Mathematical Models and Methods in Applied Sciences
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A DOMAIN DECOMPOSITION METHOD FOR A BONDED STRUCTURE

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We show that the transmission conditions through two elastic bodies bonded by a thin adhesive layer can be written as Robin type conditions, well suited for using a domain decomposition algorithm for which we prove convergence. Numerical approximation by means of finite element methods is also presented and analyzed. Convergence of the discrete algorithm is proven as well as optimal error estimates.

1. Introduction

We introduce and analyze a numerical approach to deal with a simplified model of a bonded structure. The equations describing this model are such that a domain decomposition type procedure applies in a very natural way. A typical example of bonding of two elastic bodies is shown in fig. 1, where two bodies, Ω^+ and Ω^- are bonded along their common surface $S = \partial\Omega^+ \cap \partial\Omega^-$. The bonding is obtained by a very thin adhesive layer, and the adhesive is more flexible than the adherents. From a mechanical point of view, the problem is the transmission of stresses through three elastic bodies, i.e., the thin adhesive and the two adherents. However, the differences in both geometrical and constitutive properties between adhesive and adherents are so important that the numerical solution of the problem as it is becomes delicate and may produce undesirable instabilities. This motivates the introduction of a simplified model which permits the effective computation of the solution with a good approximation. Since a pioneering work by Goland and Reissner[13] in 1944, the bonding of two elastic three dimensional structures by an adhesive layer is treated with asymptotic analysis. With this approximation, the adhesive is treated as a material surface: it disappears from a geometrical point of view, but it is represented by an energy of adhesion. The limit problem becomes then a non usual transmission problem between two bodies, the adherents only. Two complementary approaches have been used to obtain these simplified models. We refer for instance to Refs. 15,12,8,6,5,16,11 for the multiscale asymptotic developments approach, and to Refs. 3,17,20,9 for the energy methods approach.

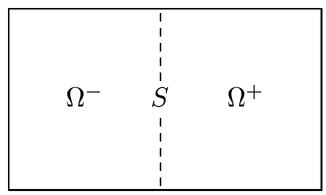

Fig. 1.

It is known (see the above Refs.) that, for the limit problem, the conormal derivative of the solution (i.e., the normal component of the stresses) is continuous at the interface, while the solution (i.e., the displacement) is discontinuous and has a jump proportional to the conormal derivative. These transmission conditions can then be reinterpreted as Robin type conditions, thus giving rise to a new formulation, well suited for applying a domain decomposition procedure between two subdomains, the adherents. More traditional numerical methods can be found in Refs. 6,7,18,2.

The paper is organized as follows. In Sect. 2, to fix ideas, we recall the formulation of the limit problem for the heat conduction case, and show equivalence with a suitable multidomain formulation. Very briefly, at the end of the section we present an application to linear elasticity problems. In Sect. 3 we introduce an iterative procedure between subdomains for which we prove convergence. In Sect. 4 we study the finite element approximation and the discrete analog of the iterative algorithm, for which we prove convergence. We also prove that the sequence of iterates converges to the finite element solution of the multidomain formulation, thus giving optimal error estimates. Finally, in Sect. 5 we present some numerical results.

2. Position of the problem

For the sake of simplicity we shall consider a model problem, such as the heat conduction through two bodies, bonded by a thin adhesive layer whose thermal conductivity coefficients are small with respect to those of the adherents. Let Ω^+ and Ω^- denote the two bodies, that we assume to be open connected subsets of $\mathbf{R}^3$ with boundaries $\partial\Omega^+$ and $\partial\Omega^-$ piecewise of class C^2, and let $S = \partial\Omega^+ \cap \partial\Omega^-$ be a non empty projectable regular surface of positive measure. Let Ω be the union of Ω^+ and Ω^-, with boundary $\partial\Omega$. For simplicity, assume that homogeneous Dirichlet conditions are taken on $\partial\Omega$. For a function w defined on Ω, let w^+ (resp. w^-) denote the restriction of w to Ω^+ (resp. Ω^-). Let $f_{,i}$ denote the $i-th$ partial derivative of a function f. Using the summation convention of repeated indices, the local equations are (see Refs. 10,11)

$$\begin{cases} -\left(a^+_{ij}u^+_{,j}\right)_{,i} = f^+ & \text{in } \Omega^+ \\ -\left(a^-_{ij}u^-_{,j}\right)_{,i} = f^- & \text{in } \Omega^- \\ u^+ = 0 & \text{on } \partial\Omega^+ \cap \partial\Omega \\ u^- = 0 & \text{on } \partial\Omega^- \cap \partial\Omega \end{cases} \tag{2.1}$$

with the transmission condition on S

$$\begin{cases} a^+_{ij}u^+_{,j}n^+_j = -K(u^+ - u^-) & \text{on } S \\ a^-_{ij}u^-_{,j}n^-_j = K(u^+ - u^-) & \text{on } S \end{cases} \tag{2.2}$$

where $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) is the outward unit normal to Ω^+ (resp. Ω^-). Existence and uniqueness of the solution of (2.1)-(2.2) is proved under the usual regularity assumptions on the data: $f^+ \in L^2(\Omega^+)$, $f^- \in L^2(\Omega^-)$, $\mathbf{a}^+$ and $\mathbf{a}^-$ are symmetric positive definite matrices with smooth and bounded coefficients; $K = K(x)$ is a function defined on S verifying $0 < K_* \le K \le K^*$ (see refs. 10, 11.) In order to apply a domain decomposition type procedure, we observe that the boundary conditions (2.2) can be rewritten as

