MIXED FINITE ELEMENT METHODS WITH CONTINUOUS STRESSES

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We present a modified mixed formulation for second order elliptic equations and linear elasticity problems which automatically satisfies the "ellipticity on the kernel" condition, i.e., one of the two compatibility conditions necessary to ensure stability and optimal error bounds (the other being the Inf-Sup condition). This modification differs from similar ones introduced by other authors in that it is independent of the mesh size. Moreover, it allows the use of continuous stresses.

1. Introduction

In the mixed finite element approximations of second order elliptic problems and of linear elasticity problems one has to deal (see e.g. Ref. 4) with two sets of variables: stresses and displacements. (This terminology is taken from elasticity, but its meaning in the context of second order elliptic equations should still be rather clear).

It is well known that the discretization of these two sets of variables has to be made in a "compatible" way in order to avoid instabilities. For the two problems under consideration, these compatibility conditions are particularly difficult. Actually, one has to fulfill *two* conditions which are, in a sense, "fighting" each other. (The situation is somewhat easier for other mixed formulations such as, for instance, that for the Stokes problem where there is essentially only one condition to deal with. See always Ref. 4 for a general treatment). Let us consider, to make things clearer, the simplest second order elliptic problem: $\Delta u = f$ in Ω, with $u \in H^1_0(\Omega)$.

Introducing the "stress" $\underline{\sigma} = \underline{\nabla} u$, the mixed formulation is

$$\begin{cases} (\underline{\sigma}, \underline{\tau}) \; + \; (u, \operatorname{div} \underline{\tau}) \; = \; 0 & \forall \underline{\tau} \in H(\operatorname{div}; \Omega), \\ (\operatorname{div} \underline{\sigma}, v) \; = \; (f, v) & \forall v \in L^2(\Omega). \end{cases} \tag{1.1}$$

where $(.\,,.)$ represents, as usual, the inner product in the space $L^2(\Omega)$ (or in $(L^2(\Omega))^2$) and $H(\operatorname{div}; \Omega) = \{\underline{\tau} \in (\mathrm{L}^2(\Omega))^2 \;,\; \operatorname{div} \underline{\tau} \in \mathrm{L}^2(\Omega)\}$ with the usual graph norm

$$||\underline{\tau}||_H^2 \;=\; ||\underline{\tau}||_0^2 \;+\; ||\operatorname{div} \underline{\tau}||_0^2, \tag{1.2}$$

($||.||_0$ is the usual $L^2(\Omega)$ or $(L^2(\Omega))^2$ norm). Assume that we take finite element spaces $\Sigma_h \subset H(\operatorname{div}; \Omega)$ and $U_h \subset L^2(\Omega)$ in order to discretize our mixed formulation. We have

$$\begin{cases} \text{find } (\underline{\sigma}_h,\, u_h) \in \Sigma_h \times U_h \text{ such that} \\ (\underline{\sigma}_h, \underline{\tau}_h) \;+\; (u_h, \operatorname{div} \underline{\tau}_{\mathrm{h}}) \;=\; 0 & \forall \underline{\tau}_h \in \Sigma_h, \\ (\operatorname{div} \underline{\sigma}_h, v_h) \;=\; (f, v_h) & \forall v_h \in U_h. \end{cases} \tag{1.3}$$

We are now able to write explicitly the two compatibility conditions:

$$(C1) \qquad \begin{cases} \exists \alpha > 0 \text{ such that} \\ ||\underline{\tau}_h||_0^2 \;\geq\; \alpha ||\underline{\tau}_h||_H^2 \quad \forall \underline{\tau}_h \in K_h, \end{cases} \tag{1.4}$$

where

$$K_h \;=\; \{\underline{\tau}_h \in \Sigma_h \;:\; (\operatorname{div} \underline{\tau}_h, v_h) \;=\; 0 \;\; \forall v_h \in U_h\}, \tag{1.5}$$

$$(C2) \qquad \begin{cases} \exists \beta > 0 \text{ such that} \\ \displaystyle \operatorname*{Sup}_{\underline{\tau}_h \in \Sigma_h} \frac{(\operatorname{div} \underline{\tau}_h, v_h)}{||\underline{\tau}_h||_H} \geq \beta ||v_h||_0 \quad \forall v_h \in U_h. \end{cases} \tag{1.6}$$

It is also clear that α and β in $(C1)$ and $(C2)$ respectively have to be independent of h. Let us see in which sense $(C1)$ and $(C2)$ are "fighting" each other. It is clear that $(C2)$ demands for a large choice of Σ_h (compared with U_h): the larger is Σ_h, the bigger is the supremum in $(C2)$. In a sense, for a given choice of U_h, it is sufficient to take Σ_h "large enough" and $(C2)$ will hold. Let us now turn to $(C1)$. Inequality (1.4) has no hopes to be true, unless K_h is small enough: for instance, if every $\underline{\tau}_h$ in K_h satisfies $\operatorname{div} \underline{\tau}_h = 0$, then (1.4) holds trivially with $\alpha = 1$; on the other hand, if K_h contains vectors $\underline{\tau}_h$ with $\operatorname{div} \underline{\tau}_h \neq 0$, then (1.4) becomes very difficult, since it requires to bound the L^2 norm of $\operatorname{div} \underline{\tau}_h$ by means of the L^2 norm of $\underline{\tau}_h$ which is, in general, impossible (unless you take $\alpha = O(h)$). Hence, $(C1)$ requires a "small" K_h: but this is the same as requiring a large U_h (compared with Σ_h); for instance, if $\operatorname{div}(\Sigma_h) \subseteq U_h$ then K_h is made only of divergence-free vectors, and $(C1)$ (as we have seen) holds trivially with $\alpha = 1$. We conclude that, whereas

