$$

$$\text{- } |f|^{-1}(m, v_h)_{0,f} \forall m \in \mathcal{M}_{k-2}(f) \text{ on each face } f \text{ of } K, \quad (31)$$

$$\text{- } |K|^{-1}(m, v_h)_{0,K} \forall m \in \mathcal{M}_{k-2}(K). \quad (32)$$

It is not difficult to check that the dimension of  $\mathbb{U}_k(K)$ , computed in (28), equals the total number of degrees of freedom (29)-(32), and that the degrees of freedom (29)-(32) are unisolvent.

**Remark 7.** *We note that on each face  $f \in \partial K$  the degrees of freedom (29)-(30) uniquely determine an "edgewise" polynomial of degree  $\leq k$  on the boundary of  $f$ . Following the two-dimensional theory of the previous section we know that adding (31) is then equivalent to prescribe  $v_h$  on  $f$ , and that, moreover, out of the degrees of freedom (29)-(31) we can construct, always on each face  $f$ , the projector  $\Pi_k^{\nabla,f}$  (that is, the operator  $\Pi_k^{\nabla}$ , as defined in (12)-(14), this time on the face  $f$ ) and the  $L^2(f)$  projection operator  $\Pi_k^{0,f}$  onto the set  $\mathbb{P}_k(f)$  of polynomials of degree  $\leq k$  on the face  $f$ . In turn, these can be assembled to construct projection operators  $\Pi_k^{\nabla,\partial K}$  and  $\Pi_k^{0,\partial K}$  whose restriction to each face  $f$  coincides with  $\Pi_k^{\nabla,f}$  and  $\Pi_k^{0,f}$ , respectively.  $\square$*

At this point we can introduce the three-dimensional operator  $\Pi_k^\nabla \equiv \Pi_k^{\nabla, K}$ , that mimicking the two-dimensional case we define through

$$(\nabla \Pi_k^\nabla v_h, \nabla q)_{0,K} = (\nabla v_h, \nabla q)_{0,K} \quad \forall v_h \in \mathbb{U}_k(K), \quad \forall q \in \mathbb{P}_k(K) \quad (33)$$

with

$$\int_{\partial K} \Pi_k^\nabla v_h \, dS = \int_{\partial K} \Pi_k^{\nabla, \partial K} v_h \, dS \quad \text{for } k = 1, \quad (34)$$

or

$$\int_K \Pi_k^\nabla v_h \, d\mathbf{x} = \int_K v_h \, d\mathbf{x} \quad \text{for } k \geq 2. \quad (35)$$

Let us see the possible problems in the computation on  $\Pi_k^\nabla$ . We consider again the Green formula, for  $p_k \in \mathbb{P}_k(K)$  and for  $v_h \in \mathbb{U}_h$

$$\int_K \nabla p_k \cdot \nabla v_h \, d\mathbf{x} = - \int_K \Delta p_k v_h \, d\mathbf{x} + \int_{\partial K} \frac{\partial p_k}{\partial n} v_h \, ds. \quad (36)$$

In the two-dimensional case we observed that the first term in the right-hand side of (10) was computable out of the degrees of freedom (8). Here we have an identical situation and the degrees of freedom (32) allow us to compute the first term in the right-hand side of (36). On the other hand the second term in the right-hand side of (10) was computable because  $v_h$  could be computed exactly on the whole boundary  $\partial K$  out of the degrees of freedom (6) and (7), since on each edge  $v_h$  was a polynomial. Here however  $v_h$ , on each face, is not, in general, a polynomial, and the degrees of freedom (31) are not enough, since the normal derivative of  $p_k$  is a polynomial of degree  $k-1$  and from (31) we only know the moments of  $v_h$  up to the order  $k-2$ . Our life-saver is that on each face  $f$  our  $v_h$  is taken in  $W_k(f)$ , and therefore, from the two-dimensional theory (as already observed in Remark 7), we can compute  $\Pi_k^{0, \partial K}$  that is, for each face  $f$ , *all the moments of  $v_h|_f$  up to the order  $k$*  (and therefore, in particular, the moments of order up to  $k-1$  that are necessary to compute the last term of (36)).

It is also easy to check that whenever  $v_h \in \mathbb{P}_k(K)$  then  $\Pi_k^\nabla v_h \equiv v_h$ , showing that  $\Pi_k^\nabla$  is indeed a projection onto  $\mathbb{P}_k(K)$ .

Once we have defined (and constructed) the operator  $\Pi_k^\nabla$  we can finally consider the finite dimensional space  $\mathbb{W}_k(K)$  defined as

$$\begin{aligned} \mathbb{W}_k(K) &:= \{v \in H^1(K) : v|_{\partial K} \in W_k(\partial K), \text{ such that } \Delta v|_K \in \mathbb{P}_k(K) \\ &\text{and } (v - \Pi_k^\nabla v, m)_{0,K} = 0 \text{ for all } m \in \mathcal{M}_{k-1}^*(K) \cup \mathcal{M}_k^*(K)\}. \end{aligned} \quad (37)$$

As for the two-dimensional case, in the space  $\mathbb{W}_k(K)$  we can construct the  $L^2(K)$  projection operator  $\Pi_k^0$  using the degrees of freedom (32) for the moments up to the order  $k-2$  and using (37) for the moments of order  $k-1$  and  $k$ .