$$a^+_{ij}u^+_{,j}n^+_j = -a^-_{ij}u^-_{,j}n^-_j \quad \text{on } S, \tag{2.3}$$

$$a^+_{ij}u^+_{,j}n^+_j + 2Ku^+ = a^-_{ij}u^-_{,j}n^-_j + 2Ku^- \quad \text{on } S, \tag{2.4}$$

where (2.3) is the continuity condition of the conormal derivative, while (2.4) is a Fourier-Robin condition. Next, for $g \in L^2(S)$, consider the following problems

$$\begin{cases} -\left(a^+_{ij}u^+_{,j}\right)_{,i} = f^+ & \text{in } \Omega^+ \\ u^+ = 0 & \text{on } \partial\Omega^+ \cap \partial\Omega \\ a^+_{ij}u^+_{,j}n^+_j + 2Ku^+ = g & \text{on } S \end{cases} \tag{2.5}$$

$$\begin{cases} -\left(a^-_{ij}u^-_{,j}\right)_{,i} = f^- & \text{in } \Omega^- \\ u^- = 0 & \text{on } \partial\Omega^- \cap \partial\Omega \\ a^-_{ij}u^-_{,j}n^-_j + 2Ku^- = g & \text{on } S \end{cases} \tag{2.6}$$

For any given $g \in L^2(S)$ problems (2.5)-(2.6) have a unique solution $u^+ \in H^1(\Omega^+)$ and $u^- \in H^1(\Omega^-)$ respectively. (Moreover, note that the boundary conditions on

S in (2.5)-(2.6) actually induce more regularity on the solutions.) Due to linearity, these solutions can be split as

$$u^+ \;=\; u_f^+ + u_g^+, \qquad\qquad u^- \;=\; u_f^- + u_g^-, \tag{2.7}$$

with u_f^+ solution of (2.5) with $g = 0$, and u_g^+ solution of (2.5) with $f^+ = 0$. (The same applies to u_f^-, u_g^-.) We can then define the linear continuous operators T_f^+, T_f^-, T_g^+, T_g^-

$$f^+ \in L^2(\Omega^+) \longrightarrow u_f^+ = T_f^+(f^+), \qquad f^- \in L^2(\Omega^-) \longrightarrow u_f^- = T_f^-(f^-), \tag{2.8}$$

$$g \in L^2(S) \longrightarrow u_g^+ = T_g^+(g), \qquad u_g^- = T_g^-(g), \tag{2.9}$$

so that (2.7) becomes

$$u^+ = T_f^+(f^+) + T_g^+(g) \qquad u^- = T_f^-(f^-) + T_g^-(g). \tag{2.10}$$

Next, let $\mathcal{A}$ be the operator from $L^2(S)$ in itself defined as

$$g \in L^2(S) \longrightarrow \mathcal{A}g \;=\; (u_g^+ + u_g^-)_{|S} \equiv (T_g^+(g) + T_g^-(g))_{|S}. \tag{2.11}$$

It is immediate to check that $\mathcal{A}$ is linear and continuous. Moreover, thanks to the trace theorem (see, e.g., Ref. 14), we have $u_g^+{}_{|S} \in H^{1/2}(S)$, $u_g^-{}_{|S} \in H^{1/2}(S)$, so that $\mathcal{A}$ is linear and continuous from $L^2(S)$ into $H^{1/2}(S)$, and

$$\exists\, C > 0 \text{ such that} \qquad ||\mathcal{A}g||_{H^{1/2}(S)} \leq C\, ||g||_{0,S}. \tag{2.12}$$

Going back to formulation (2.5)-(2.6), note that the continuity condition (2.3) is not taken into account. Hence, we must find a suitable g such that the solutions of (2.5)-(2.6) verify (2.3). From (2.5)-(2.6) we deduce $a_{ij}^+ u_{,j}^+ n_j^+ + a_{ij}^- u_{,j}^- n_j^- \;=\; 2(g - K(u^+ + u^-))$. Therefore, such a g will be the solution of the following minimization problem

$$\text{Find } g^* \in L^2(S) \;:\; 0 = J(g^*) \leq J(g) \qquad \forall g \in L^2(S), \tag{2.13}$$

for the quadratic functional

$$J(g) := ||g - K(u^+ + u^-)||_{0,S}^2. \tag{2.14}$$

Using the notation introduced in (2.10)-(2.11) and setting

$$F := K(T_f^+(f^+) + T_f^-(f^-))_{|S}, \tag{2.15}$$

we have

$$K(u^+ + u^-)_{|S} \;=\; F + K\mathcal{A}g, \tag{2.16}$$

so that (2.14) can be written as

$$J(g) = ||g - (F + K\mathcal{A}g)||^2_{0,S}. \tag{2.17}$$

Existence and uniqueness of the solution of the original formulation (2.1)-(2.2) imply that problem (2.13) has a unique solution g^*, which verifies

$$g^* \;=\; F + K\mathcal{A}g^*. \tag{2.18}$$

In order to write the variational formulation of (2.5)-(2.6) we set

$$V^+ := \{v \in H^1(\Omega^+)\ , v = 0 \text{ on } \partial\Omega^+ \cap \partial\Omega\}, \tag{2.19}$$

$$V^- := \{v \in H^1(\Omega^-)\ , v = 0 \text{ on } \partial\Omega^- \cap \partial\Omega\}, \tag{2.20}$$

$$a^+(v,w) = \int_{\Omega^+} a^+_{ij} v_{,j} w_{,i}\, dx \;+\; \int_S 2Kv\,w\,ds \qquad v,\, w \in V^+, \tag{2.21}$$

$$a^-(v,w) = \int_{\Omega^-} a^-_{ij} v_{,j} w_{,i}\, dx \;+\; \int_S 2Kv\,w\,ds \qquad v,\, w \in V^-. \tag{2.22}$$