$(C2)$ is demanding a Σ_h large enough (compared with U_h), $(C1)$ is demanding a U_h large enough (compared with Σ_h). Starting with a given pair (Σ_h, U_h), it is always possible to fulfill $(C1)$ by enlarging U_h conveniently *or* to fulfill $(C2)$ by enlarging Σ_h conveniently. But we need *both* conditions to hold at the same time... For common choices of Σ_h and U_h conditions $(C1)$ and $(C2)$ practically amount to

$$(S1) \qquad \text{div } (\Sigma_h) \;=\; U_h$$

and

$$(S2) \qquad \begin{cases} \exists \beta > 0 \text{ such that} \\ \forall v_h \in U_h \quad \exists \underline{\tau}_h \in \Sigma_h \text{ with :} \\ \text{div}\, \underline{\tau}_h = v_h \quad ||\underline{\tau}_h||_H \le (1/\beta)||v_h||_0. \end{cases}$$

In other words, $(S2)$ means that the operator *div*, from Σ_h to U_h, has a continuous lifting with norm $\le \beta^{-1}$. It is very easy to check that $(S1)$ and $(S2)$ are sufficient conditions for $(C1)$ and $(C2)$ to hold. As we said, they are practically necessary for usual choices of Σ_h and U_h.

Our model problem and, in general, second order elliptic problems, are however so simple that it has been possible to construct several families of methods satisfying $(S1)$ and $(S2)$ (Refs. 10, 2, 3). On the other hand, for linear elasticity problems such a construction is yet unachieved and looks rather difficult (see Refs. 12, 13 for attempts in this direction). We still refer to Ref. 4 for a detailed description of the available families for elliptic equations and for a discussion of the difficulties related to elasticity problems.

It has to be pointed out , however, that all the families satisfying $(S1)$ and $(S2)$ available in the literature so far have a discontinuous stress field. To be more precise, the space Σ_h is made of vectors whose *normal* component is continuous across the interelement boundaries, while the *tangential* component is, a-priori, discontinuous when passing from one element to another. This is very reasonable, since we want to construct subspaces of $H(\text{div}\,;\Omega)$. On the other hand, there are several applications where, for various reasons (typically when the equation has to be coupled with other equations in a bigger system), one needs to work with a continuous stress field $\underline{\sigma}_h$. In such a case, no reasonable choice is available. More specifically, no reasonable example is known of pairs (Σ_h, U_h) of finite element spaces satisfying $(S1)-(S2)$ together with $\Sigma_h \subset (C^0(\overline{\Omega}))^2$, unless one takes polynomials of a rather high degree (see Ref. 11), and not for an arbitrary mesh.

We propose here a modification of the problem such that: 1) the new problem has exactly the same solution as the old one, and 2) for the approximation of the new problem condition $(C1)$ is always (automatically) satisfied. It is clear that, to approximate the new problem, one has to deal only with $(C2)$, and a large enough choice of Σ_h will do the job. Moreover, in cases where one is interested in working with a continuous stress field (i.e., with $\Sigma_h \subset (C^0(\overline{\Omega}))^2$, one can profitably use all the results and methodologies already developed for Stokes problems, in order to find pairs (Σ_h, U_h) fulfilling $(C2)$.

As we shall see, our modification is strictly related to a whole class of similar tricks introduced by other authors for a wider class of problems (see e.g., Ref. 8 and the references therein). However, on one hand our modification can be seen directly at the level of the continuous problem. On the other hand, it is independent of the mesh size (in contrast with the others quoted above). Moreover, we want to make use of $\Sigma_h \subset (C^0(\overline{\Omega}))^2$ while Ref. 8 is taylored for discontinuous stresses.

Let us also remark that we shall obtain optimal error bounds in spaces of the type $H(\mathrm{div}\,;\Omega)$. Such estimates can be seen as non-optimal estimates in $(L^2(\Omega))^2$. Although this way of looking is not unreasonable, we feel that in many cases, and in particular in those cases where one *has to* work with continuous stresses, the "loss of one order of convergence" is affordable.

Finally, we point out explicitely that, in our opinion, the use of a C^0 discretization for the stress field should, in general, be avoided. The main reason for this is the difficulty in the numerical solution of the linear system of equations. However, we accept that there are cases where the use of continuous stresses presents other additional advantages that justify its choice. For these cases, or for other situations where more traditional elements cannot be used (see e.g., Ref. 9), we think that our trick can help in the construction of new stable approximations.