**Remark 8.** *As we had in the two-dimensional case, the degrees of freedom in  $\mathbb{U}_k(K)$  and in  $\mathbb{W}_k(K)$  are the same. Hence the discussion made in Remark 6 applies to the present case as well.  $\square$*

**Remark 9.** We point out that the natural extension of the original VEM to the three-dimensional case would correspond to use on each face, instead of (26),

$$V_k(\partial K) := \{v \in C^0(\partial K) : v|_f \in V_k(f) \ \forall \text{ face } f \subset \partial K\}, \quad (38)$$

(where  $V_k(f)$  is defined as in (5)), and then set

$$\mathbb{V}_k(K) := \{v \in H^1(K) : v|_{\partial K} \in V_k(\partial K), \ \Delta v|_K \in \mathbb{P}_{k-2}(K)\}. \quad (39)$$

However, as we already saw discussing (36), we could not, with this choice, compute the second term in the right-hand side of (36), as the normal derivative of  $p_k$  has degree  $k-1$  while in  $V_k(f)$  we have, as degrees of freedom, only the moments up to the degree  $k-2$ . The only possibility, sticking to the original version of VEM, would then be to add  $k$  additional degrees of freedom on each face, as it was done for the higher order Mimetic Finite Differences in [6].  $\square$

As we already saw in the two-dimensional case (and we shall see again in a while even for the three-dimensional case) in a certain number of cases the knowledge of the projector  $\Pi_k^\nabla$  would be sufficient to construct the numerical scheme. In these cases we could consider the possibility of using, locally, only the space  $\mathbb{U}_k(K)$  instead of  $\mathbb{W}_k(K)$ . However, we point out that from the practical point of view (meaning the choice of the degrees of freedom and the actual computations) there is no difference in the two choices, in particular if one doesn't need to use the operator  $\Pi_k^0$ . Finally, following Remark 9, we point out once more that the choice (39) with the degrees of freedom (29)-(32) does not allow the construction of  $\Pi_k^\nabla$  (see (33)-(35)). Indeed, as we said already, the original MFD version [6] needed to use moments up to the order  $k-1$  in (31), with a considerable increase in the total number of degrees of freedom (of the order of  $k+1$  times the number of faces in the decomposition).

## 5. The three-dimensional Poisson Problem

### 5.1. The continuous problem

Let us see how these new virtual elements can be used to deal with three-dimensional problems. We consider the model problem

$$-\Delta u = g \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma = \partial\Omega, \quad (40)$$

where  $\Omega \subset \mathbb{R}^3$  is a polyhedral domain and  $g \in L^2(\Omega)$ . The variational formulation reads

$$\begin{cases} \text{find } u \in V := H_0^1(\Omega) \text{ such that} \\ a(u, v) = (g, v)_0 \quad \forall v \in V, \end{cases} \quad (41)$$

with  $a(u, v) = (\nabla u, \nabla v)_0$ . It is well known that problem (41) has a unique solution, since

$$a(u, v) \leq M |u|_1 |v|_1, \quad a(v, v) \geq \alpha |v|_1^2 \quad \forall u, v \in V, \quad (42)$$

with  $\alpha = M = 1$  in our simplified case.

### 5.2. The decompositions

We discuss now the discretized version of the above problem.

Let  $\{\mathcal{T}_h\}_h$  be a sequence of decompositions of  $\Omega$  into polyhedral elements  $K$ . On the sequence of decompositions we make the following assumption.

**A0** - We assume that there exists a positive real number  $\gamma$  such that

- for every element  $K$ , for every face  $f$  of  $K$ , and for every edge  $e$  of  $f$

$$h_e \geq \gamma h_f \geq \gamma^2 h_K;$$

- every element  $K$  is starshaped with respect to all the points of a sphere of radius  $\geq \gamma h_K$ ;
- every face  $f$  is starshaped with respect to all the points of a disk having radius  $\geq \gamma h_f$ .

In all the sequel, for every decomposition  $\mathcal{T}_h$  we set

$$|h| := \max_{K \in \mathcal{T}_h} h_K \quad (43)$$

□

**Remark 10.** Assumption **A0** is, at the same time, not very demanding (allowing the use of very general decompositions), and more restrictive than necessary. Actually, we could get away with even more general assumptions, but then it would be long and boring to make precise (among many possible crazy decompositions that nobody will ever use) the ones that are allowed and the ones that are not. □

**Remark 11.** It can be shown that the above conditions imply the existence of an integer number  $\mathcal{N}$  such that every polyhedron has less than  $\mathcal{N}$  faces and every face has less than  $\mathcal{N}$  edges. □

The bilinear form  $a(\cdot, \cdot)$  and the norm  $|\cdot|_1$  can obviously be split as

$$a(u, v) = \sum_{K \in \mathcal{T}_h} a^K(u, v) \quad \forall u, v \in V, \quad |v|_1 = \left( \sum_{K \in \mathcal{T}_h} |v|_{1,K}^2 \right)^{1/2} \quad \forall v \in V. \quad (44)$$

Since in what follows we shall also deal with functions belonging to the space  $H^1(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} H^1(K)$ , we need to define a broken  $H^1$ -seminorm:

$$|v|_{h,1} := \left( \sum_{K \in \mathcal{T}_h} |v|_{1,K}^2 \right)^{1/2}. \quad (45)$$

Note that, for discontinuous functions, this is *really* a seminorm and not a norm: for instance,  $|c_h|_{h,1} \equiv 0$  for every piecewise constant function  $c_h$ .

At the abstract level, for a given order of accuracy  $k \geq 1$ , we consider, as in [4], discretizations that satisfy the following assumptions.

**A1** - We assume to have, for each  $h$ ,

- for each  $K \in \mathcal{T}_h$  a space  $\mathbb{W}_k(K) \subset H^1(K)$ ;
- a space  $\mathbb{W}_h \subseteq V \cap \prod_{K \in \mathcal{T}_h} \mathbb{W}_k(K)$ ;
- a bilinear form  $a_h$  from  $\mathbb{W}_h \times \mathbb{W}_h$  to  $\mathbb{R}$  which can be split as

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} a_h^K(u_h, v_h) \quad \forall u_h, v_h \in \mathbb{W}_h, \quad (46)$$

where each  $a_h^K(\cdot, \cdot)$  is a symmetric, positive semidefinite bilinear form on  $W_k(K) \times W_k(K)$ ;

- an element  $g_h \in \mathbb{W}'_h$  (dual space of  $\mathbb{W}_h$ ). □

Together with **A1** we further assume the following crucial properties.