The variational formulation of problems (2.5)-(2.6) is then

$$\left\{ \begin{array}{l} \text{Find } u^+ \in V^+ \text{ such that :} \\ a^+(u^+,v) \;=\; (f^+,v) + (g,v)_S \qquad \forall v \in V^+, \end{array} \right. \tag{2.23}$$

$$\left\{ \begin{array}{l} \text{Find } u^- \in V^- \text{ such that :} \\ a^-(u^-,v) \;=\; (f^-,v) + (g,v)_S \qquad \forall v \in V^-. \end{array} \right. \tag{2.24}$$

In (2.23)-(2.24) (f^+,v), (f^-,v) denote the scalar product in $L^2(\Omega^+)$, $L^2(\Omega^-)$ respectively, and $(g,v)_S$ is the scalar product in $L^2(S)$. Existence, uniqueness and a-priori error bounds for the solutions of (2.23) and (2.24) are ensured by the continuity and coercivity properties of the bilinear forms:

$$\exists\, M^+ > 0 \text{ such that } |a^+(v,w)| \le M^+||v||_{1,\Omega^+}||w||_{1,\Omega^+} \quad v,w \in V^+, \tag{2.25}$$

$$\exists\, M^- > 0 \text{ such that } |a^-(v,w)| \le M^-||v||_{1,\Omega^-}||w||_{1,\Omega^-} \quad v,w \in V^-, \tag{2.26}$$

$$\exists\, \alpha^+ > 0 \text{ such that } a^+(v,v) \ge \alpha^+||v||^2_{1,\Omega^+} \quad \forall v \in V^+, \tag{2.27}$$

$$\exists\, \alpha^- > 0 \text{ such that } a^-(v,v) \ge \alpha^-||v||^2_{1,\Omega^-} \quad \forall v \in V^-. \tag{2.28}$$

(As usual, $||.||_{1,D}$ denotes the norm in $H^1(D)$.)
We conclude this section with an application to linear elasticity problems. The statement of the problem being essentially the same, we shall write directly the multidomain variational formulation.

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}^+ \in V^+ := \{\mathbf{v} \in [H^1(\Omega^+)]^3\ , \mathbf{v} = 0 \text{ on } \partial\Omega^+ \cap \partial\Omega\} \text{ such that :} \\ a^+(\mathbf{u}^+,\mathbf{v}) \;=\; (\mathbf{f}^+,\mathbf{v}) + (\mathbf{g},\mathbf{v})_S \qquad \forall \mathbf{v} \in V^+, \end{array} \right. \tag{2.29}$$

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}^- \in V^- := \{\mathbf{v} \in [H^1(\Omega^-)]^3\ , \mathbf{v} = 0 \text{ on } \partial\Omega^- \cap \partial\Omega\} \text{ such that :} \\ a^-(\mathbf{u}^-,\mathbf{v}) \;=\; (\mathbf{f}^-,\mathbf{v}) + (\mathbf{g},\mathbf{v})_S \qquad \forall \mathbf{v} \in V^-, \end{array} \right. \tag{2.30}$$

having used the following notation:

$$a^+(\mathbf{u}^+,\mathbf{v}) = \int_{\Omega^+} A^+_{ijkl}\gamma_{kl}(\mathbf{u}^+)\gamma_{ij}(\mathbf{v})\,dx \;+\; \int_S 2\mathbf{K}\mathbf{u}^+\mathbf{v}\,ds,$$
$$a^-(\mathbf{u}^-,\mathbf{v}) = \int_{\Omega^-} A^-_{ijkl}\gamma_{kl}(\mathbf{u}^-)\gamma_{ij}(\mathbf{v})\,dx \;+\; \int_S 2\mathbf{K}\mathbf{u}^-\mathbf{v}\,ds,$$

where $\mathbf{A}^+$ (resp. $\mathbf{A}^-$) is the anisotropic elasticity tensor in Ω^+ (resp. Ω^-), bounded and coercive; γ is the linearized strain tensor; $\mathbf{K}$ is a second order tensor assumed to be bounded and strictly positive, $(\mathbf{f}^+,\mathbf{v})$ (resp. $(\mathbf{f}^-,\mathbf{v})$) denotes the scalar product in $[L^2(\Omega^+)]^3$ (resp. $[L^2(\Omega^-)]^3$), and $(\mathbf{g},\mathbf{v})_S$ the scalar product in $[L^2(S)]^3$. As in the previous case, we can define the functional $J(\mathbf{g})$ (obviously in the $[L^2(S)]^3$ framework), and study the minimum problem for $J(\mathbf{g})$ as in (2.13).

3. The algorithm

We shall now present a domain decomposition type algorithm, based on the variational formulation (2.23)-(2.24) and the minimum problem (2.13), for which we shall prove convergence.
Compute $u_f^+ = T_f^+(f^+)$, $u_f^- = T_f^-(f^-)$ solutions of

$$u_f^+ \in V^+ : \quad a^+(u_f^+,v) \;=\; (f^+,v) \qquad \forall v \in V^+, \tag{3.1}$$
$$u_f^- \in V^- : \quad a^-(u_f^-,v) \;=\; (f^-,v) \qquad \forall v \in V^-, \tag{3.2}$$

and set

$$g^0 \;=\; K(u_f^+ + u_f^-)_{|S} \; (= F). \tag{3.3}$$

For $m \geq 0$ compute the solutions $u_m^+ = T_g^+(g^m)$, $u_m^- = T_g^-(g^m)$ of the problems

$$u_m^+ \in V^+ \;: a^+(u_m^+,v) \;=\; (g^m,v)_S \qquad \forall v \in V^+, \tag{3.4}$$
$$u_m^- \in V^- \;: a^-(u_m^-,v) \;=\; (g^m,v)_S \qquad \forall v \in V^-. \tag{3.5}$$

