New mixed finite element schemes for current continuity equations

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1. Introduction

In this paper we present examples of mixed finite element schemes for the current continuity equations of the drift-diffusion semiconductor model. We recall that, in the drift-diffusion model, a Poisson equation for the electric potential is coupled with two continuity equations for negative and positive charge densities. For a description of different iterative procedures to decouple the global system see, e.g. [6]. For simplicity, we shall deal only with the equation for the positive charge density p. After a suitable scaling [6], this equation has the form

$$\begin{cases} Find\ p \in H^1(\Omega)\ such\ that \\ -div(\underline{\nabla} p + p\underline{\nabla}\psi) = R(p,n) & in\ \Omega\ \subset R^2 \\ p = g & on\ \Gamma_0 \subset \partial\Omega \\ \dfrac{\partial p}{\partial n} + p\dfrac{\partial \psi}{\partial n} = 0 & on\ \Gamma_1 = \partial\Omega\backslash\Gamma_0 \end{cases} \tag{1.1}$$

where p and n are positive and negative charge densities respectively, ψ is the electric potential, R is the recombination-generation term, and the current density $\underline{J}$ is given by

$$\underline{J} = -\underline{\nabla} p - p\underline{\nabla}\psi. \tag{1.2}$$

We recall that, since $|\underline{\nabla}\psi|$ is quite large in some parts of the domain, equation (1.1) is an advection dominated equation, for which classical discretization methods may fail. We shall deal with discretizations of (1.1) assuming that ψ and n are known. Moreover, ψ is assumed to be piecewise linear (stemming from a discretization of the Poisson equation). During the iterative solution process, equation (1.1) is usually linearized in such a way that (1.1) becomes

$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{Find\ p \in H^1(\Omega)\ such\ that} \\ -div(\underline{\nabla} p + p\underline{\nabla}\psi)\ +\ cp & = & f & in\ \Omega\ \subset R^2 \\ p & = & g & on\ \Gamma_0 \subset \partial\Omega \\ \dfrac{\partial p}{\partial n} + p\dfrac{\partial \psi}{\partial n} & = & 0 & on\ \Gamma_1 = \partial\Omega \backslash \Gamma_0 \end{array}\right. \tag{1.3}$$

In equation (1.3) f is a function independent of p, and c a non negative function independent of p, which can be assumed piecewise constant. Using the classical change of variable from the charge density p to the Slotboom variable ρ

$$p \ = \ \rho e^{-\psi}\ , \tag{1.4}$$

equation (1.3) can be written in the symmetric form

$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{Find\ \rho\ \in H^1(\Omega)\ such\ that} \\ -div(e^{-\psi}\underline{\nabla}\rho)\ +\ ce^{-\psi}\rho & = & f & in\ \Omega \\ \rho\ =\ \chi & := & e^{\psi}g & on\ \Gamma_0 \\ \dfrac{\partial \rho}{\partial n} & = & 0 & on\ \Gamma_1 \end{array}\right. \tag{1.5}$$

and the current is now given by

$$\underline{J} \ = \ -e^{-\psi}\underline{\nabla}\rho\ . \tag{1.6}$$

The idea is to discretize equation (1.5) with mixed finite element methods, go back to the original variable p by using a discrete version of the transformation (1.4), and then solve for p.

For the case $c = 0$, a mixed scheme (based on lowest order Raviart-Thomas element [7]) has been introduced, extensively discussed in [2] for the case $f = 0$, and in [3] for $f \neq 0$. The scheme provides an approximate current with continuous normal component at the interelement boundaries. Moreover, the matrix associated with the scheme can be proved to be an M-matrix, if a weakly acute triangulation is used (every angle of every triangle is $\leq \pi/2$). This property guarantees a discrete maximum principle and, in particular, a non-negative solution if the boundary data are non-negative. Moreover, when going back to the variable p, this structure property of the matrix is preserved. Unfortunately, the M-matrix property does not hold anymore if $c > 0$. For that, we introduce here two new mixed finite elements which provide M-matrices for all non negative function c (if the triangulation is of weakly acute type). For the first presented scheme (see example 1 in section 3), the continuity at the interelement boundaries is slightly relaxed, in the sense that the jumps of the normal component of the approximate current have zero mean value. For the second scheme (see example 2 in section 3), strong continuity of the normal component of the current at the interelement boundaries is guaranteed. The two new elements are constructed according to the abstract theory of [5], which we refer to for the error analysis. In the present paper we describe the elements in detail, and exhibiting the elementary matrix associated with the problem.