2. Modified formulations for linear elliptic equations.

For the sake of simplicity, we shall deal with the following problem

$$\begin{cases} \text{find } u \in H^1_0(\Omega) \text{ such that} \\ \mathrm{div}\,(a\underline{\nabla} u) \;=\; f \qquad \text{in } \Omega. \end{cases} \tag{2.1}$$

In (2.1) the functions $a(x)$ and $f(x)$, $(x \in \Omega)$ are supposed to be given and "well behaved"; for instance, $f \in L^2(\Omega)$, $a \in L^\infty(\Omega)$ and $0 < \overline{a} \le a(x)$ a.e. in Ω. As we have seen, the usual mixed formulation of (2.1) starts by setting

$$\underline{\sigma} = a\underline{\nabla} u \quad \text{in } \Omega, \tag{2.2}$$

which gives

$$a^{-1}\underline{\sigma} = \underline{\nabla} u \quad \text{in } \Omega, \tag{2.3}$$

and then the variational formulation

$$\begin{cases} \text{find } \underline{\sigma} \in H(\mathrm{div}\,;\Omega) \text{ and } u \in L^2(\Omega) \text{ such that} \\ (a^{-1}\underline{\sigma},\underline{\tau}) \;+\; (u, \mathrm{div}\,\underline{\tau}) \;=\; 0 \qquad \forall \underline{\tau} \in H(\mathrm{div}\,;\Omega), \\ (\mathrm{div}\,\underline{\sigma}, v) \;=\; (f, v) \qquad \forall v \in L^2(\Omega). \end{cases} \tag{2.4}$$

It is clear however that, using the equation

$$\mathrm{div}\,\underline{\sigma} \;=\; f \quad \text{in } \Omega \tag{2.5}$$

one can consider, in place of (2.2), the alternative (and equivalent) setting

$$a^{-1}\underline{\sigma} - \underline{\nabla}\mathrm{div}\,\underline{\sigma} \;=\; \underline{\nabla} u - \underline{\nabla} f \quad \text{in } \Omega, \tag{2.6}$$

which gives rise to the alternative variational formulation

$$\begin{cases} \text{find } \underline{\sigma} \in H(\mathrm{div}\,;\Omega) \text{ and } u \in L^2(\Omega) \text{ such that} \\ (a^{-1}\underline{\sigma}, \underline{\tau}) \ + \ (\mathrm{div}\,\underline{\sigma}, \mathrm{div}\,\underline{\tau}) \ + \ (u, \mathrm{div}\,\underline{\tau}) \ = \ (f, \mathrm{div}\,\underline{\tau}) \quad \forall \underline{\tau} \in H(\mathrm{div}\,;\Omega), \\ (\mathrm{div}\,\underline{\sigma}, v) \ = \ (f, v) \qquad \forall v \in L^2(\Omega). \end{cases} \tag{2.7}$$

Discretizing (2.7) by means of $\Sigma_h \subset H(\mathrm{div}\,;\Omega)$ and $U_h \subset L^2(\Omega)$ the compatibility conditions (see always Ref.4) are now

$$(C1) \qquad \begin{cases} \exists \alpha > 0 \text{ such that} \\ (a^{-1}\underline{\tau}, \underline{\tau}) + ||\mathrm{div}\,\underline{\tau}||_0^2 \ \geq \ \alpha ||\underline{\tau}||_H^2 \quad \forall \underline{\tau} \in K_h, \end{cases} \tag{2.8}$$

with (as in (1.5))

$$K_h \ = \ \{\underline{\tau}_h \in \Sigma_h \ : \ (\mathrm{div}\,\underline{\tau}_h, v_h) \ = \ 0 \ \ \forall v_h \in U_h\}, \tag{2.9}$$

$$(C2) \qquad \begin{cases} \exists \beta > 0 \text{ such that} \\ \displaystyle\mathop{\mathrm{Sup}}_{\underline{\tau}_h \in \Sigma_h} \frac{(\mathrm{div}\,\underline{\tau}_h, v_h)}{||\underline{\tau}_h||_H} \geq \beta ||v_h||_0 \quad \forall v_h \in U_h. \end{cases} \tag{2.10}$$

Comparing (2.8) with (1.4) one can see the gain of the formulation (2.7). Actually, since $a \in L^\infty(\Omega)$, we have that (2.8) holds trivially with

$$\alpha = min\{1, (||a||_{L^\infty(\Omega)})^{-1}\} \tag{2.11}$$

for every $\underline{\tau}_h$ *in* Σ_h, *regardless* of the nature of K_h. Hence, we only have to worry about $(C2)$. However, now, to enforce $(C2)$ is relatively easy, since we only have to take Σ_h large enough. For instance, in the case $\Sigma_h \subset (C^0(\overline{\Omega}))^2$, which is of particular interest for the present paper, we can just take as (Σ_h, U_h) a velocity-pressure finite element pair that satisfies the Inf-Sup condition for the Stokes problem. Actually, if one interprets Σ_h as a velocity field and U_h as a pressure field, $(C2)$ is an immediate consequence of the Inf-Sup condition for the Stokes problem

$$\mathop{\mathrm{Inf}}_{v_h \in U_h} \mathop{\mathrm{Sup}}_{\underline{\tau}_h \in \Sigma_h} \frac{(\mathrm{div}\,\underline{\tau}_h, v_h)}{||\underline{\tau}_h||_1 ||v_h||_0} \geq \beta \tag{2.12}$$

since $||\underline{\tau}||_1^2 \ (= ||\underline{\tau}||_0^2 + ||\underline{\underline{\nabla}}\,\underline{\tau}||_0^2) \geq ||\underline{\tau}||_H^2$ for all $\underline{\tau}$.