Then set

$$\tilde{g}^m := g^m - K(u_m^+ + u_m^-)_{|S}, \tag{3.6}$$
$$g^{m+1} := g^m - \rho(\tilde{g}^m - g^0), \tag{3.7}$$

and compute the solutions u_{m+1}^-, u_{m+1}^+ of (3.4)-(3.5) with the new datum g^{m+1}. In (3.7) $\rho > 0$ is a parameter to be chosen in order to have convergence, that is,

$$\lim_{m\to\infty} g^m = g^*, \tag{3.8}$$

where g^* is defined in (2.18). In order to prove (3.8) we shall need the following theorem:

Theorem 3.1 *$K\mathcal{A}$ is a compact operator. Moreover, the eigenvalues z of $K\mathcal{A}$ are all real and verify*

$$\exists\, C_1 > 0 \text{ such that } \qquad 0 \le z \le 1 - C_1 < 1 \qquad \forall z. \tag{3.9}$$

Proof Due to (2.12) and the compact injection of $H^{1/2}(S)$ into $L^2(S)$ we deduce that $K\mathcal{A}$ is compact. Therefore its spectrum $\sigma(K\mathcal{A})$ can have only zero as an accumulation point, and all the non-zero elements of $\sigma(K\mathcal{A})$ are eigenvalues. Let then z be an eigenvalue of $K\mathcal{A}$ with $z \neq 0$. Then, there exists a $g \in L^2(S)$ $(g \neq 0)$ such that

$$K\mathcal{A}g \;=\; zg \tag{3.10}$$

or, equivalently (see definition (2.11))

$$K(u_g^+ + u_g^-)_{|S} \;=\; zg. \tag{3.11}$$

Since u_g^+ is the solution of (2.23) with $f^+ = 0$, u_g^- is the solution of (2.24) with $f^- = 0$, taking $g = (K(u_g^+ + u_g^-)_{|S})/z$ in (2.23)-(2.24), $v = u_g^+$ in (2.23), $v = u_g^-$ in (2.24) and adding the two equations we can write

$$a^+(u_g^+, u_g^+) + a^-(u_g^-, u_g^-) \;=\; \frac{1}{z}\int_S K(u_g^+ + u_g^-)^2\, ds, \tag{3.12}$$

and deduce that z is real and positive. Recalling definitions (2.21)-(2.22) we have

$$\begin{aligned} a^+(u_g^+,\ u_g^+) + a^-(u_g^-, u_g^-) &\ge C(||u_g^+||^2_{1,\Omega^+} + ||u_g^-||^2_{1,\Omega^-}) + \textstyle\int_S 2K((u_g^+)^2 + (u_g^-)^2)\, ds \\ &= C(||u_g^+||^2_{1,\Omega^+} + ||u_g^-||^2_{1,\Omega^-}) + \textstyle\int_S K(u_g^+ - u_g^-)^2\, ds + \int_S K(u_g^+ + u_g^-)^2\, ds. \end{aligned} \tag{3.13}$$

Using (3.13) in (3.12) we deduce that

$$C(||u_g^+||^2_{1,\Omega^+} + ||u_g^-||^2_{1,\Omega^-}) + \int_S K(u_g^+ - u_g^-)^2\, ds \le (\frac{1}{z} - 1)\int_S K(u_g^+ + u_g^-)^2\, ds. \tag{3.14}$$

Poincaré's inequality on the right-hand side of (3.14), applied in Ω^+ and Ω^- (with constants $c_+,\ c_-$ respectively) gives

$$z \le 1 - C_1 < 1 \qquad \text{with } C_1 = \frac{C}{2K^* \max\{c_+^2, c_-^2\} + C}, \tag{3.15}$$

and the proof is completed □.

We can now prove the following convergence theorem.

Theorem 3.2 *For all* ρ *with* $0 < \rho < 2$ *we have*

$$\lim_{m\to\infty} g^m = g^*, \tag{3.16}$$

where g^m *is the sequence defined in (3.1)-(3.7), and* g^* *is defined in (2.18).*

Proof Note that, according to definition (2.11), (3.6) can be rewritten as

$$\tilde{g}^m \ = \ (I - K\mathcal{A})g^m. \tag{3.17}$$

From (3.7) and (3.17), using (3.3) and (2.18) we then have

$$\begin{aligned} g^{m+1} - g^* \quad &= \ (1-\rho)g^m + \rho K\mathcal{A}g^m + \rho g^0 - g^* + \rho g^* - \rho g^* \\ &= (1-\rho)g^m + \rho K\mathcal{A}g^m + \rho(g^0 - g^*) - (1-\rho)g^* \\ &= (1-\rho)(g^m - g^*) + \rho K\mathcal{A}(g^m - g^*) \\ &= ((1-\rho)I + \rho K\mathcal{A})(g^m - g^*). \end{aligned} \tag{3.18}$$

Recursive application of (3.18) yields

$$g^{m+1} - g^* = ((1-\rho)I + \rho K\mathcal{A})^{m+1}(g^0 - g^*), \quad \text{with } g^0 - g^* = -K\mathcal{A}g^*. \tag{3.19}$$

Convergence will be proved if we can show that

$$\lim_{m\to\infty} |||((1-\rho)I + \rho K\mathcal{A})^{m+1}||| = 0, \tag{3.20}$$

where $|||L|||$ denotes the norm of the linear operator L. From a theorem by Gelfand, if L is bounded, then $\lim_{n\to\infty} |||L^n|||^{1/n} = \sup\{|\lambda|,\ \lambda \in \sigma(L)\}$, $\sigma(L)$ being the spectrum of L. Thanks to Theorem 3.1, the elements of the spectrum of the operator $(1-\rho)I + \rho K\mathcal{A}$ are : $1-\rho$ and

$$\lambda_j = (1-\rho) + \rho z_j, \tag{3.21}$$

z_j being the eigenvalues of $K\mathcal{A}$. Proving (3.20) amounts then to prove that

$$f(\rho) := \max\{|1-\rho|, \max_j |\lambda_j|\} < 1. \tag{3.22}$$

It is now immediate to see (since $0 \le z_j < 1\ \forall j$) that, for all $\rho \in]0,2[$, we have $-1 < 1-\rho < 1$ and $-1 < 1 - \rho(1-z_j) = \lambda_j < 1\ \Box$.