2. Discretization

Let T_h be a regular decomposition of Ω into triangles T [4], and let E_h be the set of the edges e of T_h. (Ω is assumed to be a polygonal domain). We define, for all $\mathrm{T} \in T_h$, the following set of polynomial vectors:

$$\Sigma(\mathrm{T}) \;=\; \mathrm{span}\;\{\underline{\tau}^1,\; \underline{\tau}^2,\; \underline{\tau}^3\}\;, \tag{2.1}$$

with

$$\underline{\tau}^1 \;=\; (1,0)\;\;,\;\; \underline{\tau}^2 \;=\; (0,1)\;\;,\;\; \underline{\tau}^3 \;=\; (\omega_1, \omega_2). \tag{2.2}$$

The polynomials $\omega_1,\ \omega_2$ have to satisfy the requirement:

$$\int_{\mathrm{T}} div\ \underline{\tau}^3 dxdy \;\neq\; 0\;. \tag{2.3}$$

In the next section we shall specify proper choices of $\omega_1,\ \omega_2$. Here, we only remark that

$$dim\ (div\ \Sigma(\mathrm{T})) \;=\; 1\;. \tag{2.4}$$

Then, we construct our finite element spaces as follows:

$$V_h \;=\; \{\underline{\tau} \in [L^2(\Omega)]^2 : \underline{\tau}_{|\mathrm{T}} \in \Sigma(\mathrm{T}),\ \forall \mathrm{T} \in T_h\}, \tag{2.5}$$

$$W_h = \{\phi \in L^2(\Omega) : \phi_{|\mathrm{T}} \in P_0(\mathrm{T})\ \forall \mathrm{T} \in T_h\}, \tag{2.6}$$

$$\Lambda_{h,\xi} = \{\mu \in L^2(E_h) : \mu_{|e} \in P_0(e)\ \forall e \in E_h\ ;\ \int_e (\mu - \xi) ds = 0\ \forall e \subset \Gamma_0\}\ , \tag{2.7}$$

where ξ is any given function in $L^2(\Gamma_0)$ and $P_0(K)$ denotes the space of constants on the set K. The mixed-hybrid formulation of (1.5) is then:

$$\begin{cases} \textit{Find } \underline{J}_h \in V_h\ ,\ \rho_h \in W_h\ ,\ \lambda_h \in \Lambda_{h,\chi} \textit{ such that :} \\ \int_\Omega e^{\overline{\psi}} \underline{J}_h \cdot \underline{\tau} dxdy - \sum_{\mathrm{T}} \int_{\mathrm{T}} div\ \underline{\tau}\ \rho_h dxdy + \sum_{\mathrm{T}} \int_{\partial \mathrm{T}} \lambda_h \underline{\tau} \cdot \underline{n}\ ds = 0 & \underline{\tau} \in V_h, \\ \sum_{\mathrm{T}} \int_{\mathrm{T}} div\ \underline{J}_h\ \phi dxdy + \int_\Omega c e^{-\widetilde{\psi}} \rho_h \phi dxdy = \int_\Omega f \phi dxdy & \phi \in W_h, \\ \sum_{\mathrm{T}} \int_{\partial \mathrm{T}} \mu \underline{J}_h \cdot \underline{n}\ ds = 0 & \mu \in \Lambda_{h,0}. \end{cases} \tag{2.8}$$