Hence we have immediately the following results

Theorem 2.1 Assume that (Σ_h, U_h) is a pair of finite element spaces $\Sigma_h \subset H(\mathrm{div}\,;\Omega)$ and $U_h \subset L^2(\Omega)$ such that (2.10) holds with β independent of h. Then the discrete problem

$$\begin{cases} \text{find } \underline{\sigma}_h \in \Sigma_h \text{ and } u_h \in U_h \text{ such that} \\ (a^{-1}\underline{\sigma}_h, \underline{\tau}_h) + (\mathrm{div}\,\underline{\sigma}_h, \mathrm{div}\,\underline{\tau}_h) + (u_h, \mathrm{div}\,\underline{\tau}_h) = (f, \mathrm{div}\,\underline{\tau}_h) \quad \forall \underline{\tau}_h \in \Sigma_h, \\ (\mathrm{div}\,\underline{\sigma}_h, v_h) = (f, v_h) \qquad \forall v_h \in U_h. \end{cases} \tag{2.13}$$

has a unique solution. Moreover,

$$||\underline{\sigma} - \underline{\sigma}_h||_H + ||u - u_h||_0 \;\leq\; \gamma\{\inf_{\underline{\tau}_h \in \Sigma_h} ||\underline{\sigma} - \underline{\tau}_h||_H + \inf_{v_h \in U_h} ||u - v_h||_0\} \qquad (2.14)$$

with γ independent of h. ∎

Corollary 2.1 Assume that $\Sigma_h \subset (H^1(\Omega))^2$ and $U_h \subset L^2(\Omega)$ are a pair of finite element spaces satisfying (2.12) (that is, they are a good velocity-pressure approximation for the Stokes problem). Then (2.10) holds and the conclusions of Theorem 2.1 follow.

Example. Assume, to fix the ideas, that we take, as a Stokes pair, the Hood-Taylor element. We then have that Σ_h is made of piecewise quadratic continuous vectors and U_h is made of piecewise linear continuous functions. With the notation of Ref. 4,

$$\Sigma_h \;=\; (\mathcal{L}^1_2)^2 \quad ; \quad U_h \;=\; \mathcal{L}^1_1. \qquad (2.15)$$

Since (2.15) satisfies (2.12) (see e.g. Ref. 4) then we deduce (2.14). If u and $\underline{\sigma}$ are smooth we get

$$||\underline{\sigma} - \underline{\sigma}_h||_H + ||u - u_h||_0 \;\leq\; Ch^2(||\underline{\sigma}||_3 + ||u||_2) \qquad (2.16)$$

with the usual notation. Note that (2.16) can be seen as suboptimal since, in the left-hand side, we bound $\underline{\sigma} - \underline{\sigma}_h$ in the $H(\mathrm{div}\,;\Omega)$-norm and not in $(H^1(\Omega))^2$ ∎

A partial remedy to the suboptimality in (2.16) (and, essentially, in all the other examples that can be constructed starting from Corollary 2.1) can be sought as follows. We set

$$|||\underline{\tau}|||^2 = ||a^{-1}\underline{\tau}||_0^2 + ||\mathrm{div}\,\underline{\tau}||_0^2 + ||\mathrm{curl}\,(\mathrm{a}^{-1}\underline{\tau})||_0^2 \qquad (2.17)$$

where, as usual, $\mathrm{curl}\,\underline{\phi} \equiv \partial\phi_2/\partial \mathrm{x}_1 \;-\; \partial\phi_1/\partial \mathrm{x}_2$. Notice that (2.17) defines a norm which is much closer to the $(H^1(\Omega))^2$-norm than $||.||_H$. In particular, if either the normal or the tangential component of $\underline{\tau}$ vanishes on the boundary $\partial\Omega$, then the norms $|||\underline{\tau}|||$ and $||\underline{\tau}||_1$ are equivalent. We also point out explicitly that the two norms are not equivalent in general. To believe this, take a function $\psi \in H^1(\Omega)$ with $\Delta\psi = 0$; you can easily construct ψ such that it does not belong to $H^2(\Omega)$. Take now $\underline{\tau} = \underline{\nabla}\psi$ (and $a(x) \equiv 1$). Then $\mathrm{div}\,\underline{\tau} = 0$ and $\mathrm{curl}\,(\mathrm{a}^{-1}\underline{\tau}) = 0$, so that $|||\underline{\tau}||| = ||\underline{\tau}||_0 \leq ||\psi||_1$ is finite, while $||\underline{\tau}||_1 \simeq ||\psi||_2$ is infinite. In general, however, we consider the norm $|||.|||$ to be better than the norm $||.||_H$.

If we consider now the bilinear form

$$A(\underline{\sigma}, \underline{\tau}) \;=\; (a^{-1}\underline{\sigma}, \underline{\tau}) \;+\; (\mathrm{div}\,\underline{\sigma}, \mathrm{div}\,\underline{\tau}) \;+\; (\mathrm{curl}\,(\mathrm{a}^{-1}\underline{\sigma}), \mathrm{curl}\,(\mathrm{a}^{-1}\underline{\tau})) \qquad (2.18)$$

we have that the problem

$$\begin{cases} A(\underline{\sigma}, \underline{\tau}) \;+\; (u, \mathrm{div}\,\underline{\tau}) \;=\; (f, \mathrm{div}\,\underline{\tau}) & \forall \underline{\tau} \in \tilde{H}, \\ (\mathrm{div}\,\underline{\sigma}, v) \;=\; (f, v) & \forall v \in L^2 \end{cases} \qquad (2.19)$$

(where $\tilde{H}$ is the set of $\underline{\tau}$'s such that $|||\underline{\tau}|||$ is finite) has a unique solution which coincides with that of (2.7). Similarly to Corollary 2.1 we have now