Remark 3.1 The optimal value for ρ is the minimizing argument of the function $f(\rho)$ in (3.22). A simple computation gives $\rho_{opt} = \dfrac{2}{2-z_{max}} > 1$, and $f(\rho_{opt}) = \dfrac{z_{max}}{2-z_{max}}$. Hence, the reduction factor per iteration, though obviously smaller than one, can be close to one even in the optimal case $\rho = \rho_{opt}$, whenever z_{max} is close to one. This fact can influence the performance of the procedure in terms of number of iterations required to achieve convergence (see the results of Sect. 5.) $\square$.

4. The finite element approximation

We shall assume both Ω^+ and Ω^- to be convex polygonal domains in $\mathbf{R^2}$, and S to be a straight line. This will allow us to use the H^2 regularity of the solutions of (2.5)-(2.6), thus avoiding further technicalities in the derivation of the error estimates. Let $\mathcal{T}_h$ be a regular decomposition (see Ref. 4) of Ω into triangles T not crossing the interface S. Thus, each element is either contained in $\overline{\Omega}^+$ or $\overline{\Omega}^-$. Let $\mathcal{T}_h^+$ (resp. $\mathcal{T}_h^-$) be the restriction of $\mathcal{T}_h$ to Ω^+ (resp. Ω^-). Define the conforming Lagrangian finite element spaces

$$V_h^+ = \{v \in C^0(\overline{\Omega}^+) : v_{|T} \in P_1(T)\ \forall T \in \mathcal{T}_h^+,\ v = 0 \text{ on } \partial\Omega \cap \partial\Omega^+\} \subset V^+, \quad (4.1)$$

$$V_h^- = \{v \in C^0(\overline{\Omega}^-) : v_{|T} \in P_1(T)\ \forall T \in \mathcal{T}_h^-,\ v = 0 \text{ on } \partial\Omega \cap \partial\Omega^-\} \subset V^-, \quad (4.2)$$

where $P_1(T)$ denotes the space of polynomials of degree ≤ 1 on T. Denoting by Σ_h the decomposition of S (induced by $\mathcal{T}_h$) into intervals I, define

$$\Phi_h = \{\phi \in C^0(S) :\ \phi_{|I} \in P_1(I)\ \forall I \in \Sigma_h,\ \phi_{|\partial S} = 0\} \subset L^2(S). \quad (4.3)$$

The finite element approximation of (2.23)-(2.24) is then: for $g_h \in \Phi_h$ given, let $u_h^+,\ u_h^-$ be the solutions of

$$u_h^+ \in V_h^+ \ :\ a^+(u_h^+, v) = (f^+, v) + (g_h, v) \qquad \forall v \in V_h^+, \quad (4.4)$$

$$u_h^- \in V_h^- \ :\ a^-(u_h^-, v) = (f^-, v) + (g_h, v) \qquad \forall v \in V_h^-. \quad (4.5)$$

As in the continuous case, the solutions of (4.4)-(4.5) can be split as

$$u_h^+ = u_{f,h}^+ + u_{g,h}^+, \qquad u_h^- = u_{f,h}^- + u_{g,h}^-,$$

with $u_{f,h}^+,\ u_{f,h}^-$ solutions of (4.4)-(4.5) with $g_h = 0$, and $u_{g,h}^+,\ u_{g,h}^-$ solutions of (4.4)-(4.5) with $f^+ = 0,\ f^- = 0$ respectively. In analogy with the notation of Sect. 2 we can introduce discrete operators $T_{f,h}^+,\ T_{f,h}^-,\ T_{g,h}^+,\ T_{g,h}^-$, and write

$$u_{f,h}^+ = T_{f,h}^+(f^+) + T_{g,h}^+(g_h), \qquad u_{f,h}^- = T_{f,h}^-(f^-) + T_{g,h}^-(g_h). \quad (4.6)$$

Similarly, the discrete analog of the operator $\mathcal{A}$ defined in (2.11) will be $\mathcal{A}_h$ defined as

$$g \in L^2(S) \longrightarrow \mathcal{A}_h g = (T_{g,h}^+(g) + T_{g,h}^-(g))_{|S}. \quad (4.7)$$

In particular, we will have

$$g_h \in \Phi_h \subset L^2(S) \longrightarrow \mathcal{A}_h g_h \;=\; (T^+_{g,h}(g_h) + T^-_{g,h}(g_h))_{|S}. \tag{4.8}$$

We can easily prove the following error estimates:

$$\begin{aligned} ||u^+ - u_h^+||_{1,\Omega^+} &\leq C(h|u^+|_{2,\Omega^+} + ||g - g_h||_{0,S}), \\ ||u^- - u_h^+||_{1,\Omega^-} &\leq C(h|u^-|_{2,\Omega^-} + ||g - g_h||_{0,S}), \end{aligned} \tag{4.9}$$

where, here and in the following, C denotes a constant independent of h, and $|.|_{2,D}$ is the norm in $H^2(D) \cap H^1_0(D)$. Estimates (4.9) can be obtained with the following arguments: let $\widetilde{u}_h^+$, $\widetilde{u}_h^-$ be the finite element solutions of

$$\widetilde{u}_h^+ \in V_h^+ \;:\; a^+(\widetilde{u}_h^+, v) \;=\; (f^+, v) + (g, v) \qquad \forall v \in V_h^+, \tag{4.10}$$

$$\widetilde{u}_h^- \in V_h^- \;:\; a^-(\widetilde{u}_h^-, v) \;=\; (f^-, v) + (g, v) \qquad \forall v \in V_h^-. \tag{4.11}$$