In the first equation of (2.8) $\overline{\psi}$ denotes the piecewise constant function defined in each triangle T by

$$e^{\overline{\psi}}{}_{|\mathrm{T}} = (\int_{\mathrm{T}} e^{\psi} dxdy)/|\mathrm{T}|\ . \tag{2.9}$$

In the second equation of (2.8) $\widetilde{\psi}$ denotes the piecewise constant function defined in each triangle T via a harmonic average:

$$e^{-\widetilde{\psi}}{}_{|\mathrm{T}} = |\tilde{e}| / \int_{\tilde{e}} e^{\psi} ds\ , \tag{2.10}$$

(where $\tilde{e}$ is an edge where ψ reaches its maximum. More precisely, since ψ is assumed linear in each T, two possibilities arise (apart from the trivial case $\psi = constant$ on T). If ψ reaches its maximum on an edge, that edge is chosen in (2.10). If the maximum is attained at one vertex, any of the two edges having that vertex in commom can be taken. The reason for this choice will be clear in the next section. Due to (2.3) and (2.4), the choice of the spaces V_h, W_h, and $\Lambda_{h,\chi}$ is such that the abstract theory of [5] applies. Hence, problem (2.8) has a unique solution ($\underline{J}_h$, ρ_h, λ_h). Moreover, $\underline{J}_h$ is an approximation of the current $\underline{J}$, ρ_h is an approximation of ρ, and λ_h is an approximation of ρ at the interelement boundaries, as proved in [1], [5]. The first equation of (2.8) is a weak discrete version of (1.6). The second equation is the discrete version of $div\underline{J} + ce^{-\psi}\rho = f$. Finally, the third equation of (2.8) imposes a continuity requirement of the normal component of $\underline{J}_h$ at the interelement boundaries. More precisely, since $\mu \in \Lambda_{h,0}$ is constant on each edge e, the jump of $\underline{J}_h \cdot \underline{n}$ across e has zero mean value. If then $\underline{J}_h \cdot \underline{n}$ itself is constant (and this is the case for the element described in example 2), the normal component is continuous across the interelement boundaries. For error estimate results see, e.g., [5] and the references therein.

The linear system associated with (2.8) can be written in matrix form as follows:

$$\begin{pmatrix} A & -B & C \\ -B^* & -D & 0 \\ C^* & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{J}_h \\ \rho_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} 0 \\ -F \\ 0 \end{pmatrix} . \tag{2.11}$$

In (2.11) the notation $\underline{J}_h, \rho_h, \lambda_h$ is used also for the vectors of the nodal values of the corresponding functions. The matrix in (2.11) is not positive definite. However, A is block- diagonal (each block being a 3x3 matrix corresponding to a single element T) and can be easily inverted at the element level. Hence, the variable $\underline{J}_h$ can be eliminated by static condensation, leading to the new system

$$\begin{pmatrix} B^*A^{-1}B + D & -B^*A^{-1}C \\ & \\ -C^*A^{-1}B & C^*A^{-1}C \end{pmatrix} \begin{pmatrix} \rho_h \\ \\ \lambda_h \end{pmatrix} = \begin{pmatrix} F \\ \\ 0 \end{pmatrix} . \tag{2.12}$$

Now the matrix is symmetric and positive definite. Moreover, the matrix $B^*A^{-1}B + D$ is diagonal, so that the variable ρ_h can also be eliminated by static condensation. This leads to a final system, acting on the unknown λ_h only, of the form

$$M\, \lambda_h = G , \tag{2.13}$$

where M and G are given by:

$$M = C^*A^{-1}C - C^*A^{-1}B(B^*A^{-1}B + D)^{-1}B^*A^{-1}C , \tag{2.14}$$

$$G = C^*A^{-1}B(B^*A^{-1}B + D)^{-1}F , \tag{2.15}$$

and M is symmetric and positive definite.