**A2** - For all  $h$ , and for all  $K$  in  $\mathcal{T}_h$ , we have  $\mathbb{P}_k(K) \subseteq \mathbb{W}_k(K)$  and

- **k-Consistency**: for all  $p \in \mathbb{P}_k(K)$  and for all  $v_h \in \mathbb{W}_k(K)$ ,

$$a_h^K(p, v_h) = a^K(p, v_h); \quad (47)$$

- **Stability**: there exist two positive constants  $\alpha_*$  and  $\alpha^*$ , independent of  $h$  and of  $K$ , such that

$$\forall v_h \in \mathbb{W}_k(K) \quad \alpha_* a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* a^K(v_h, v_h). \quad (48)$$

□

We notice that the symmetry of  $a_h$ , property (48), and the definition of  $a^K$  easily imply the uniform continuity of  $a_h$  with

$$\begin{aligned} a_h^K(u, v) &\leq \left( a_h^K(u, u) \right)^{1/2} \left( a_h^K(v, v) \right)^{1/2} \leq \alpha^* \left( a^K(u, u) \right)^{1/2} \left( a^K(v, v) \right)^{1/2} \\ &= \alpha^* |u|_{1,K} |v|_{1,K} \quad \text{for all } u, v \in \mathbb{W}_k(K). \end{aligned} \quad (49)$$

### 5.3. An abstract convergence theorem

The following convergence theorem has been proved in [4] for the two-dimensional case, but the arguments given therein do not depend on the space dimension.

**Theorem 1.** *Under Assumptions **A1-A2**, the discrete problem:*

$$\text{Find } u_h \in \mathbb{W}_h \text{ such that} \quad a_h(u_h, v_h) = \langle g_h, v_h \rangle \quad \forall v_h \in \mathbb{W}_h, \quad (50)$$

has a unique solution  $u_h$ . Moreover, for every approximation  $u_I \in \mathbb{W}_h$  of  $u$  and for every approximation  $u_\pi$  of  $u$  that is piecewise in  $\mathbb{P}_k$ , we have

$$|u - u_h|_1 \leq C \left( |u - u_I|_1 + |u - u_\pi|_{h,1} + \|g - g_h\|_{\mathbb{W}'_h} \right),$$

where  $C$  is a constant depending only on  $\alpha_*$  and  $\alpha^*$ , and  $\|g - g_h\|_{\mathbb{W}'_h}$  is defined as

$$\|g - g_h\|_{\mathbb{W}'_h} := \sup_{v_h \in \mathbb{W}_h} \frac{\langle g - g_h, v_h \rangle}{\|v_h\|_V}. \quad (51)$$

#### 5.4. Projection error

According to the classical Scott-Dupont theory (see e.g. [10]) we have then the following result.

**Proposition 3.** *Assume that Assumption **A0** is satisfied. Then there exists a constant  $C$ , depending only on  $k$  and  $\gamma$ , such that for every  $s$  with  $1 \leq s \leq k+1$ , for every  $K$ , and for every  $w \in H^s(K)$  there exists a  $w_\pi \in \mathbb{P}_k(K)$  such that*

$$\|w - w_\pi\|_{0,K} + h_K |w - w_\pi|_{1,K} \leq C h_K^s |w|_{s,K}. \quad (52)$$

**Remark 12.** *Always following [10] we note that we could take the weaker assumption that (roughly speaking) every  $K$  is the union of a finite (and uniformly bounded) number of star-shaped domains, each satisfying **A0**.  $\square$*

#### 5.5. Construction of $\mathbb{W}_h$

We can now use what we learned on individual polyhedra in order to design a Virtual Element space on the whole  $\Omega$ . In particular: for every decomposition  $\mathcal{T}_h$  of  $\Omega$  into polyhedra  $K$ , for every integer  $k \geq 1$ , and for every  $K$  in  $\mathcal{T}_h$  we define  $\mathbb{W}_k(K)$  as in (37). Then we set, as natural:

$$\mathbb{W}_h := \{v \in H_0^1(\Omega) : v|_K \in \mathbb{W}_k(K) \text{ for each element } K \text{ in } \mathcal{T}_h\}. \quad (53)$$

Arguing as we did in the case of a single polyhedron (but remembering that on  $\partial\Omega$  we set homogeneous Dirichlet boundary conditions), we can easily see that the dimension of the whole space  $\mathbb{W}_h$  is given by

$$N^{tot} \equiv \dim \mathbb{W}_h = N_V + N_E(k-1) + N_F k(k-1)/2 + N_P k(k^2-1)/6, \quad (54)$$

where  $N_V$ ,  $N_E$ ,  $N_F$ , and  $N_P$  are, respectively, the total number of *internal Vertices*, *internal Edges*, *internal Faces*, and *elements (polyhedra)* in  $\mathcal{T}_h$ .

In agreement with the local choice of the degrees of freedom (29)-(32), in  $\mathbb{W}_h$  we choose the following *degrees of freedom*:

$$\text{- the values of } v_h \text{ at the } \textit{internal vertices}, \quad (55)$$

and for  $k \geq 2$  the moments

$$\text{- } |e|^{-1} (m, v_h)_{0,e} \quad \forall m \in \mathcal{M}_{k-2}(e) \text{ on each } \textit{internal edge } e, \quad (56)$$

$$- |f|^{-1}(m, v_h)_{0,f} \quad \forall m \in \mathcal{M}_{k-2}(f) \text{ on each internal face } f, \quad (57)$$

$$- |K|^{-1}(m, v_h)_{0,K} \quad \forall m \in \mathcal{M}_{k-2}(K) \text{ on each element } K. \quad (58)$$

We explicitly recall that the request  $\mathbb{W}_h \subset V$  implies  $v_h = 0$  on the nodes, on the edges and on the faces belonging to the boundary  $\partial\Omega$ .

It is not difficult to check that, here too, the dimension  $N^{tot}$  of  $\mathbb{W}_h$ , computed in (54), equals the total number of degrees of freedom (55)-(58). The *local unisolvence* will now easily imply that the global degrees of freedom are unisolvent for the global space  $\mathbb{W}_h$ . Exactly as it happens for the usual Finite Element spaces.