Standard error estimates results apply to $\widetilde{u}_h^+$, $\widetilde{u}_h^-$, so that

$$\begin{aligned} ||u^+ - \widetilde{u}_h^+||_{1,\Omega^+} &\leq C \inf\nolimits_{v_h \in V_h^+} ||u^+ - v_h||_{1,\Omega^+} \leq Ch|u^+|_{2,\Omega^+}, \\ ||u^+ - \widetilde{u}_h^+||_{1,\Omega^+} &\leq C \inf\nolimits_{v_h \in V_h^+} ||u^+ - v_h||_{1,\Omega^+} \leq Ch|u^-|_{2,\Omega^-}. \end{aligned} \tag{4.12}$$

By subtracting (4.10) from (4.4), (4.11) from (4.5) and using ellipticity we get

$$||u_h^+ - \widetilde{u}_h^+||_{1,\Omega^+} \leq C||g - g_h||_{0,S}, \qquad ||u_h^- - \widetilde{u}_h^-||_{1,\Omega^-} \leq C||g - g_h||_{0,S}. \tag{4.13}$$

Estimates (4.9) follow then by triangle inequality. In particular, estimates (4.12) imply

$$||T_f^+(f^+) - T_{f,h}^+(f^+)||_{1,\Omega^+} \leq Ch|u_f^+|_{2,\Omega^+}, \tag{4.14}$$

$$||T_f^-(f^-) - T_{f,h}^-(f^-)||_{1,\Omega^-} \leq Ch|u_f^-|_{2,\Omega^-}, \tag{4.15}$$

$$||T_g^+(g) - T_{g,h}^+(g)||_{1,\Omega^+} \leq Ch|u_g^+|_{2,\Omega^+}, \tag{4.16}$$

$$||T_g^-(g) - T_{g,h}^-(g)||_{1,\Omega^-} \leq Ch|u_g^-|_{2,\Omega^-}. \tag{4.17}$$

In view of (4.14)-(4.17) it will be convenient to introduce

$$[u]_2 := |u_f^+|_{2,\Omega^+} + |u_g^+|_{2,\Omega^+} + |u_f^-|_{2,\Omega^-} + |u_g^-|_{2,\Omega^-},$$

to be used, for simplicity, in the right-hand side of the various estimates. From (4.16)-(4.17) and Poincaré's inequality we then have

$$||K\mathcal{A}g - K\mathcal{A}_h g||_{0,S} \leq Ch(|u_g^+|_{2,\Omega^+} + |u_g^-|_{2,\Omega^-}) \leq Ch[u]_2. \tag{4.18}$$

The discrete analog of the minimum problem (2.13) becomes

$$\text{Find } g_h^* \in \Phi_h \;:\; J(g_h^*) \leq J(g) \qquad \forall g \in \Phi_h. \tag{4.19}$$

The discrete algorithm reads exactly as (3.1)-(3.7). More precisely:
Compute $u_{f,h}^+ = T_{f,h}^+(f^+)$, $u_{f,h}^- = T_{f,h}^-(f^-)$ solutions of

$$u_{f,h}^+ \in V_h^+ : \quad a^+(u_{f,h}^+, v) \;=\; (f^+, v) \qquad \forall v \in V_h^+, \tag{4.20}$$

$$u_{f,h}^- \in V_h^- : \quad a^-(u_{f,h}^-, v) \;=\; (f^-, v) \qquad \forall v \in V_h^-, \tag{4.21}$$

and set

$$g^0{}_h \;=\; K(u_{f,h}^+ + u_{f,h}^-)_{|\Sigma_h} \equiv K(T_{f,h}^+(f^+) + T_{f,h}^-(f^-))_{|\Sigma_h} (=: F_h). \tag{4.22}$$

For $m \geq 0$ compute the solutions $u_{m,h}^+ = T_{g,h}^+(g_h^m)$, $u_{m,h}^- = T_{g,h}^-(g_h^m)$ of the problems

$$u_{m,h}^+ \in V_h^+ \;: a^+(u_{m,h}^+, v) \;=\; (g_h^m, v) \qquad \forall v \in V_h^+, \tag{4.23}$$

$$u_{m,h}^- \in V_h^- \;: a^-(u_{m,h}^-, v) \;=\; (g_h^m, v) \qquad \forall v \in V_h^-. \tag{4.24}$$

Then set

$$\tilde{g}_h^m := g_h^m - K(u_{m,h}^+ + u_{m,h}^-)_{|\Sigma_h}, \tag{4.25}$$

$$g_h^{m+1} := g_h^m - \rho(\tilde{g}_h^m - g^0{}_h), \tag{4.26}$$

and compute the solutions $u_{m+1,h}^-$, $u_{m+1,h}^+$ of (4.23)-(4.26) with the new datum g_h^{m+1}. Exactly as for the continuous case we prove that, for all $\rho \in]0, 2[$,

$$\lim_{m \to \infty} g_h^m = g_h^*, \qquad g_h^* = F_h + K\mathcal{A}_h g_h^*, \tag{4.27}$$

following step by step the proof of Theorem 3.2, with the operator $\mathcal{A}$ replaced by $\mathcal{A}_h$ defined in (4.7). It remains now to prove that g_h^* converges to g^* for $h \to 0$.