Corollary 2.2 Let $\Sigma_h \subset (H^1(\Omega))^2$ and $U_h \subset L^2(\Omega)$ be a pair of finite element spaces satisfying (2.12). Then the discrete problem

$$\begin{cases} \text{Find } \underline{\sigma}_h \in \Sigma_h \text{ and } u_h \in U_h \text{ such that} \\ A(\underline{\sigma}_h, \underline{\tau}_h) \ + \ (u_h, \operatorname{div} \underline{\tau}_h) \ = \ (f, \operatorname{div} \underline{\tau}_h) \quad \forall \underline{\tau} \in \Sigma_h, \\ (\operatorname{div} \underline{\sigma}_h, v_h) \ = \ (f, v_h) \qquad\qquad\qquad\qquad \forall v_h \in U_h \end{cases} \tag{2.20}$$

has a unique solution and

$$|||\underline{\sigma} - \underline{\sigma}_h||| + \|u - u_h\|_0 \ \leq \ \gamma\{ \inf_{\underline{\tau}_h \in \Sigma_h} |||\underline{\sigma} - \underline{\tau}_h||| + \inf_{v_h \in U_h} \|u - v_h\|_0 \} \tag{2.21}$$

with γ independent of h. ∎

Remark 2.1 The idea of using the bilinear form $A(\underline{\sigma}, \underline{\tau})$ is also used in a paper in preparation by K.J.Bathe and F.Brezzi. ∎

Remark 2.2 The modification by L.Franca and T.Hughes (Ref. 8) for this problem leads to

$$\begin{cases} (a^{-1}\underline{\sigma}_h, \underline{\tau}_h) - (\underline{\nabla} u_h, \underline{\tau}_h) + \delta h^2 (\operatorname{div} \underline{\sigma}_h, \operatorname{div} \underline{\tau}_h)_h = \delta h^2 (f, \operatorname{div} \underline{\tau}_h)_h & \forall \underline{\tau}_h \in \Sigma_h, \\ (\underline{\sigma}_h, \underline{\nabla} v_h) = (f, v_h) & \forall v_h \in U_h. \end{cases} \tag{2.22}$$

where $(.,.)_h$ is the sum over all the triangles of the L^2-inner pruduct on every triangle. We notice two major differences between (2.22) and (2.13): first, in (2.22) one assumes to work with totally (a-priori) discontinuous stresses and continuous displacements (that is, $\Sigma_h \subset (L^2(\Omega))^2$, $U_h \subset H^1(\Omega)$); second, the perturbation to the first equation is added with a coefficient ($= \delta h^2$) which depends on the mesh size h. We also point out that numerical experiments in Ref. 7 show that a bigger δ leads to a better accuracy. ∎

Remark 2.3 The use of $\alpha(h) = O(h)$ in (1.4) (attainable without our modification) can produce a much worse suboptimality than that hidden in (2.14) or (2.21). See Ref. 4 for more details. ∎

3. Elasticity Problems

As pointed out in the introduction, the main interest of our construction is however for elasticity problems, where the question of coercivity, (condition (1.4)), becomes harder and harder as the material approaches incompressibility and for which we do not have totally satisfactory constructions for symmetric approximations of the stress field. Indeed, we are now looking for a **symmetric** stress tensor

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \in H_S(\underline{\operatorname{div}}; \Omega) =: \Sigma \tag{3.1}$$

where

$$\Sigma := H_S(\underline{\mathrm{div}};\Omega) = \{\underline{\underline{\sigma}} \mid \sigma_{ij} \in L^2(\Omega), \underline{\mathrm{div}}\,\underline{\underline{\sigma}} \in (L^2(\Omega))^2,\ \sigma_{12} = \sigma_{21}\} \tag{3.2}$$

with the norm $||\underline{\underline{\tau}}||^2_\Sigma = ||\underline{\underline{\tau}}||^2_0 + ||\underline{\mathrm{div}}\,\underline{\underline{\tau}}||^2_0$. We consider the bilinear form

$$a(\underline{\underline{\sigma}};\underline{\underline{\tau}}) = \frac{1}{2\mu}\int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\tau}}^D\,dx + \frac{1}{(\lambda+\mu)}\int_\Omega tr(\underline{\underline{\sigma}})tr(\underline{\underline{\tau}})\,dx \tag{3.3}$$

where $\underline{\underline{\sigma}}^D = \underline{\underline{\sigma}} - \frac{1}{2}\ tr\,\underline{\underline{\sigma}}$ is the deviatoric of $\underline{\underline{\sigma}}$. We then have

$$a(\underline{\underline{\sigma}},\underline{\underline{\sigma}}) \geq \inf(\frac{1}{2\mu},\frac{1}{\lambda+\mu})||\underline{\underline{\sigma}}||^2_0 \tag{3.4}$$

with a coercitivity constant vanishing for large λ, that is, for nearly incompressible materials. Fortunately, things are not so bad, at least in the continuous case, because we have (see e.g. Ref. 4) that, if $\int_\Omega tr\,\underline{\underline{\tau}}\,dx = 0$, then

$$||\underline{\underline{\tau}}||^2_0 \leq C(||\underline{\underline{\tau}}^D||^2_0 + ||\underline{\mathrm{div}}\,\underline{\underline{\tau}}||^2_0), \tag{3.5}$$

so that, for $\underline{\mathrm{div}}\,\underline{\underline{\tau}} = 0$, that is in the kernel of $\underline{\mathrm{div}}$, we can have coercivity, independently of λ. Inequality (3.5) also implies that, considering the modified problem

$$\operatorname*{Inf}_{\underline{\underline{\tau}}}\ \operatorname*{Sup}_{\underline{v}}\ \frac{1}{2}a(\underline{\underline{\tau}},\underline{\underline{\tau}}) + \frac{\delta}{2}||\underline{\mathrm{div}}\,\underline{\underline{\tau}} + \underline{f}||^2_0 + (\underline{v}, \underline{\mathrm{div}}\,\underline{\underline{\tau}} + \underline{f}), \tag{3.6}$$

we now have, for any λ, coercivity in the norm $||\underline{\underline{\tau}}||_\Sigma$ with a constant depending on δ (hence the importance of choosing $\delta \approx 1$) and not $O(h)$).