In order to go back to the original unknown p we recall that λ_h is an approximation of ρ and we can use a discrete version of the inverse transform of (1.4):

$$\lambda_h = (e^{\psi})^I p_h. \tag{2.16}$$

In (2.16) $(e^\psi)^I$ is given edge by edge by the meanvalue of e^ψ. The transformation (2.16) amounts to multiplying the matrix M columnwise by the value of $(e^\psi)^I$ on the corresponding edge. The final system in the unknown p_h will be of the type

$$\widetilde{M} p_h \; = \; G \; . \tag{2.17}$$

The matrix $\widetilde{M}$ is not symmetric anymore, but it is an M-matrix if the matrix (2.14) is an M-matrix, which holds true if the triangulation is of weakly acute type, and if ω_1, ω_2 are properly chosen.

3. Examples

We give here two possible choices of $\Sigma(\mathrm{T})$, that is, of $\underline{\tau}^3$. Computations will be carried out directly on the current triangle T. For the notation see fig.1. $\underline{n}^i$ is the outward unit normal to the edge e_i, and $\underline{t}^i$ is the unit tangent: $(t_1^i, t_2^i) = (-n_2^i, n_1^i)$. Given e_1, the edges denoted by e_2 and e_3 are determined by the counterclockwise numbering.

fig. 1

Example 1 . Let us denote by e_1 the edge where ψ assumes its maximum, according to (2.10). Then, choose $\underline{\tau}^3 \; = (\omega_1, \omega_2)$ with $\omega_1, \; \omega_2 \in P_1(\mathrm{T})$ determined by the following degrees of freedom

$$\int_{e_2} \underline{\tau}^3 \cdot \underline{n}^2 ds \; = \; \int_{e_3} \underline{\tau}^3 \cdot \underline{n}^3 ds \; = \; 0 \; , \qquad \int_{e_1} \underline{\tau}^3 \cdot \underline{n}^1 ds \; = \; |e_1| \; , \tag{3.1}$$

$$\int_{\mathrm{T}} \omega_1 \; dxdy \; = \; \int_{\mathrm{T}} \omega_2 \; dxdy \; = 0 \; . \tag{3.2}$$

The 5 degrees of freedom (3.1)-(3.2) determine a one dimensional manifold. Namely, $\underline{\tau}^3$ can be written in terms of the barycentric coordinates $\lambda_i, (i = 1, 3)$ in the form

$$\underline{\tau}^3 \;=\; a(\lambda_2 - \lambda_1)\underline{t}^2 \;+\; \frac{2 - a(\underline{t}^2 \cdot \underline{n}^1)}{\underline{t}^3 \cdot \underline{n}^1}(\lambda_3 - \lambda_1)\underline{t}^3 \quad . \tag{3.3}$$

Then, $\underline{\tau}^3$ can be chosen, for instance, as the element of minimum norm. This choice yields

$$a \;=\; \frac{2(\underline{t}^2 \cdot \underline{n}^1) - (\underline{t}^3 \cdot \underline{n}^1)(\underline{t}^2 \cdot \underline{t}^3)}{(\underline{t}^3 \cdot \underline{n}^1)^2 + (\underline{t}^2 \cdot \underline{n}^1)^2 - (\underline{t}^2 \cdot \underline{n}^1)(\underline{t}^3 \cdot \underline{n}^1)(\underline{t}^2 \cdot \underline{t}^3)} \quad . \tag{3.4}$$

Example 2 . As for the previous case, let e_1 be the edge where ψ reaches its maximum, according to (2.10). Then, choose $\underline{\tau}^3 = (\omega_1, \omega_2)$ with $\omega_1,\ \omega_2 \in P_2(\mathrm{T})$ determined by the following degrees of freedom

$$\underline{\tau}^3 \cdot \underline{n}^2{}_{|e_2} \;=\; \underline{\tau}^3 \cdot \underline{n}^3{}_{|e_3} \;=\; 0\ , \qquad \underline{\tau}^3 \cdot \underline{n}^1{}_{|e_1} \;=\; 1\ , \tag{3.5}$$

$$\int_{\mathrm{T}} \omega_1 \; dxdy \;=\; \int_{\mathrm{T}} \omega_2 \; dxdy \;= 0 \tag{3.6}$$