Theorem 4.1 *Let g^* be defined in (2.18), and let g_h^* be defined in (4.27). The following result holds*

$$||g^* - g_h^*||_{0,S} \leq Ch[u]_2, \tag{4.28}$$

with C a constant independent of h.

Proof Using definitions (2.18), (4.27) and adding and subtracting $K\mathcal{A}_h g^*$ we have

$$\begin{aligned} &||g^* - g_h^*||_{0,S} \;= ||K\mathcal{A}g^* + F - K\mathcal{A}_h g_h^* - F_h||_{0,S} \\ &\leq ||K\mathcal{A}g^* - K\mathcal{A}_h g^*||_{0,S} + ||F - F_h||_{0,S} + ||K\mathcal{A}_h g^* - K\mathcal{A}_h g_h^*||_{0,S}. \end{aligned} \tag{4.29}$$

The first term in the right-hand side of (4.29) is bounded as in (4.18). For the second term we can use (4.14)-(4.15), giving

$$\begin{aligned} \|F - F_h\|_{0,S} &= \|K(T_f^+(f^+) - T_{f,h}^+(f^+)) + K(T_f^-(f^-) - T_{f,h}^-(f^-))\|_{0,S} \\ &\leq C(\|T_f^+(f^+) - T_{f,h}^+(f^+)\|_{1,\Omega^+} + \|T_f^-(f^-) - T_{f,h}^-(f^-)\|_{1,\Omega^-}) \\ &\leq Ch(|u_f^+|_{2,\Omega^+} + |u_f^-|_{2,\Omega^-}) \leq Ch[u]_2. \end{aligned} \tag{4.30}$$

For the last term we use (3.15), that holds also for the eigenvalues of $K\mathcal{A}_h$, independently of h. Hence,

$$\begin{aligned} \|K\mathcal{A}_h g^* - K\mathcal{A}_h g_h^*\|_{0,S} &\leq |||K\mathcal{A}_h|||\ \|g^* - g_h^*\|_{0,S} \\ &\leq (1 - C_1)\|g^* - g_h^*\|_{0,S}. \end{aligned} \tag{4.31}$$

Combining (4.18),and (4.30)-(4.31) in (4.29) we have the result □.

Remark 4.1 We point out that the arguments of this section apply straightforward to conforming finite element approximations of degree $r \geq 1$, provided that an H^{r+1} regularity in each subdomain can be used □.

5. Numerical results

Few general comments are in order for the actual implementation of the algorithm presented in Sect. 4:

- The finite element approximation (4.1)-(4.5) is a standard finite element formulation of linear elasticity problems, where the classical variational form is modified in order to take into account the Robin boundary conditions on S (see (2.21)-(2.22).) This amounts to add to the stiffness matrix of the boundary elements the contribution of a boundary integral.
- The algorithm described in (4.20)-(4.26) is a domain decomposition type algorithm which iteratively computes the "value of the jump of normal stresses on the interface" starting from an initial guess. The same data structures on the interface as in a standard domain decomposition algorithm can be used. At each iteration the local problems (4.23)-(4.24) have to be solved in each subdomain. We chose to use a direct method, namely the Cholesky factorization. As the matrix does not change from one iteration to another the factorization can be done once and for all. Then, during the iterative procedure, the solution of a linear system is reduced to a forward-backward substitution.

The finite element approximation of Sect. 4 has been implemented within the Modulef library. The numerical experiments have been performed on a plane strain linear elasticity problem. We study a typical single lap joint (see Fig. 2).
The materials are isotropic and the two adherents have the same material characteristics ($\nu = .3$ and $E = 200000MPa$). The bonding tensor $\mathbf{K}$ is given by:

$$\mathbf{K} = \frac{E^*}{2(1+\nu^*)} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2(1-\nu^*)}{(1-2\nu^*)} \end{pmatrix}$$

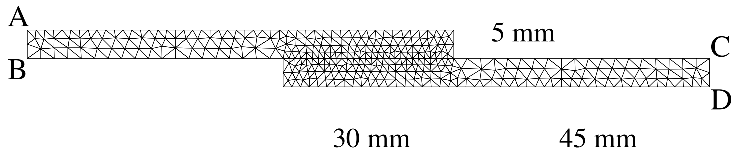

Fig. 2. Single lap joint. Mesh sample.

where $E^* = 1700MPa$ and $\nu^* = .35$. All over the boundary, except the edges AB and CD, stress free boundary conditions ($\sigma \cdot n = 0$) are taken. On the edges AB and CD the following conditions are considered: $(\sigma \cdot n)_y = 0$, $(\sigma \cdot n)_x = 1MPa$, and $u_y = 0$. (The additional condition $u_x = 0$ at the origin was imposed to guarantee uniqueness of the solution of the global problem.)

Our first aim is to evaluate the performance of the domain decomposition algorithm in terms of rate of convergence. To this end, the first test was performed on the discretization of Fig. 2. The mesh contains 630 elements and 418 nodes (836 degrees of freedom). As already pointed out in Remark 3.1, the optimal value for ρ is related to z_{max}, the (unknown) maximum eigenvalue of $K\mathcal{A}$. An estimate of z_{max} can be obtained by running the code with $\rho = 1$, and using the computed reduction factor $f(1)$ as an approximation of z_{max}. (See (3.22): $f(1) = z_{max}$.) Then we take $\rho = \frac{2}{2-f(1)}$ as an approximation of ρ_{opt}. Indeed, this is essentially equivalent to using the power method for computing the maximum eigenvalue of the iteration operator. It is however less expensive, in that z_{max} is computed within the same code, and the iterative procedure goes on with the updated value of ρ once z_{max} is computed. For the test case we found $z_{max} \simeq 0.97$, and consequently $\rho_{opt} \simeq 1.977$, giving a reduction factor per iteration $f(\rho_{opt}) \simeq 0.941$.