### 5.6. Interpolation error

Numbering the  $N^{tot}$  degrees of freedom (55)-(58) from 1 to  $N^{tot}$ , we can denote by  $\chi_i$ ,  $i = 1, \dots, N^{tot}$  the operator that to each smooth enough function  $\varphi$  associates its  $i$ -th degree of freedom  $\chi_i(\varphi)$ . It follows easily from the above construction that for every smooth enough  $w$  vanishing on  $\partial\Omega$  there exists a unique element  $w_I$  of  $\mathbb{W}_h$  such that

$$\chi_i(w - w_I) = 0, \quad i = 1, \dots, N^{tot}. \quad (59)$$

More generally, always following for instance [10], it is not difficult to see that the following result holds.

**Proposition 4.** *Assume that Assumption **A0** is satisfied, and that the space  $\mathbb{W}_h$  has been constructed following the above procedure. Then there exists a constant  $C$ , depending only on  $k$  and  $\gamma$ , such that for every  $s$  with  $2 \leq s \leq k + 1$ , for every  $h$ , for all  $K \in \mathcal{T}_h$  and for every  $w \in H^s(K)$  the interpolant  $w_I \in \mathbb{W}_h$  defined in (59) satisfies*

$$\|w - w_I\|_{0,K} + h_K |w - w_I|_{1,K} \leq C h_K^s |w|_{s,K}. \quad (60)$$

### 5.7. Construction of $a_h$

At this point we can follow in a rather slavish way the procedure applied for the two-dimensional case in [4]. We summarize it briefly. We have to construct a computable  $a_h$  that satisfies (47) and (48). For this, for every element  $K$  we use the operator  $\Pi_k^\nabla : \mathbb{W}_k(K) \rightarrow \mathbb{P}_k(K) \subseteq \mathbb{W}_k(K)$  defined in (33)-(35), and we choose

$$a_h^K(u, v) := a^K(\Pi_k^\nabla u, \Pi_k^\nabla v) + S^K(u - \Pi_k^\nabla u, v - \Pi_k^\nabla v) \quad \forall u, v \in \mathbb{W}_k(K) \quad (61)$$

where  $S^K(u, v)$  is, in the canonical basis induced by the degrees of freedom (29)-(32), the identity matrix multiplied by  $h_K$ . As in the two-dimensional case [4], one can easily verify that with the choice (61) Assumption **A2** is satisfied.

**Remark 13.** *An approach similar to that used in (61) (although only for the case of quadrilaterals) can be found in [23].*  $\square$

**Remark 14.** *Most of the use of Assumption **A0** is actually hidden here. What we are using is that the square of the  $H^1(K)$ -norm of each basis function scales (in three dimensions) like  $h_K$ , which would not be the case if, for instance, two vertices were too close.*  $\square$



### 5.8. Treatment of the right-hand side

For  $k \geq 2$  the treatment of the right-hand side can be easily dealt with in the classical VEM framework [4]. We can simply take  $g_h := \Pi_{k-2}^0 g$  since the integral

$$\int_K (\Pi_{k-2}^0 g) v_h dK \quad (62)$$

is always computable directly from the degrees of freedom (32). With this choice we have

$$\begin{aligned} (g - g_h, v_h)_{0,K} &= (g - \Pi_{k-2}^0 g, v_h)_{0,K} \\ &= (g - \Pi_{k-2}^0 g, v_h - \Pi_0^0 v_h)_{0,K} \leq C h_K^{k-1} |g|_{k-1,K} h_K |v_h|_{1,K} \end{aligned} \quad (63)$$

that easily implies

$$\|g - g_h\|_{\mathbb{W}'_h} \leq C |h|^k \|g\|_{k-1,\Omega}. \quad (64)$$

For  $k = 1$  instead we can take  $g_h = \Pi_0^0 g$ , since

$$\int_K (\Pi_0^0 g) v_h dK \equiv \int_K (\Pi_0^0 g) \Pi_1^\nabla v_h dK$$

for  $v_h$  in  $\mathbb{W}_1(K)$  (as defined in (37)). Proceeding as before, we easily obtain

$$\|g - g_h\|_{\mathbb{W}'_h} \leq C |h| \|g\|_{0,\Omega}, \quad (65)$$

or, alternatively,

$$\|g - g_h\|_{\mathbb{W}'_h} \leq C |h|^2 \|g\|_{1,\Omega}. \quad (66)$$

**Remark 15.** We note that for  $k \geq 2$  and  $v_h \in \mathbb{W}_k(K)$  we could have taken  $g_h = \Pi_{k-1}^0 g$ , and gain an extra power of  $h$  as in (66):

$$\|g - g_h\|_{\mathbb{W}'_h} \leq C |h|^{k+1} \|g\|_{k,\Omega}. \quad (67)$$

□

**Theorem 2.** Let  $u$  be the solution of problem (41), and let  $u_h \in \mathbb{W}_h$  be the solution of the discretized problem (50), with  $\mathbb{W}_h$  defined in (53),  $a_h$  defined in (61), and  $g_h := \Pi_{k-2}^0 g$  for  $k \geq 2$  and  $g_h := \Pi_0^0 g$  for  $k = 1$ . Assume further that the right-hand side  $g$  belongs to  $H^{k-1}(\Omega)$ , and that the exact solution  $u$  of (41) belongs to  $H^{k+1}(\Omega)$ . Then

$$\|u - u_h\|_{1,\Omega} \leq C |h|^k |u|_{k+1,\Omega}, \quad (68)$$

with  $C$  a positive constant independent of  $h$ .

*Proof.* The result is an immediate consequence of the abstract Theorem 1 and the estimates (52), (60), and (64) (or (65) for  $k = 1$ ). □

## 6. Additional applications

We shall now briefly highlight some other applications of these elements. It is clear that the range of possible applications is *much wider*.