The 11 degrees of freedom (3.5)-(3.6) determine a one dimensional manifold. As in the previous case, $\underline{\tau}^3$ can be written in terms of barycentric coordinates $\lambda_i, (i = 1, 3)$

$$\begin{aligned} \underline{\tau}^3 \;=\; & \left[\frac{1}{\underline{t}^2 \cdot \underline{n}^1}\lambda_3^2 + a\lambda_1\lambda_3 - (\frac{2}{\underline{t}^2 \cdot \underline{n}^1} + a)\lambda_2\lambda_3) \right] \underline{t}^2 \\ & + \left[\frac{1}{\underline{t}^3 \cdot \underline{n}^1}\lambda_2^2 \;-\; (\frac{6}{\underline{t}^3 \cdot \underline{n}^1} + \frac{\underline{t}^2 \cdot \underline{n}^1}{\underline{t}^3 \cdot \underline{n}^1}a)\lambda_1\lambda_2 \;+\; (\frac{4}{\underline{t}^3 \cdot \underline{n}^1} + \frac{\underline{t}^2 \cdot \underline{n}^1}{\underline{t}^3 \cdot \underline{n}^1}a)\lambda_2\lambda_3 \right] \underline{t}^3 \end{aligned} \tag{3.7}$$

The last *d.o.f.* can be determined again by choosing among all elements of the form (3.7) that with minimum norm. Then, we obtain

$$a \;=\; -\frac{1}{\underline{t}^2 \cdot \underline{n}^1} \left[\frac{5(\underline{t}^2 \cdot \underline{n}^1)^2 - 3(\underline{t}^2 \cdot \underline{n}^1)(\underline{t}^3 \cdot \underline{n}^1)(\underline{t}^2 \cdot \underline{t}^3) + (\underline{t}^3 \cdot \underline{n}^1)^2}{(\underline{t}^3 \cdot \underline{n}^1)^2 + (\underline{t}^2 \cdot \underline{n}^1)^2 - (\underline{t}^2 \cdot \underline{n}^1)(\underline{t}^3 \cdot \underline{n}^1)(\underline{t}^2 \cdot \underline{t}^3)} \right] \quad . \tag{3.8}$$

■

As already pointed out, the theory for both elements is set up in [5]. Here we present in detail the structure of the matrix M defined in (2.14). We construct the elementary matrix M^{T} and the elementary right hand side G^{T} associated with the current element T. As basis functions $\phi \in W_h$ and $\mu \in \Lambda_{h,0}$, we make the natural choice: $\phi = 1$ in T and $\phi = 0$ elsewhere; $\mu = 1$ on one edge e and $\mu = 0$ on the others. We introduce the following notation

$$\underline{\nu}^i = \underline{n}^i |e_i| \ , \tag{3.9}$$

$$\alpha = \int_{\mathrm{T}} (\omega_1^2 + \omega_2^2)\ dxdy \ , \tag{3.10}$$

$$\gamma = \int_{e_1} \underline{\tau}^3 \cdot \underline{n}^1 ds \tag{3.11}$$

$$\beta(c) = \gamma^2 |\mathrm{T}| / (\gamma^2 + \alpha c |\mathrm{T}| e^{\overline{\psi} - \widetilde{\psi}}) \quad . \tag{3.12}$$