The optimality conditions associated with (3.6) are,

$$\begin{aligned}\frac{1}{2\mu}\int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\tau}}^D\,dx + \frac{1}{(\lambda+\mu)}\int_\Omega tr\,\underline{\underline{\sigma}}\,tr\,\underline{\underline{\tau}}\,dx + \delta\int_\Omega (\underline{\mathrm{div}}\,\underline{\underline{\sigma}} + \underline{f})\cdot\underline{\mathrm{div}}\,\underline{\underline{\tau}}\,dx \\ + \int_\Omega \underline{u}\cdot\underline{\mathrm{div}}\,\underline{\underline{\tau}}\,dx = 0 \qquad \forall\underline{\underline{\tau}} \in \Sigma,\end{aligned} \tag{3.7}$$

$$\int_\Omega (\underline{\mathrm{div}}\,\underline{\underline{\sigma}} + \underline{f})\cdot\underline{v}\,dx = 0 \qquad \forall\underline{v} \in U, \tag{3.8}$$

where obviously U is now $(L^2(\Omega))^2$. Similar to Theorem 2.1 we have now the following result.

Theorem 3.1 Let $\underline{\underline{\sigma}} \in \Sigma$ and $\underline{u} \in U$ be the solution of (3.7), (3.8). Let $\Sigma_h \subset \Sigma$ and $U_h \subset U$ be finite dimensional subspaces satisfying

$$\operatorname*{Inf}_{\underline{v}_h}\ \operatorname*{Sup}_{\underline{\underline{\tau}}_h}\ \frac{\int_\Omega \underline{v}_h\cdot\underline{\mathrm{div}}\,\underline{\underline{\tau}}_h\,dx}{||\underline{\underline{\tau}}_h||_\Sigma|\underline{v}|_0} \geq k_0, \tag{3.9}$$

with k_0 independent of h, and let $\underline{\underline{\sigma}}_h \in \Sigma_h$, $\underline{u}_h \in U_h$ be the solution of

$$\begin{cases} \text{find } \underline{\underline{\sigma}}_h \in \Sigma_h \text{ and } \underline{u}_h \in U_h \text{ such that} \\ \dfrac{1}{2\mu}\displaystyle\int_\Omega \underline{\underline{\sigma}}_h{}^D : \underline{\underline{\tau}}_h{}^D\,dx + \dfrac{1}{(\lambda+\mu)}\displaystyle\int_\Omega tr\,\underline{\underline{\sigma}}_h tr\,\underline{\underline{\tau}}_h\,dx + \\ \delta\displaystyle\int_\Omega (\underline{\operatorname{div}}\,\underline{\underline{\sigma}}_h + \underline{f})\cdot \underline{\operatorname{div}}\,\underline{\underline{\tau}}_h\,dx + \displaystyle\int_\Omega \underline{u}_h \cdot \underline{\operatorname{div}}\,\underline{\underline{\tau}}_h\,dx = 0 \qquad \forall \underline{\underline{\tau}}_h \in \Sigma_h \\ \displaystyle\int_\Omega (\underline{\operatorname{div}}\,\underline{\underline{\sigma}}_h + \underline{f})\cdot \underline{v}_h\,dx = 0 \qquad \forall \underline{v}_h \in U_h. \end{cases} \tag{3.10}$$

Then we have the following error estimate

$$||\,\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h||_\Sigma + ||\underline{u} - \underline{u}_h||_0 \le C\left\{\inf_{\underline{\underline{\tau}}_h} ||\,\underline{\underline{\sigma}} - \underline{\underline{\tau}}_h||_\Sigma + \inf_{\underline{v}_h} ||\underline{u} - \underline{v}_h||_0\right\}, \tag{3.11}$$

with C independent of h. ∎

From this point, we may let ourselves be guided by the previous examples and try to employ finite elements similar to those built for the Stokes problem, in order to satisfy the Inf-Sup condition:

$$\operatorname*{Inf}_{\underline{v}_h}\operatorname*{Sup}_{\underline{\underline{\tau}}_h} \frac{\int_\Omega \underline{v}_h \cdot \underline{\operatorname{div}}\,\underline{\underline{\tau}}_h\,dx}{||\underline{\underline{\tau}}_h||_1\,|\underline{v}|_0} \ge k_0 \tag{3.12}$$

As in Corollary 2.1 we have again that (3.12) immediately implies (3.9) (since $||\,\underline{\underline{\tau}}\,||_\Sigma \le ||\,\underline{\underline{\tau}}\,||_1$), and hence (3.11). The discussion of the previous section on the quality of estimate (3.11) can be transferred verbatim except that we know of no norm similar to (2.17) for tensors.

Let us now consider a few examples.