### 6.1. $L^2$ error estimates

We consider now the problem of optimal error estimates in the  $L^2(\Omega)$  norm.

**Theorem 3.** *Let  $u$  be the solution of problem (41), and let  $u_h \in \mathbb{W}_h$  be the solution of the discretized problem (50), with  $\mathbb{W}_h$  defined in (53),  $a_h$  defined in (61), and  $g_h := \Pi_{k-1}^0 g$  for  $k \geq 1$ . Assume further that  $\Omega$  is convex, that the right-hand side  $g$  belongs to  $H^k(\Omega)$ , and that the exact solution  $u$  of (41) belongs to  $H^{k+1}(\Omega)$ .*

*Then the following estimate holds:*

$$\|u - u_h\|_{0,\Omega} + |h| \|u - u_h\|_{1,\Omega} \leq C |h|^{k+1} |u|_{k+1,\Omega}, \quad (69)$$

with  $C$  a constant independent of  $h$ .

*Proof.* The  $H^1$  estimate follows from Theorem 1. To prove the  $L^2$  estimate we employ the usual duality argument. Let therefore  $\psi$  be the solution of

$$-\Delta\psi = u - u_h \quad \text{in } \Omega \quad \text{with } \psi \in H_0^1(\Omega) \quad (70)$$

that, due to the convexity assumption, satisfies

$$\|\psi\|_{2,\Omega} \leq C \|u - u_h\|_{0,\Omega} \quad (71)$$

for a constant  $C$  that depends only on  $\Omega$ . Let  $\psi_I$  be an interpolant of  $\psi$  in  $\mathbb{W}_h$ , for which it holds

$$\|\psi - \psi_I\|_{1,\Omega} \leq C|h| \|\psi\|_{2,\Omega} \leq C|h| \|u - u_h\|_{0,\Omega} \quad (72)$$

(having used estimate (60) and then (71)).

Then, using (70), adding and subtracting  $\psi_I$ , and using (41) and (50) we have

$$\begin{aligned} \|u - u_h\|_0^2 &= (u - u_h, -\Delta\psi) = a(u - u_h, \psi) \\ &= a(u - u_h, \psi - \psi_I) + a(u - u_h, \psi_I) \\ &= a(u - u_h, \psi - \psi_I) + \langle g, \psi_I \rangle - a(u_h, \psi_I) \\ &= a(u - u_h, \psi - \psi_I) + \langle g - g_h, \psi_I \rangle + a_h(u_h, \psi_I) - a(u_h, \psi_I) \\ &=: I + II + III. \end{aligned} \quad (73)$$

Obviously we have first, from (42) and (72),

$$I = a(u - u_h, \psi - \psi_I) \leq C \|u - u_h\|_{1,\Omega} |h| \|u - u_h\|_{0,\Omega}. \quad (74)$$

Then we observe that for every element  $K$  we have

$$(g - g_h, q)_{0,K} \equiv (g - \Pi_{k-1}^0 g, q)_{0,K} = 0 \quad \forall q \in \mathbb{P}_{k-1}(K), \quad (75)$$

so that using (52) (for  $\Pi_{k-1}^0$  and  $\Pi_0^0$ ) and again (60)

$$\begin{aligned} II &= \langle g - g_h, \psi_I \rangle = \langle g - \Pi_{k-1}^0 g, \psi_I - \Pi_0^0 \psi_I \rangle \leq C |h|^k |g|_{k,\Omega} |h| |\psi_I|_{1,\Omega} \\ &\leq C |h|^k |g|_{k,\Omega} |h| \|u - u_h\|_{0,\Omega}. \end{aligned} \quad (76)$$

Finally, using (twice) (47)

$$\begin{aligned} III &= a_h(u_h, \psi_I) - a(u_h, \psi_I) = \sum_K \left( a_h^K(u_h, \psi_I) - (a^K(u_h, \psi_I)) \right) \\ &= \sum_K \left( a_h^K(u_h - \Pi_k^0 u, \psi_I) - a^K(u_h - \Pi_k^0 u, \psi_I) \right) \\ &= \sum_K \left( a_h^K(u_h - \Pi_k^0 u, \psi_I - \Pi_1^0 \psi) - a^K(u_h - \Pi_k^0 u, \psi_I - \Pi_1^0 \psi) \right). \end{aligned} \quad (77)$$

By adding and subtracting  $\psi$  and using (52) and (72) we easily have

$$\begin{aligned} \|\psi_I - \Pi_1^0 \psi\|_{1,\Omega} &\leq \|\psi_I - \psi\|_{1,\Omega} + \|\psi - \Pi_1^0 \psi\|_{1,\Omega} \leq C |h| |\psi|_{2,\Omega} \\ &\leq C |h| \|u - u_h\|_{0,\Omega}, \end{aligned} \quad (78)$$

that inserted in (77) gives

$$III \leq C \left( \|u_h - u\|_{1,\Omega} + \|u - \Pi_k^0 u\|_{1,\Omega} \right) |h| \|u - u_h\|_{0,\Omega}. \quad (79)$$

Using now (74), (76), and (79) in (73) we have then

$$\|u - u_h\|_0^2 \leq C \left( \|u - u_h\|_{1,\Omega} + \|u - \Pi_k^0 u\|_{1,\Omega} + |h|^k |g|_{k,\Omega} \right) |h| \|u - u_h\|_{0,\Omega} \quad (80)$$

and the result follows from (68) and (52).  $\square$

**Remark 16.** We point out that for  $k \geq 3$  we could have taken  $g_h = \Pi_{k-2}^0 g$  as in (62), and still obtain optimal estimates. Indeed,  $k - 2 \geq 1$  for  $k \geq 3$ , so that the estimate (76) of  $II$  can be made as