Then the initial mesh of Fig. 2 was successively refined taking mesh sizes $h/2$, $h/4$, $h/8$, h being the mesh size of the initial discretization. This produced meshes with 2520, 10080, and 40320 elements respectively. We observed that the number of iterations required to reduce by a factor 10^{-6} the L^2- norm of the initial residual $g_h^0 - KAg_h^0$ is virtually independent of h. In Table 1 the number of iterations is reported versus the number of unknowns on the interface. Note that a simple extrapolation procedure gives 487 iterations for the continous problem.

Table 1. Iterations versus unknowns

interface nodes	iterations
28	453
57	473
115	481
231	484

In Figs. 3-5 the Von Mises stresses on the four meshes are represented. The triangles are "coloured" differently according to the intensity of the stresses (constant on each triangle): light colour corresponds to high stresses. (For reasons of visibility different blow up around the critical region are shown.)

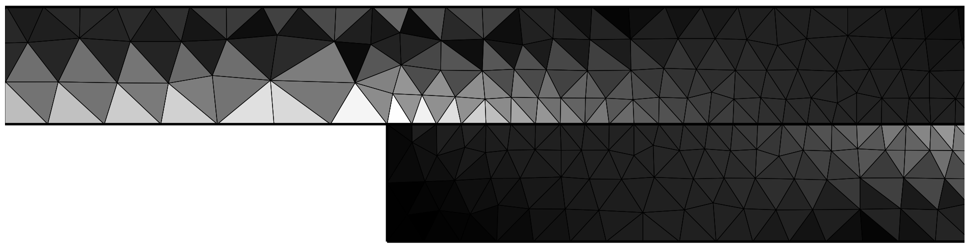

Fig. 3.

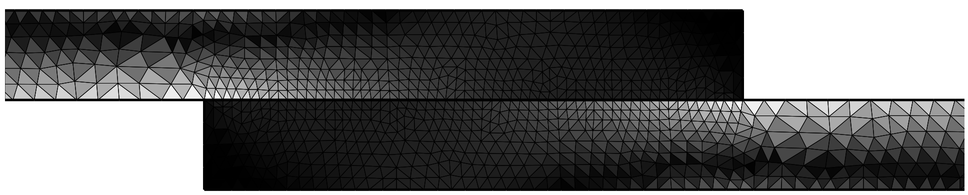

Fig. 4.

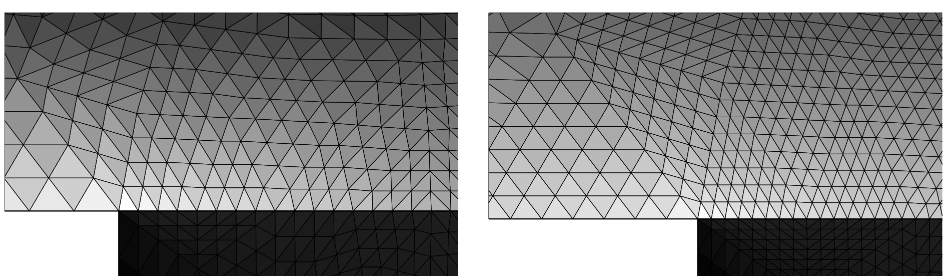

Fig. 5.

We notice that the coarse mesh gives already an accurate solution (the stresses and general behaviour do not change too much).
A second point of interest is the comparison of our domain decomposition algorithm with standard finite element methods applied to the whole real structure containing a thin adhesive layer of $0.1mm$ with material characteristics $E^* = 1700MPa$ and

$\nu^* = .35$. The presence of the thin adhesive layer can produce ill-conditioning (see, e.g., Refs. 19,21), generally avoided with special interface elements. Boundary conditions and material characteristics of the two adhering bodies are the same as before.

For this structure a mesh of 20268 elements (20700 d.o.f) was considered. The results obtained are shown in Figure 6 and are in good agreement with those obtained by the domain decomposition method. The mesh considered here is the coarsest we could obtain due to the different aspect ratios that we have to take into account: the width of the adhesive layer is 1/1200 of the total length of the lap joint, and 1/50 of the width of the adherents.

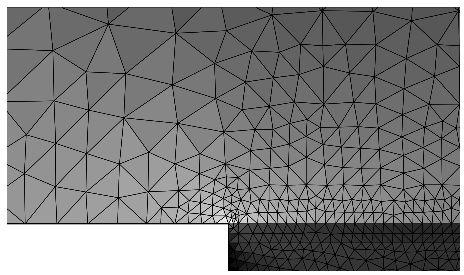

Fig. 6.

As expected, (see e.g. Ref. 1), all the computations show a stress concentration in the adherents at the end of the adhesive layer. This corresponds to a weak (logarithmic) singularity, as predicted for the limit (simplified) structure in Refs. 10 and 11. The boundary layer analysis of 10,11 also predicts that, in the real structure, the highest stress exists at the corner of the adhesive adjacent to the loaded adherent. It should howewer be noted that, because of the constraint imposed by the different aspect ratios, it is difficult to determine very accurately the stresses just below the surface. ∎

6. Conclusions

The method presented in this paper is very efficient and less expensive than a standard one where the adhesive is modeled with a thin layer. The implementation and use is straightforward. It is perhaps possible to improve the performance of the algorithm by properly preconditioning the interface system. Nevertheless, even if the number of iterations is rather large, the computational cost is very low compared to that of the standard method as far as CPU time is concerned.

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