Example 3.1: *The MINI element.*

This example has been presented in Ref.4. It is based on the MINI element of Ref. 1 which satisfies the Inf-Sup condition for Stokes. In particular, in this case we have $\Sigma_h = (\mathcal{L}_1^1 + \mathcal{B}_3)_s^4$, $U_h = (\mathcal{L}_1^1)^2$, that is, Σ_h is made of piecewise linear elements enriched by bubble functions, and U_h is made of piecewise linear continuous functions. It is straightforward to extend the proof for Stokes problem to the present case, that is, following Ref. 4, to build an interpolation operator Π_h such that

$$\int_\Omega \underline{\operatorname{div}}(\underline{\underline{\sigma}} - \Pi_h\,\underline{\underline{\sigma}})\cdot \underline{v}_h\,dx = 0 \qquad \forall \underline{v}_h \in U_h, \tag{3.13}$$

$$||\Pi_h\,\underline{\underline{\sigma}}\,||_1 \le C||\,\underline{\underline{\sigma}}\,||_1. \tag{3.14}$$

This is classically done, element by element, by seeing that on Ω one has:

$$\int_\Omega \underline{\operatorname{div}}(\underline{\underline{\sigma}} - \Pi_h\,\underline{\underline{\sigma}})\cdot \underline{v}_h\,dx = -\int_\Omega (\underline{\underline{\sigma}} - \Pi_h\,\underline{\underline{\sigma}}) : \underline{\underline{\varepsilon}}(\underline{v}_h)\,dx \tag{3.15}$$

From (3.15), and the symmetry of $\underline{\underline{\varepsilon}}(\underline{v}_h)$, (3.13) can be verified by adjusting inside each element the bubble part of $\underline{\underline{\sigma}}_h$. This construction yields in (3.11) an $O(h)$ error estimate, independently of λ, still valid for incompressible materials. ∎

Example 3.2: *The Crouzeix-Raviart element.*

In this case we take $\Sigma_h = (\mathcal{L}_2^1 + \mathcal{B}_3)_s^4$, $U_h = (\mathcal{L}_1^0)^2$, that is, Σ_h is made of piecewise quadratic (continuous) functions, enriched by bubbles, and U_h of piecewise linear discontinuous functions. The proof of the Inf-Sup condition mimics again the proof for Stokes problem.It makes an essential use of bubbles which can be adjusted element by element to satisfy (3.13). This element produces an $O(h^2)$ estimate in (3.11). ∎

Example 3.3: *The Hood-Taylor element*

The technique of proof described above does not apply to this classical choice for Stokes problem which would be here $\Sigma_h = (\mathcal{L}_2^1)_s^4, U_h = (\mathcal{L}_1^1)^2$. It is not known if this element is stable ∎

4. Incompressible elasticity and the need for $\delta > 0$

We have stated in the previous section that the error estimate (3.11) will hold independently of λ, even for incompressible materials. It is worth to give a look to this limiting case and to see what kind of stabilization mechanism has been introduced.

Let us therefore consider (3.7) when λ becomes large. The limiting case of (3.7)-(3.8) is then,

$$\frac{1}{2\mu}\int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\tau}}^D \, dx + \delta \int_\Omega (\underline{\mathrm{div}}\, \underline{\underline{\sigma}} + \underline{f}) \cdot \underline{\mathrm{div}}\, \underline{\underline{\tau}} \, dx + \int_\Omega \underline{u} \cdot \underline{\mathrm{div}}\, \underline{\underline{\tau}} \, dx = 0 \quad \forall \underline{\underline{\tau}}, \qquad (4.1)$$

$$\int_\Omega (\underline{\mathrm{div}}\, \underline{\underline{\sigma}} + \underline{f}) \cdot \underline{v} \, dx = 0 \qquad \forall \underline{v}. \qquad (4.2)$$

If we write $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^D + p\,\underline{\underline{I}}$, $\underline{\underline{\tau}} = \underline{\underline{\tau}}^D + q\,\underline{\underline{I}}$, we may write $\underline{\mathrm{div}}\, \underline{\underline{\sigma}} = \underline{\mathrm{div}}\, \underline{\underline{\sigma}}^D + \underline{\nabla} p$, $\underline{\mathrm{div}}\, \underline{\underline{\tau}} = \underline{\mathrm{div}}\, \underline{\underline{\tau}}^D + \underline{\nabla} q$, and we get from (4.1)-(4.2)

$$\begin{aligned} \frac{1}{2\mu}\int_\Omega \underline{\underline{\sigma}}^D : \underline{\underline{\tau}}^D \, dx + \delta \int_\Omega (\underline{\mathrm{div}}\, \underline{\underline{\sigma}}^D + \underline{\nabla} p + \underline{f}) \cdot \underline{\mathrm{div}}\, \underline{\underline{\tau}}^D \, dx \\ + \int_\Omega \underline{u} \cdot \underline{\mathrm{div}}\, \underline{\underline{\tau}}^D \, dx = 0 \quad \forall \underline{\underline{\tau}}^D, \end{aligned} \qquad (4.3)$$

$$\int_\Omega \underline{u} \cdot \underline{\nabla} q \, dx + \delta \int_\Omega (\underline{\mathrm{div}}\, \underline{\underline{\sigma}}^D + \underline{\nabla} p + \underline{f}) \cdot \underline{\nabla} q \, dx = 0 \qquad \forall q, \qquad (4.4)$$

$$\int_\Omega (\underline{\mathrm{div}}\, \underline{\underline{\sigma}}^D + \underline{\nabla} p + \underline{f}) \cdot \underline{v} \, dx = 0 \qquad \forall \underline{v}. \qquad (4.5)$$

For $\delta = 0$, (4.4) yields $\mathrm{div}\, \underline{u} = 0$ and $\underline{u} \cdot \underline{\nu}_{|\partial\Omega} = 0$. For $\delta > 0$ we have an expression similar to those introduced by Brezzi-Pitkäranta (Ref. 5) or Franca-Hughes (Ref. 8) for the stabilization of standard velocity-pressure approximations of Stokes problem. It is then normal and reasonable to expect good properties of our approximation for the limit problem. We shall now consider the impact of the stabilization procedure on some of the examples presented in section 3.