$$\begin{aligned} \langle g - g_h, \psi_I \rangle &= \langle g - \Pi_{k-2}^0 g, \psi_I \rangle = \langle g - \Pi_{k-2}^0 g, \psi_I - \Pi_1^0 \psi \rangle \\ &\leq C |h|^{k-1} \|g\|_{k-1,\Omega} |h|^2 \|\psi\|_{2,\Omega} \\ &\leq C |h|^k \|g\|_{k-1,\Omega} |h| \|u - u_h\|_{0,\Omega}. \end{aligned} \quad (81)$$

and then we can proceed exactly as before.  $\square$

## 6.2. Reaction-diffusion problems

As a further example of application we consider the case of a reaction-diffusion problem of the type:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \quad -\Delta u + \alpha u = g \quad \text{in } \Omega \quad (82)$$

where  $g \in L^2(\Omega)$  as before, and  $\alpha$  is a positive constant. It could be convenient to split the associated bilinear form  $a(u, v)$  as

$$a(u, v) = a_{\nabla}(u, v) + \alpha a_0(u, v) \equiv \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x} + \alpha \int_{\Omega} u v \, d\mathbf{x}. \quad (83)$$

Using the discretized space (53) (or (17) in two dimensions) we can define on each element  $K$ :

$$\begin{aligned} a_h^K(u, v) &= a_{\nabla}^K(\Pi_k^{\nabla} u, \Pi_k^{\nabla} v) + S_{\nabla}^K(u - \Pi_k^{\nabla} u, v - \Pi_k^{\nabla} v) \\ &\quad + \alpha a_0^K(\Pi_k^0 u, \Pi_k^0 v) + \alpha S_0^K(u - \Pi_k^0 u, v - \Pi_k^0 v) \quad \forall u, v \in \mathbb{W}_k(K). \end{aligned} \quad (84)$$

In the canonical basis induced by the degrees of freedom,  $S_{\nabla}^K$  can be taken as the identity matrix multiplied by  $h_K^{d-2}$ , and  $S_0^K$  can be taken as the identity matrix multiplied by  $h_K^d$  where, as above,  $d$  (typically = 2 or 3) is the dimension of the object  $K$ . It is not difficult to check that with this choice Assumption **A2** is again satisfied, and that the optimal error bound of  $O(h^k)$  can still be easily proved.

**Remark 17.** *We point out that the term  $S_0^K$  in (84) is not needed, unless the problem is reaction-dominated. We also point out that a similar approach can be used to deal with time-dependent problems.*  $\square$

### 6.3. Numerical Experiments

We present two numerical experiments to exploit the behavior of the method when a reaction term is present. In the first test we show that the presence of the reaction stabilization term  $S_0^K(\cdot, \cdot)$  is inessential when the problem is diffusion-dominated; in the second one we show instead that this term is crucial when the problem is reaction-dominated.

#### Test 1: diffusion-dominated case.

We consider the problem

$$\begin{cases} -\Delta u + u = g & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \quad (85)$$

where  $\Omega$  is the unit square and the load term  $g$  and the Dirichlet boundary data  $u_0$  are chosen in such a way that the exact solution is

$$u_e(x, y) = \sin(2x + 0.5) \cos(y + 0.3) + \log(1 + xy).$$

We approximate problem (85) with the sequence of the four polygonal meshes shown in Fig. 1. The order of approximation is  $k = 2$ . In Fig. 2 we show the convergence curves of the error measured in a discrete  $L^2$  norm against the mean value of the mesh size  $h$  in the two cases: with and without the reaction stabilization term  $S_0^K(\cdot, \cdot)$ . The two error curves are virtually indistinguishable; note the  $O(h^3)$  optimal convergence rate.

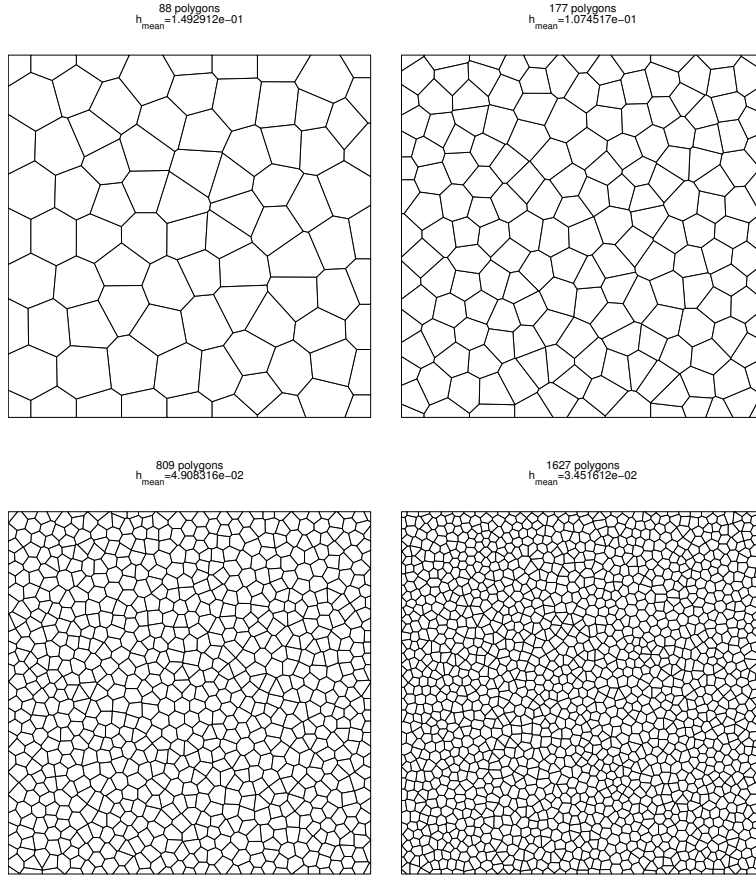


Figure 1: Sequence of polygonal meshes for the diffusion-dominated case

**Test 2: reaction-dominated case.**

We consider the problem

$$\begin{cases} -\varepsilon \Delta u + u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (86)$$

where  $\Omega$  is the unit square. When  $\varepsilon$  becomes small, the exact solution tends to be 1 inside the domain and develops a diffusive boundary-layer close to the whole boundary. We study the behavior of the VEM approximation of problem (86) for  $k = 2$  on a fixed mesh (the second of the sequence of Fig. 1) for  $\varepsilon$  decreasing from  $10^{-1}$  to  $10^{-6}$ . We show the results in Figs 3 and 4; in both figures, the first row corresponds to the approximation *without* the reaction stabilization term  $S_0^K(\cdot, \cdot)$ , while the results in the second row have been obtained *with* the term  $S_0^K(\cdot, \cdot)$ .