Then we have:

$$A^{\mathrm{T}} = e^{\overline{\psi}} \begin{pmatrix} |\mathrm{T}| & 0 & 0 \\ 0 & |\mathrm{T}| & 0 \\ 0 & 0 & \alpha \end{pmatrix} \qquad B^{\mathrm{T}} = \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} \tag{3.13}$$

$$C^{\mathrm{T}} = \begin{pmatrix} \nu_1^1 & \nu_1^2 & \nu_1^3 \\ \nu_2^1 & \nu_2^2 & \nu_2^3 \\ \gamma & 0 & 0 \end{pmatrix} \qquad D^{\mathrm{T}} = c|\mathrm{T}| e^{-\widetilde{\psi}} \qquad F^{\mathrm{T}} = \int_{\mathrm{T}} f dxdy \tag{3.14}$$

The coefficients of the matrix M^{T} are then given by:

$$m_{ij}^{\mathrm{T}} = \begin{cases} e^{-\overline{\psi}} \frac{\underline{\nu}^1 \cdot \underline{\nu}^1}{|\mathrm{T}|} \ + \ e^{-\widetilde{\psi}} c \beta(c) & if \ i = j = 1, \\ e^{-\overline{\psi}} \frac{\underline{\nu}^i \cdot \underline{\nu}^j}{|\mathrm{T}|} & otherwise, \end{cases} \tag{3.15}$$

and the coefficients of the right-hand side are:

$$g_i^{\mathrm{T}} = \begin{cases} \beta(c) \frac{1}{|\mathrm{T}|} \int_{\mathrm{T}} f dxdy & if \ i = 1, \\ 0 & if \ i = 2, 3 \ . \end{cases} \tag{3.16}$$

It is easy to check that $\beta(c) > 0$, for $c \geq 0$. Hence, if the triangulation is of weakly acute type, we have

$$m_{ii}^{\mathrm{T}} > 0 \quad , \quad m_{ij}^{\mathrm{T}} \leq 0 \quad , i \neq j \quad i, j = 1, 3 \ . \tag{3.17}$$

Therefore, the matrix M^{T} is a symmetric positive definite diagonally dominant M-matrix. Then, the final matrix obtained by summing the contributions of the elementary matrices over the triangles is still a symmetric positive definite diagonally dominant M-matrix.

When going back to the variable p_h, then the transformation (2.16) is applied. Notice that the definition (2.10) of $\widetilde{\psi}$ implies

$$\left(e^{\psi}\right)^{I}_{|e_1} \equiv e^{\widetilde{\psi}} \quad . \tag{3.18}$$

Then, the coefficients of the elementary stiffness matrix have the form

$$\widetilde{m}^{\mathrm{T}}_{ij} = \begin{cases} \left(e^{\psi}\right)^{I}_{|e_1} \; e^{-\overline{\psi}} \left[\frac{\underline{\nu}^1 \cdot \underline{\nu}^1}{|\mathrm{T}|}\right] + c\beta(c) & if\ i = j = 1, \\ \left(e^{\psi}\right)^{I}_{|e_j} \; e^{-\overline{\psi}} \left[\frac{\underline{\nu}^i \cdot \underline{\nu}^j}{|\mathrm{T}|}\right] & otherwise. \end{cases} \tag{3.19}$$

As already pointed out, $|\underline{\nabla}\psi|$ can be quite large in a part of the domain, so that the presence of exponentials in the coefficients might be a source of numerical problems. In order to exploit the behaviour of the coeffients it is more convenient to set

$$\psi = \frac{\psi_0}{l} \tag{3.20}$$

and assume that $\underline{\nabla}\psi_0$ is smooth everywhere and l is a small number. Accordingly, equation (1.3) becomes

$$-div(\underline{\nabla}p + p\frac{\underline{\nabla}\psi_0}{l}) + cp = f \quad . \tag{3.21}$$