Example 4.1: *The MINI element*

We come back to the case of Example 3.1 and we try to interpret condition (4.4) for $\delta = 0$. We get

$$-\int_{\Omega} \mathrm{div}\underline{u}_h\, q_h\, dx + \int_{\partial\Omega} \underline{u}_h \cdot \underline{\nu}\, q_h\, ds = 0 \qquad \forall q_h \in (\mathcal{L}_1^1 + \mathcal{B}_3). \tag{4.6}$$

Taking $q_h = b_K$, that is, a bubble on K and zero elsewhere and remembering that $\mathrm{div}\underline{u}_h$ is constant on K, we conclude that $\mathrm{div}\underline{u}_h$ must be exactly zero everywhere. It is well known that this implies that $\underline{u}_h$ is constant (except for special meshes) and therefore the use of $\delta > 0$ is necessary to obtain a reasonable approximation. ∎

Example 4.2: *The Crouzeix-Raviart element*

We consider now Example 3.2 and again $\delta = 0$. We now have, as $\underline{u}_h$ is discontinuous,

$$\sum_K \{-\int_K \mathrm{div}\, \underline{u}_h\, q_h\, dx + \int_{\partial K} \underline{u}_h \cdot \underline{\nu}\, q_h\, ds\} = 0 \qquad \forall q_h \in (\mathcal{L}_2^1 + \mathcal{B}_3). \tag{4.7}$$

Taking again $q_h = b_K$, the bubble function on K, we again conclude that

$$\mathrm{div}\underline{u}_{h|K} = 0. \tag{4.8}$$

Taking now for q_h the piecewise quadratic basis function associated with the mid-side node M on the interface S between two elements K_1 and K_2, we get,

$$\int_S [\underline{u}_h \cdot \underline{\nu}] ds = 0 \tag{4.9}$$

where $[\underline{u}_h \cdot \underline{\nu}]$ denotes the jump of $[\underline{u}_h \cdot \underline{\nu}]$ on S. As $\underline{u}_h$ is piecewise linear, this implies that the normal component of $\underline{u}_h$ (up to the orientation of $\underline{\nu}$) is continuous at M. Finally, taking q_h to be the basis function Φ_V associated with a vertex V we have from (4.4) (always with $\delta = 0$):

$$\sum_{K \in N(V)} \int_K \underline{u}_h \cdot \underline{\nabla}\Phi_V\, dx = 0 \tag{4.10}$$

where $N(V)$ is the set of elements having V as a vertex.

Contrarily to what happened in the previous example, the set of vector valued functions in $(\mathcal{L}_1^0)^2$ satifying (4.8)-(4.9)(4.10) is not empty. It can even be characterized. In order to do so, we introduce the non-conforming bubble

$$b_{NC,K}(x) = 2 - 3(\lambda_1^2(x) + \lambda_2^2(x) + \lambda_3^2(x)) \tag{4.11}$$

on every element K, λ_i being the barycentric coordinates, and the space

$$B_{NC} = \{v \mid v_{|K} = b_{NC,K}\}. \tag{4.12}$$

Let $W_{0h} = \mathcal{L}_2^1 + \mathcal{B}_{NC}$ be the non conforming quadratic approximation of $H_0^1(\Omega)$ defined in Fortin-Soulié (Ref. 6). For $v_h \in W_{0h}$ we denote in a standard way $\underline{\mathrm{curl}}_h v_h \in (\mathcal{L}_1^0)^2$ to be the non conforming $\underline{\mathrm{curl}}$ operator, computed elementwise. We then have

Proposition 4.1: The set $V_h \subset (\mathcal{L}_1^0)^2$ of discrete divergence-free functions defined by (4.8)-(4.9)-(4.10) is equal to $\underline{\mathrm{curl}}_h W_{0h}$.

Proof: One easily ckecks that $\underline{\mathrm{curl}}_h v_h$ satisfies (4.8)-(4.10) $\forall v_h \in W_{0h}$ so that we have $\underline{\mathrm{curl}}_h W_{0h} \subset V_h$. To complete the proof, we check that the two spaces have the same dimension ($= 2e_{int}$, where e_{int} is the number of internal edges). ∎

One may wonder in this case if the regularization is truly necessary to ensure a good approximation for large λ as the space V_h appears to be reasonably large. To prove this, we would need to check a discrete analogue of (3.5). Finally, a last way around would be to use a space of tensors enriched by a non conforming bubble (instead of the cubic bubble). Then the discrete kernel

$$\ker B_h = \{\underline{\underline{\sigma}}_h \mid \sum_K \int_K \underline{\mathrm{div}}\, \underline{\underline{\sigma}}_h \cdot \underline{v}_h \, dx = 0 \qquad \forall \underline{v}_h \in U_h\} \tag{4.13}$$

is elementwise divergence-free. The question of error estimates for such a non conforming method is still an open question. First attempts seem to indicate that they are not optimal with respect to the order of the polynomials employed.

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