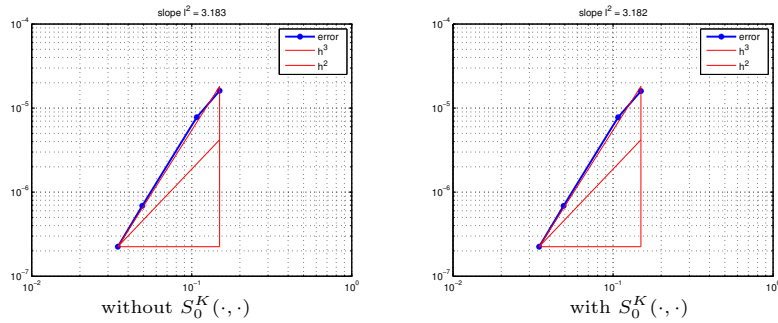


Figure 2: Convergence curves for the diffusion-dominated case

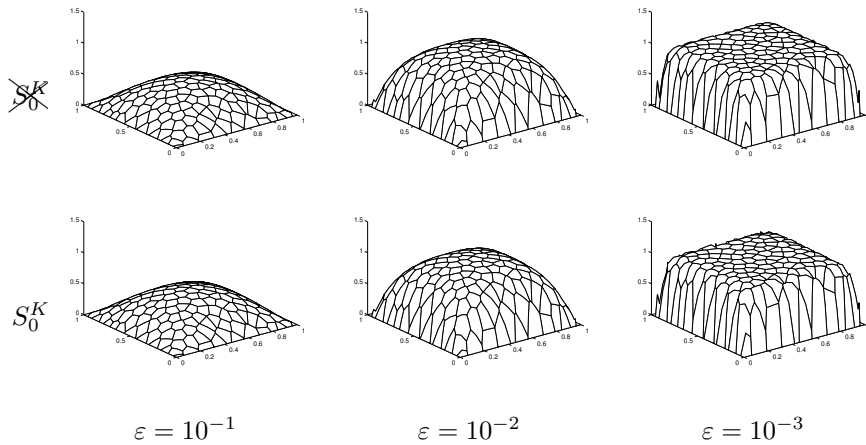


Figure 3: Reaction-dominated case,  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$

In Fig. 3 we see that when  $\varepsilon$  is moderately small the results in the two cases are very similar, in conformity with what we have obtained in the previous experiment. On the contrary, when  $\varepsilon$  becomes smaller we see clearly from Fig. 4 that the reaction stabilization term is needed in order to produce a reasonable solution.

**Remark 18.** *No adjustments of the methods have been made to actually reduce the over- and undershoots due to the diffusion layer, which are also typical of the classical finite element approximations. Indeed, in these circumstances higher order finite elements exhibit the same oscillations near the boundary, and the lowest order ones ( $k = 1$ ) require some mass lumping (at least near the boundary) to be fixed. Hence we might say that with the reaction stabilization term the Virtual Elements behave just like Finite Elements, although on much more general geometries.*

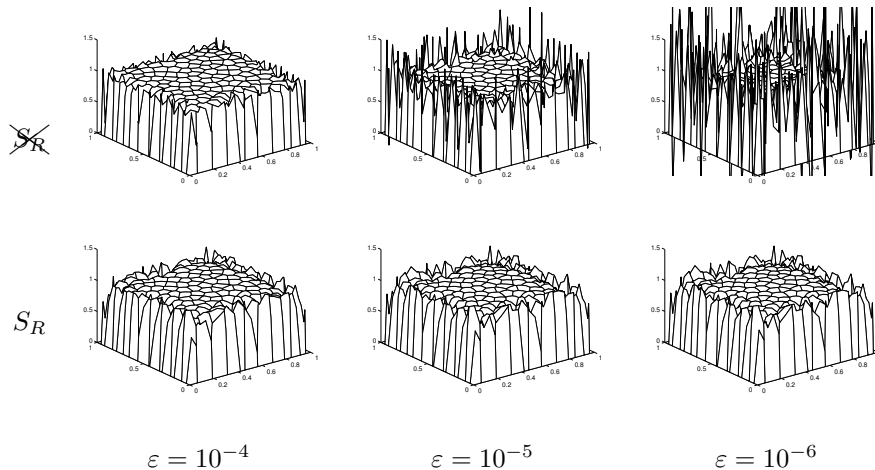


Figure 4: Reaction-dominated case,  $\varepsilon = 10^{-4}, 10^{-5}, 10^{-6}$

## 7. Conclusions

We presented a different point of view on the Virtual Element Methods that allows the exact computation of the local  $L^2$  projection of trial and test functions on polynomials of degree  $k$  (where  $k$  is the maximum integer such that all polynomials of degree  $\leq k$  are contained in the local space). From the computational point of view, we could briefly summarize the result by saying: *compute the  $H_0^1(K)$ -projection  $\Pi_k^\nabla v$  as you do for the original VEM, and then use it (for the moments of degree strictly higher than  $k - 2$ ) as if it was the  $L^2$  projection.* The new approach allows to get away with this “mistake”, showing that what looks like a mistake is instead perfectly correct if we assume that we are working in a slightly different space (that however still preserve all the optimal approximation properties). The trick (of using  $\Pi_k^\nabla v$  as if it was the  $L^2$  projection), could be used in a number of different circumstances.