The nature of equation (3.21) is such that, as $l \to 0$, the higher order term behaves like l^{-1}, while the zero order term is of order 1. Hence, for very small l (say $l \ll |\underline{\nabla}\psi_0|h_{\mathrm{T}}$), our discrete scheme must reproduce the behaviour of the continuous equation (3.21). To check that, recall that ψ (and then ψ_0) is assumed piecewise linear, and denote by ψ^M the maximum of ψ on T. We only consider the generic case where the maximum is reached at one vertex. When $l \ll |\underline{\nabla}\psi_0|h_{\mathrm{T}}$, a simple computation shows that

$$e^{\overline{\psi}} = \frac{1}{|\mathrm{T}|}\int_{\mathrm{T}} e^{\psi_0/l}dxdy \simeq l^2 e^{\psi_0^M/l} = l^2 e^{\psi^M} \tag{3.22}$$

$$\left(e^{\psi}\right)^{I}_{|e_j} = \frac{1}{|e_j|}\int_{e_j} e^{\psi_0/l}ds \simeq l e^{\psi_0^{M_j}/l} = l e^{\psi^{M_j}} \tag{3.23}$$

where ψ^{M_j} denotes the maximum of ψ on the edge e_j. Then we have

$$\left(e^{\psi}\right)^I_{|e_j} \; e^{-\overline{\psi}} \;\simeq\; \begin{cases} l^{-1} & if \; \psi^{M_j} = \psi^M \\ 0 & otherwise \end{cases} \tag{3.24}$$

Hence, recalling the definition of the edge e_1 as the edge where ψ assumes its maximum, the coefficients (3.19) behave like

$$\widetilde{m}^{\mathrm{T}}_{11} \;\simeq\; l^{-1}\frac{\underline{\nu}^1 \cdot \underline{\nu}^1}{|\mathrm{T}|} \;+\; c\beta(c) \quad , \tag{3.25}$$

and, for $i \neq 1, \; j \neq 1$,

$$\widetilde{m}^{\mathrm{T}}_{ij} \;=\; \begin{cases} l^{-1}\frac{\underline{\nu}^i \cdot \underline{\nu}^j}{|\mathrm{T}|} & if \; \psi^{M_j} = \psi^M, \\ 0 & otherwise. \end{cases} \tag{3.26}$$

Moreover, due to (3.24), (3.18) and to the choice of e_1, notice that

$$e^{\overline{\psi}-\widetilde{\psi}} \simeq l \quad , \tag{3.27}$$

hence

$$\beta(c) \;\simeq\; \gamma^2|\mathrm{T}|/(\gamma^2 + \alpha c|\mathrm{T}|l) \quad . \tag{3.28}$$

The reason of the choice (2.10) is now clear. The expected behaviour in terms of the order of magnitude with respect to l is preserved, and, moreover, no bad blow-up occurs. Different (and maybe more natural) choices for $e^{-\widetilde{\psi}}$ could lead to a coefficient for the zero order term in which $\left(e^{\psi}\right)^I_{|e_1} \; e^{-\widetilde{\psi}}$ does not cancel. Then, the presence of this factor could give rise to a scheme whose structure does not fit the structure of the continuous problem, and whose solution does not converge to the solution of the continuous problem.

The expression (3.24) tells us that whenever $|\underline{\nabla}\psi|$ is large, the coefficient corresponding to the node on the edge where ψ does not reach its maximum is zero (with respect to the machine precision). Such a node can be regarded as downwind node (wind=-$\underline{\nabla}\psi$) and the scheme as an upwind scheme. In a sense, the scheme automatically adapts to the changed nature of the problem when advection becomes bigger than diffusion, and chooses the upwind nodes with no extra computational cost.

Remark Since ψ is assumed linear in each triangle T, the integrals which define $e^{\overline{\psi}}$, $e^{\widetilde{\psi}}$ in (2.9)-(2.10) can be computed exactly. In the computations, the definition of $\tilde{e}$ as the edge which connects the vertex with the largest potential value and the vertex with the second largest potential value allows a unique definition. For the computation of $\int_{\mathrm{T}} f dxdy$ in (3.16) a quadrature formula which is exact for constant f can be used.

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5. References

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