## References

- [1] J.E. Aarnes, S. Krogstad, and K.-A. Lie. Multiscale mixed/mimetic methods on corner-point grids. *Comput. Geosci.*, 12:297–315, 2008. Special Issue on Multiscale Methods.
- [2] L. Beirão da Veiga. A residual based error estimator for the mimetic finite difference method. *Numer. Math.*, 108(3):387–406, 2008.
- [3] L. Beirão da Veiga. A mimetic finite difference method for linear elasticity. *M2AN: Math. Model. Numer. Anal.*, 44(2):231–250, 2010.
- [4] L. Beirão da Veiga, F. Brezzi, A. Cangiani, L.D. Marini, G. Manzini, and A. Russo. The basic principles of Virtual Elements Methods. *Math. Models Methods Appl. Sci.* 23(1):199–214, 2013.

- [5] L. Beirão da Veiga, F. Brezzi, L.D. Marini. Virtual Elements for linear elasticity problems. To appear in *SIAM J. Num. Anal.*
- [6] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. Arbitrary-Order Nodal Mimetic Discretizations of Elliptic Problems on Polygonal Meshes. *SIAM J. Numer. Anal.*, 49(5):1737–1760, 2011.
- [7] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. Convergence analysis of the high-order mimetic finite difference method. *Numer. Math.*, 113(3):325–356, 2009.
- [8] L. Beirão da Veiga and G. Manzini. A higher-order formulation of the mimetic finite difference method. *SIAM, J. Sci. Comput.*, 31(1):732–760, 2008.
- [9] P. Bochev and J. M. Hyman. Principle of mimetic discretizations of differential operators. In D. Arnold, P. Bochev, R. Lehoucq, R. Nicolaides, and M. Shashkov, editors, *Compatible discretizations. Proceedings of IMA hot topics workshop on compatible discretizations*, IMA Volume 142. Springer-Verlag, 2006.
- [10] S. C. Brenner and R. L. Scott. *The mathematical theory of finite element methods*. Texts in Applied Mathematics, 15. Springer-Verlag, New York, 2008.
- [11] F. Brezzi and A. Buffa. Innovative mimetic discretizations for electromagnetic problems. *J. Comput. Appl. Math.*, 234(6):1980–1987, 2010.
- [12] F. Brezzi, A. Buffa, and K. Lipnikov. Mimetic finite differences for elliptic problems. *M2AN: Math. Model. Numer. Anal.*, 43:277–295, 2009.
- [13] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, 1991.
- [14] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Num. Anal.*, 43:1872–1896, 2005.
- [15] F. Brezzi, K. Lipnikov, M. Shashkov, and V. Simoncini. A new discretization methodology for diffusion problems on generalized polyhedral meshes. *Comput. Meth. Appl. Mech. Engrg.*, 196:3682–3692, 2007.
- [16] F. Brezzi, K. Lipnikov, and V. Simoncini. A family of mimetic finite difference methods on polygonal and polyhedral meshes. *Math. Models Methods Appl. Sci.*, 15:1533–1553, 2005.
- [17] F. Brezzi and L.D. Marini. Virtual elements for plate bending problems. *Comput. Meth. Appl. Mech. Engrg.*, 253:155–462, 2013.



- [18] J. C. Campbell and M. J. Shashkov. A tensor artificial viscosity using a mimetic finite difference algorithm. *J. Comput. Phys.*, 172(2):739–765, 2001.
- [19] A. Cangiani and G. Manzini. Flux reconstruction and pressure post-processing in mimetic finite difference methods. *Comput. Methods Appl. Mech. Engrg.*, 197/9-12:933–945, 2008.
- [20] A. Cangiani, G. Manzini, and A. Russo. Convergence analysis of the mimetic finite difference method for elliptic problems. *SIAM J. Numer. Anal.*, 47(4):2612–2637, 2009.
- [21] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, 1978.
- [22] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods. *Math. Models Methods Appl. Sci. (M3AS)*, 20(2):265–295, 2010.
- [23] C. A. Felippa. Supernatural QUAD4: A template formulation. *Comput. Methods Appl. Mech. Engrg.*, 195(41-43):5316–5342, 2006.
- [24] T.-P. Fries and T. Belytschko. The extended/generalized finite element method: An overview of the method and its applications. *Int. J. Numer. Meth. Engrg.*, 84: 253-304, 2010
- [25] V. Gyrya and K. Lipnikov. High-order mimetic finite difference method for diffusion problems on polygonal meshes. *J. Comput. Phys.*, 227:8841–8854, 2008.
- [26] J. M. Hyman and M. Shashkov. The orthogonal decomposition theorems for mimetic finite difference methods. *SIAM J. Numer. Anal.*, 36(3):788–818, 1999.
- [27] J. Hyman and M. Shashkov. Mimetic discretisations of Maxwell’s equations and the equations of magnetic diffusion. *Progress in Electromagnetic Research*, 32:89–121, 2001.
- [28] Yu. Kuznetsov and S. Repin. New mixed finite element method on polygonal and polyhedral meshes. *Russ. J. Numer. Anal. Math. Modelling*, 18(3):261–278, 2003.
- [29] S. Rjasanow and S. Weißer. Higher order BEM-based FEM on polygonal meshes. Preprint Nr. 297, Fachrichtung 6.1 - Mathematik, Saarbrücken 2011.
- [30] M. Shashkov and S. Steinberg. Solving diffusion equations with rough coefficients in rough grids. *J. Comput. Phys.*, 129(2):383–405, 1996.
- [31] A. Tabarraei and N. Sukumar. Conforming polygonal finite elements. *Int. J. Numer. Meth. Engrg.*, 61(12):2045–2066, 2004.

- [32] A. Tabarraei and N. Sukumar. Extended finite element method on polygonal and quadtree meshes. *Comput. Methods Appl. Mech. Engrg.*, 197(5):425–438, 2007.
- [33] K.A. Trapp. Inner products in covolume and mimetic methods. *M2AN: Math. Model. Numer. Anal.*, 42:941–959, 2008.
- [34] E. Wachspress. *A rational Finite Element Basis*. Academic Press, New York, 1975.